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On Strongly m-Convex Functions

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Abstract

The main purpose of this research is to introduce the definition of a strongly m-convex function. To achieve this goal, we generalize the well known notion of a strongly convex function by following a similar procedure to the one employed to generate the notion of functional m-convexity from the classical functional convexity. Several properties of this new class of functions are established as well as some inequalities of Jensen type in the discrete case. In the course of this study, we prove some additional and interesting results for the bigger class of m-convex functions.

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1 Introduction

The concepts of m-convex and strongly convex functions were introduced in [10] and [8], respectively. There are several papers ([1, 2, 4, 5, 10]) in which we can find some results such as algebraic properties, inequalities of different type, among others. In this research we combine both definitions in one, and we establish and prove some properties for this type of functions.

Definition 1.1 *([1, 2, 10]) A function $f : [0, b] \to \mathbb{R}$ is called m-convex, $0 \le m \le 1$, if for any x, $y \in [0, b]$ and $t \in [0, 1]$ we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1}$$

Remark 1.2 *It is important to point out that the above definition is equivalent to $f(mtx + (1-t)y) \le mtf(x) + (1-t)f(y)$, x, y and t as before.*

Remark 1.3 *If f is an m-convex function, with $m \in [0, 1)$ then we take $x = y = 0$ in (1) and get $f(0) \le 0$.*

In the same way, we may define the concept of a strongly convex function.

Definition 1.4 *([5]) Let $I \subset \mathbb{R}$ be an interval and c be a positive real number. A function $f : I \to \mathbb{R}$ is said to be strongly convex with modulus c if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2, \tag{2}$$

with $x, y \in I$ and $t \in [0, 1]$.

Strongly convex functions have been introduced by Polyak in [8]. Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just "stronger versions" of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient type algorithm for minimizing a function. They play an important role in optimization theory and mathematical economics ([7, 9]).

Now we introduce a new definition which combine the two given above, let $I \subseteq [0, +\infty)$, c be a positive real number and $m \in [0, 1]$.

Definition 1.5 *A function $f : I \to \mathbb{R}$ is called strongly m-convex with modulus c if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) - cmt(1-t)(x-y)^2, \tag{3}$$

with $x, y \in I$ and $t \in [0, 1]$.

Remark 1.6 *Notice that for $m = 1$ the definition of strongly convex function is recasted. Moreover, if f is strongly m-convex with modulus c, then f is strongly m-convex with modulus k, for any constant $0 < k < c$.*

Remark 1.7 *Any strongly m-convex function is, in particular, m-convex. However, there are m-convex functions, which are not strongly m-convex with modulus c, for some $c > 0$.*

Example 1.8 *The function $f : [0, +\infty) \to \mathbb{R}$, given by*

$$f(x) = \frac{1}{12}(x^4 - 5x^3 + 9x^2 - 5x)$$

is $\frac{16}{17}$-convex ([6, 11]); therefore, f is $\frac{1}{2}$-convex (Lemma 2, [1]). Nevertheless, for any $c > \frac{1}{3}$, f is not strongly $\frac{1}{2}$-convex with modulus c. In fact, if f were strongly $\frac{1}{2}$-convex with modulus c, then for all $x, y \geq 0$ and $t \in [0, 1]$,

$$f(tx + \frac{1}{2}(1-t)y) \leq tf(x) + \frac{1}{2}(1-t)f(y) - c\frac{1}{2}t(1-t)(x-y)^2;$$

in particular, by taking $x = 1, y = 2$ and $t = \frac{1}{2}$, we get

$$0 = f(1) \leq \frac{1}{2}f(1) + \frac{1}{4}f(2) - \frac{1}{8}c = \frac{1}{24} - \frac{1}{8}c,$$

contradicting the fact that $c > \frac{1}{3}$.

2 Main Results

In [4], some basic properties for m-convex functions are proven. Now we state some others results for them.

Proposition 2.1 *If $f_1, f_2 : [0, b] \to \mathbb{R}$ are m-convex functions, then the function given by $f(x) = \max\limits_{x \in [0,b]}\{f_1(x), f_2(x)\}$ is also m-convex.*

Proof. If $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f_1(tx + m(1-t)y) \leq tf_1(x) + m(1-t)f_1(y) \leq tf(x) + m(1-t)f(y)$$

and

$$f_2(tx + m(1-t)y) \leq tf_2(x) + m(1-t)f_2(y) \leq tf(x) + m(1-t)f(y).$$

Whence

$$f(tx+m(1-t)y) = \max\{f_1(tx+m(1-t)y), f_2(tx+m(1-t)y)\} \leq tf(x)+m(1-t)f(y).$$

Proposition 2.2 *If $f_n : [0, b] \to \mathbb{R}$ is a sequence of m-convex functions converging pointwise to a function f on $[0, b]$, then f is m-convex.*

Proof. If $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$\begin{aligned} f(tx + m(1-t)y) &= \lim_{n\to\infty} f_n(tx + m(1-t)y) \\ &\le \lim_{n\to\infty} (tf_n(x) + m(1-t)f_n(y)) \\ &= tf(x) + m(1-t)f(y). \end{aligned}$$

We establish now some properties for strongly m-convex functions.

Proposition 2.3 *Let $f : [0, b] \to \mathbb{R}$ be a strongly m-convex function with modulus c. If $0 \le n < m < 1$, then f is strongly n-convex with modulus c.*

Proof. If $n = 0$, the proof is trivial (from Remark 1.3 follows that f is a starshaped function i.e., $f(tx) \le tf(x)$ for all $x \in [0, b]$ and for all $t \in [0, 1]$). We consider now $0 < n < m < 1$.

Let $x, y \in [0, b]$ with $y \le x$. Then,

$$\begin{aligned} f(tx + n(1-t)y) &= f\left(tx + m(1-t)\left(\frac{n}{m}y\right)\right) \\ &\le tf(x) + m(1-t)f\left(\frac{n}{m}y\right) - cmt(1-t)\left(x - \frac{n}{m}y\right)^2 \\ &\le tf(x) + m(1-t)\left(\frac{n}{m}\right)f(y) - cnt(1-t)\left(\frac{m}{n}\right)\left(x - \frac{n}{m}y\right)^2 \\ &\le tf(x) + n(1-t)f(y) - cnt(1-t)\left(x - \frac{n}{m}y\right)^2. \end{aligned}$$

Furthermore, since $n < m$, we have $x - \frac{n}{m}y \ge x - y \ge 0$. Thus, $(x - \frac{n}{m}y)^2 \ge (x-y)^2$ and hence,

$$f(tx + n(1-t)y) \le tf(x) + n(1-t)f(y) - cnt(1-t)(x-y)^2.$$

Now let us see the case $x < y$. As in Remark 1.2, it is clear that (3) in definition 1.5, is equivalent to

$$f(mtx + (1-t)y) \le mtf(x) + (1-t)f(y) - cmt(1-t)(x-y)^2. \quad (4)$$

Thus, for $x < y$,

$$\begin{aligned} f(ntx + (1-t)y) &= f\left(mt\left(\frac{n}{m}x\right) + (1-t)y\right) \\ &\le mtf\left(\frac{n}{m}x\right) + (1-t)f(y) - cmt(1-t)\left(\frac{n}{m}x - y\right)^2 \\ &\le ntf(x) + (1-t)f(y) - cnt(1-t)\left(\frac{m}{n}\right)\left(\frac{n}{m}x - y\right)^2 \\ &\le ntf(x) + (1-t)f(y) - cnt(1-t)\left(\frac{n}{m}x - y\right)^2. \end{aligned}$$

Since $n < m$, we have $\frac{n}{m}x - y \leq x - y < 0$. Therefore, $(\frac{n}{m}x - y)^2 \geq (x - y)^2$ and thus,

$$f(ntx + (1-t)y) \leq ntf(x) + (1-t)f(y) - cnt(1-t)(x-y)^2.$$

From (4) and the previous inequality follows that f is a strongly n-convex function with modulus c.

Proposition 2.4 *Let $m_1 \leq m_2 \neq 1$ and $f, g : [a, b] \to \mathbb{R}$, $a \geq 0$. If f is strongly m_1-convex with modulus c_1 and g is strongly m_2-convex with modulus c_2, then $f + g$ is strongly m_1-convex with modulus $c_1 + c_2$.*

Proof. Since g is strongly m_2-convex with modulus c_2 and $m_1 \leq m_2$, then by Proposition 2.3, g is strongly m_1-convex with modulus c_2. Thus, for $x, y \in [a, b]$ and $t \in [0, 1]$,

$$\begin{aligned}(f+g)(tx + m_1(1-t)y) &= f(tx + m_1(1-t)y) + g(tx + m_1(1-t)y)\\ &\leq tf(x) + m_1(1-t)f(y) - c_1m_1t(1-t)(x-y)^2 + tg(x) + m_1(1-t)g(y)\\ &\qquad - c_2m_1t(1-t)(x-y)^2\\ &= t(f+g)(x) + m_1(1-t)(f+g)(y) - (c_1+c_2)m_1t(1-t)(x-y)^2.\end{aligned}$$

Remark 2.5 *According to Remark 1.6 and Proposition 2.4, $f+g$ is strongly m_1-convex with modulus c_1 and c_2.*

Proposition 2.6 *If $f : [0, b] \to \mathbb{R}$ is strongly m-convex with modulus c and $\alpha > 0$, then αf is strongly m-convex with modulus αc. In particular, if $\alpha \geq 1$, αf is strongly m-convex with modulus c.*

Proof. For $x, y \in [0, b]$ and $t \in [0, 1]$,

$$\begin{aligned}(\alpha f)(tx + m(1-t)y) &\leq \alpha[tf(x) + m(1-t)f(y) - cmt(1-t)(x-y)^2]\\ &= t(\alpha f)(x) + m(1-t)(\alpha f)(y) - \alpha cmt(1-t)(x-y)^2.\end{aligned}$$

Definition 2.7 *([12]) Two functions f, g are said to be similarly ordered on I if*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0, \tag{5}$$

for all $x, y \in I$.

Proposition 2.8 *Let $f : [0, b] \to \mathbb{R}$ be an m-convex function, $g : [0, b_1] \to \mathbb{R}$ strongly m-convex with modulus c and range$(f) \subseteq [0, b_1]$. If g is nondecreasing and the functions $f - id$ and $f + id$ are similarly ordered on $[0, b]$, where id is the identity function, then $g \circ f$ is strongly m-convex with modulus c.*

Proof. Because f is m-convex, for all $x, y \in [0, b]$ and $t \in [0, 1]$,

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Since g is nondecreasing and strongly m-convex with modulus c, we have

$$\begin{aligned} g(f(tx + m(1-t)y)) &\leq g(tf(x) + m(1-t)f(y)) \\ &\leq tg(f(x)) + m(1-t)g(f(y)) - cmt(1-t)(f(x) - f(y))^2. \end{aligned}$$

Since $f - id$ and $f + id$ are similarly ordered, by (5) we get

$$[f(x) - x - (f(y) - y)][f(x) + x - (f(y) + y)] \geq 0,$$

or $(f(x) - f(y))^2 \geq (x - y)^2$. Indeed,

$$(g \circ f)(tx + m(1-t)y) \leq t(g \circ f)(x) + m(1-t)(g \circ f)(y) - cmt(1-t)(x - y)^2.$$

In [4], it was proved that if f, g are both nonnegative, increasing and m-convex functions, then the product function fg is also m-convex. However, in case of strong m-convexity, this is not necessarily true.

Example 2.9 *Let $c > 0$ and $m \in (0, 1]$ be given. The quadratic function $f : [0, b] \to \mathbb{R}$ $(b > c)$ defined by $x \mapsto x^2$ is strongly m-convex with modulus 1, but its square $g = f^2$ is not strongly m-convex with modulus c. Indeed, since the null function is m-convex, by Proposition 2.13, the function f is clearly strongly m-convex with modulus 1, whence half of the work is done. On the other hand, if g were a strongly m-convex function with modulus c, then inequality (3) would be true in particular for $t_0 = \frac{m}{m+1} \in [0, 1]$, $x_0 = 0$, $y_0 = \sqrt{ct_0} \in [0, b]$ and f replaced by g. By calculating appropriately with the mentioned values inequality (3) turns into $c^2 t_0^6 \leq 0$ which is not true. So, $g(x) = x^4$ for all $x \in [0, b]$ cannot be strongly m-convex with modulus c.*

Remark 2.10 *Example 2.9 shows among other aspects that the property of being strongly m-convex is not inherited by the basic operation of multiplication of functions. Even more, foregoing examples proves that squaring a simple function can destroy any possibility of conveying the property because the shown function satisfies the condition: for all $m \in (0, 1]$ f is strongly m-convex with modulus 1, but for all $m \in (0, 1]$ and for all $c > 0$ f^2 is not strongly m-convex with modulus c.*

Now we give, for strongly m-convex functions, similar results as Propositions 2.1 and 2.2.

Proposition 2.11 *If $f_1, f_2 : [0,b] \to \mathbb{R}$ are strongly m-convex functions with modulus c_1 and c_2, then the function given by $f(x) = \max_{x\in[0,b]}\{f_1(x), f_2(x)\}$ is strongly m-convex with modulus $c := \min\{c_1, c_2\}$.*

Proof. If $x, y \in [0,b]$ and $t \in [0,1]$, we have

$$\begin{aligned} f_1(tx+m(1-t)y) &\le tf_1(x)+m(1-t)f_1(y)-c_1mt(1-t)(x-y)^2 \\ &\le tf(x)+m(1-t)f(y)-cmt(1-t)(x-y)^2 \end{aligned}$$

and

$$\begin{aligned} f_2(tx+m(1-t)y) &\le tf_2(x)+m(1-t)f_2y)-c_2mt(1-t)(x-y)^2 \\ &\le tf(x)+m(1-t)f(y)-cmt(1-t)(x-y)^2. \end{aligned}$$

Whence,

$$\begin{aligned} f(tx+m(1-t)y) &= \max\{f_1(tx+m(1-t)y), f_2(tx+m(1-t)y)\} \\ &\le tf(x)+m(1-t)f(y)-cmt(1-t)(x-y)^2. \end{aligned}$$

Proposition 2.12 *If $f_n : [0,b] \to \mathbb{R}$ is a sequence of strongly m-convex functions with modulus c, converging pointwise to a function f on $[0,b]$, then f is strongly m-convex with modulus c.*

Proof. It is similar to the proof of Proposition 2.2.

The next result permits to obtain a strongly m-convex function from an m-convex.

Proposition 2.13 *If $f : [0,b] \to \mathbb{R}$ is an m-convex function and c any positive real constant, then the function g defined by $g(x) = f(x) + cx^2$ is strongly m-convex with modulus c.*

Proof. If $m = 1$, the result follows from Lemma 1, [5]. If $m = 0$, it is easy to verify by using Remark 1.3 that g satisfies definition 1.5. If $m \in (0,1)$ and $t \neq 0$, let $x, y \in [0,b]$. Then,

$$\begin{aligned} g(tx+m(1-t)y) &= f(tx+m(1-t)y)+c(tx+m(1-t)y)^2 \\ &\le tf(x)+m(1-t)f(y)+c(tx+m(1-t)y)^2 \\ &= t(f(x)+cx^2)+m(1-t)(f(y)+cy^2)-ct(1-t)x^2 \\ &\quad +2mct(1-t)xy-mc(1-t)\left(1-m(1-t)\right)y^2 \\ &= tg(x)+m(1-t)g(y)-cmt(1-t)\left(\frac{x^2}{m}-2xy+\frac{1-m(1-t)}{t}y^2\right) \\ &\le tg(x)+m(1-t)g(y)-cmt(1-t)(x-y)^2. \end{aligned}$$

Example 2.14 *The function* $f : [0, +\infty) \to \mathbb{R}$, *given by*

$$f(x) = ax + b$$

is clearly m*-convex (*$m \in [0, 1]$*) if* $b \leq 0$. *Therefore, by Proposition 2.13, the function* $g(x) = cx^2 + ax + b,\ b \leq 0$ *is strongly* m*-convex with modulus* $c > 0$.

Example 2.15 *The function* $f : [0, +\infty) \to (-\infty, 0]$, *given by*

$$f(x) = -\ln(x + 1)$$

is m*-convex ([2]). Thus, the function* $g(x) = x^2 - \ln(x+1)$ *is strongly* m*-convex with modulus 1.*

Corollary 2.16 *If* $f : [0, b] \to \mathbb{R}$ *is strongly* m*-convex with modulus* c, *then the function* h *defined by* $h(x) = f(x) + cx^2$ *is* m*-convex.*

Proof. If f is strongly m-convex with modulus c, then f is m-convex. Hence, by the Proposition 2.13, h is strongly m-convex with modulus c, and particularly, h is m-convex.

It is easy to see that if f is m-convex and $c > 0$, then $f(x) + \frac{c}{m}x^2$ is strongly m-convex with modulus c, because from Proposition 2.13, we have that $f(x) + \frac{c}{m}x^2$ is strongly m-convex with modulus $\frac{c}{m}$, and by Remark 1.6, it is also modulus c. Nevertheless, this argument is not applicable to check if $f(x) + mcx^2$ is strongly m-convex with modulus c, even when it is.

Proposition 2.17 *If* $f : [0, b] \to \mathbb{R}$ *is an* m*-convex function and* c *any positive real constant, then the function* g *defined by* $g(x) = f(x) + mcx^2$ *is strongly* m*-convex with modulus* c.

Proof. It is similar to the proof of Proposition 2.13.

3 Discrete Jensen type inequalities

In [3], it was proved a discrete Jensen type inequality for an m-convex function f, by considering f differentiable. We give a similar inequality without such hypothesis.

Theorem 3.1 *Let* $t_1, \ldots, t_n > 0$ *and* $T_n = \sum_{i=1}^{n} t_i$. *If* $f : [0, +\infty) \to \mathbb{R}$ *is an* m*-convex function, with* $m \in (0, 1]$, *then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right), \text{ for all } x_1, \ldots, x_n \in [0, +\infty).$$

Proof. The proof is by induction on n. If $n = 2$,

$$f\left(\frac{1}{T_2}\sum_{i=1}^{2} t_i x_i\right) = f\left(\frac{t_2}{T_2}x_2 + m\frac{t_1}{T_2}\left(\frac{x_1}{m}\right)\right).$$

Because of $\dfrac{t_1}{T_2} = 1 - \dfrac{t_2}{T_2}$ and the m-convexity of f,

$$f\left(\frac{t_2}{T_2}x_2 + m\frac{t_1}{T_2}\left(\frac{x_1}{m}\right)\right) \le \frac{t_2}{T_2}f(x_2) + m\frac{t_1}{T_2}f\left(\frac{x_1}{m}\right).$$

Therefore,

$$f\left(\frac{1}{T_2}\sum_{i=1}^{2} t_i x_i\right) \le \frac{1}{T_2}\left(mt_1 f\left(\frac{x_1}{m}\right) + t_2 f(x_2)\right) = \frac{1}{T_2}\sum_{i=1}^{2} m^{2-i} t_i f\left(\frac{x_i}{m^{2-i}}\right).$$

We assume now that the result is true for $n-1$, that is,

$$f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i x_i\right) \le \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-1-i}}\right),$$

and because $\dfrac{x_i}{m} \in [0, +\infty)$ for $i = 1, \ldots, n-1$, this inequality implies

$$f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i \frac{x_i}{m}\right) \le \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right). \tag{6}$$

Now,

$$\begin{aligned} f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) &= f\left(\frac{1}{T_n}\left(t_n x_n + \sum_{i=1}^{n-1} t_i x_i\right)\right) \\ &= f\left(\frac{t_n}{T_n}x_n + m\frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right). \end{aligned}$$

But $\dfrac{T_{n-1}}{T_n} = 1 - \dfrac{t_n}{T_n}$ and f is m-convex, therefore

$$\begin{aligned} f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) &= f\left(\frac{t_n}{T_n}x_n + m\frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right) \\ &\le \frac{t_n}{T_n}f(x_n) + m\frac{T_{n-1}}{T_n}f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right). \end{aligned}$$

By using (6) appropriately,

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \frac{t_n}{T_n}f(x_n) + \frac{m}{T_n}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) = \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right).$$

Remark 3.2 *If $T_n = m = 1$ (i.e., f is a convex function) in Theorem 3.1,*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i),$$

which is the classical discrete Jensen's inequality for a convex function.

We now present two versions of the discrete Jensen inequality for the class of strongly m-convex functions.

Theorem 3.3 *Let $t_1, \ldots, t_n$ be positive real numbers ($n \geq 2$) and $T_n = \sum_{i=1}^{n} t_i$. If $f : [0, +\infty) \to \mathbb{R}$ is a strongly m-convex function with modulus c, with $m \in (0, 1]$, then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \leq \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) - \frac{c}{T_n}\sum_{i=1}^{n-1} \frac{t_{i+1}T_i}{m^{n-i}T_{i+1}}\left(m x_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k x_k\right)^2, \tag{7}$$

for all $x_1, \ldots, x_n \in [0, +\infty)$.

Proof. The proof is by induction on n. If $n = 2$,

$$f\left(\frac{1}{T_2}\sum_{i=1}^{2} t_i x_i\right) = f\left(\frac{t_2}{T_2}x_2 + m\frac{t_1}{T_2}\left(\frac{x_1}{m}\right)\right).$$

Because of $\frac{t_1}{T_2} = 1 - \frac{t_2}{T_2}$ and the strong m-convexity of f,

$$\begin{aligned} f\left(\frac{t_2}{T_2}x_2 + m\frac{t_1}{T_2}\left(\frac{x_1}{m}\right)\right) &\leq \frac{t_2}{T_2}f(x_2) + m\frac{t_1}{T_2}f\left(\frac{x_1}{m}\right) - \frac{cmt_1t_2}{T_2^2}\left(x_2 - \frac{x_1}{m}\right)^2 \\ &= \frac{1}{T_2}\left[t_2 f(x_2) + mt_1 f\left(\frac{x_1}{m}\right)\right] - \frac{c}{T_2}\left[\frac{t_1t_2}{mT_2}(mx_2 - x_1)^2\right]. \end{aligned}$$

Therefore,

$$f\left(\frac{1}{T_2}\sum_{i=1}^{2} t_i x_i\right) \leq \frac{1}{T_2}\sum_{i=1}^{2} m^{2-i} t_i f\left(\frac{x_i}{m^{2-i}}\right) - \frac{c}{T_2}\sum_{i=1}^{2-1} \frac{t_{i+1}T_i}{m^{2-i}T_{i+1}}\left(m x_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k x_k\right)^2.$$

Let us suppose that (7) holds for $n-1$, that is,

$$\begin{aligned} f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i x_i\right) \leq{}& \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-1-i}}\right) \\ &- \frac{c}{T_{n-1}}\sum_{i=1}^{n-2} \frac{t_{i+1}T_i}{m^{n-1-i}T_{i+1}}\left(m x_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k x_k\right)^2, \end{aligned}$$

and because $\dfrac{x_i}{m} \in [0, +\infty)$ for $i = 1, \ldots, n-1$, this inequality implies

$$f\left(\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i \frac{x_i}{m}\right) \le \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) - \frac{c}{T_{n-1}}\sum_{i=1}^{n-2}\frac{t_{i+1}T_i}{m^{n-1-i}T_{i+1}}\left(x_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k \frac{x_k}{m}\right)^2. \tag{8}$$

Now,

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) = f\left(\frac{t_n}{T_n}x_n + m\frac{T_{n-1}}{T_n}\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right).$$

But $\dfrac{T_{n-1}}{T_n} = 1 - \dfrac{t_n}{T_n}$ and f is strongly m-convex, therefore

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \frac{t_n}{T_n} f(x_n) + m\frac{T_{n-1}}{T_n} f\left(\sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right) - \frac{cmt_nT_{n-1}}{T_n^2}\left(x_n - \sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right)^2.$$

By using (8) appropriately,

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \frac{t_n}{T_n} f(x_n) + m\frac{T_{n-1}}{T_n}\left[\frac{1}{T_{n-1}}\sum_{i=1}^{n-1} m^{n-1-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) - \frac{c}{T_{n-1}}\sum_{i=1}^{n-2}\frac{t_{i+1}T_i}{m^{n-1-i}T_{i+1}}\left(x_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k \frac{x_k}{m}\right)^2\right] - \frac{cmt_nT_{n-1}}{T_n^2}\left(x_n - \sum_{i=1}^{n-1}\frac{t_i}{T_{n-1}}\left(\frac{x_i}{m}\right)\right)^2.$$

Hence,

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right) \le \frac{1}{T_n}\left(t_n f(x_n) + \sum_{i=1}^{n-1} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right)\right) - \frac{c}{T_n}\sum_{i=1}^{n-2}\frac{t_{i+1}T_i}{m^{n-i}T_{i+1}}\left(mx_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k x_k\right)^2 - \frac{ct_nT_{n-1}}{mT_n^2}\left(mx_n - \frac{1}{T_{n-1}}\sum_{i=1}^{n-1} t_i x_i\right)^2$$
$$= \frac{1}{T_n}\sum_{i=1}^{n} m^{n-i} t_i f\left(\frac{x_i}{m^{n-i}}\right) - \frac{c}{T_n}\sum_{i=1}^{n-1}\frac{t_{i+1}T_i}{m^{n-i}T_{i+1}}\left(mx_{i+1} - \frac{1}{T_i}\sum_{k=1}^{i} t_k x_k\right)^2.$$

Before the second version, some basic notation is given and a lemma is shown. Let $2 \leq n \in \mathbb{N}$, $0 < b \in \mathbb{R}$ and $m \in [0,1]$ be given. Let $x_1, \ldots, x_n \in [0,b]$ and $t_1, \ldots, t_n > 0$ with $t_1 + \cdots + t_n = 1$. Let us denote with the symbols a_i and y_i the following real numbers:

$$a_i = \sum_{j=i}^{n} t_j \quad (1 \leq i \leq n) \tag{9}$$

$$y_i = \frac{1}{a_{i+1}} \sum_{j=i+1}^{n} m^{j-(i+1)} t_j x_j \quad (1 \leq i \leq n-1). \tag{10}$$

Lemma 3.4 *Let $x_1, \ldots, x_n \in [0,b]$ and $t_1, \ldots, t_n > 0$ with $t_1 + \cdots + t_n = 1$ be given. Then the following statements are true:*

1. *For $n = 2$, $\frac{mt_1a_2}{a_1}(x_1 - y_1)^2 = mt_1t_2(x_1 - x_2)^2$.*
2. *For $n = k+1$, $\frac{mt_1a_2}{a_1}(x_1 - y_1)^2 = mt_1(1-t_1)(x_1 - \sum_{i=2}^{k+1} \frac{m^{i-2}t_i}{1-t_1} x_i)^2$.*
3. *For $n = k+1$, let us set the following two k-tuples $\{\overline{x}_i\}_{i=1}^k$ and $\{\overline{t}_i\}_{i=1}^k$ by:*

$$\overline{x}_i = x_{i+1} \quad \text{and} \quad \overline{t}_i = \frac{t_{i+1}}{1-t_1}$$

for all $i = 1, \ldots, k$.

If we denote by $\overline{a}_i = \sum_{j=i}^k \overline{t}_j$ and by $\overline{y}_i = \frac{1}{\overline{a}_{i+1}} \sum_{j=i+1}^k m^{j-(i+1)} \overline{t}_j \overline{x}_j$, then

$$\overline{a}_i = \frac{1}{1-t_1} a_{i+1} \quad \text{and} \quad \overline{y}_i = y_{i+1}$$

for all $i = 1, \ldots, k$.

Proof. 1. It follows directly after replacing the corresponding values of a_1, a_2 and y_1 for $n = 2$.

2. After replacing the values of a_1, a_2 and y_1 for $n = k+1$ and by realizing that $1 - t_1 = a_2$.

3. By replacing the value of $\overline{t}_j$ in the first formula we get

$$\overline{a}_i = \sum_{j=i}^{k} \frac{t_{j+1}}{1-t_1} = \frac{1}{1-t_1} a_{i+1}$$

and by doing something similar with each pair ($\overline{t}_j$,$\overline{x}_j$) and using the previous formula for $i+1$ we obtain

$$\overline{y}_i = \frac{1-t_1}{a_{i+2}} \sum_{j=i+1}^{k} m^{j-(i+1)} \frac{t_{j+1}}{1-t_1} x_{j+1} = y_{i+1}.$$

Remark 3.5 *The k-tuple* $\{\bar{t}_i\}_{i=1}^k$ *defined above satisfies:* $\bar{t}_1 + \cdots + \bar{t}_k = 1$.

Theorem 3.6 (Special case) *If* $f : [0, b] \longrightarrow \mathbb{R}$ *is strongly m-convex with modulus c, then*

$$f\left(\sum_{i=1}^{n} m^{i-1} t_i x_i\right) \leq \sum_{i=1}^{n} m^{i-1} t_i f(x_i) - c \sum_{i=1}^{n-1} \frac{m^i t_i a_{i+1}}{a_i}(x_i - y_i)^2 \tag{11}$$

for all $x_1, \ldots, x_n \in [0, b]$, $t_1, \ldots, t_n > 0$ *with* $t_1 + \cdots + t_n = 1$ *where* a_i *and* y_i *are defined by (9) and (10) and for all* $n \geq 2$.

Proof. Let $P(n)$ be the proposition indicated by inequality (11). To prove $P(n)$ for all $n \geq 2$ requires to check the veracity of the basic step $P(2)$. It is true because by being f strongly m-convex and by part (1) from Lemma 3.4 follows

$$\begin{aligned} f\left(\sum_{i=1}^{2} m^{i-1} t_i x_i\right) &\leq t_1 f(x_1) + m t_2 f(x_2) - c m t_1 t_2 (x_1 - x_2)^2 \\ &= \sum_{i=1}^{2} m^{i-1} t_i f(x_i) - c \sum_{i=1}^{2-1} \frac{m^i t_i a_{i+1}}{a_i}(x_i - y_i)^2. \end{aligned}$$

The inductive hypothesis is the statement that $P(k)$ is true, that is, inequality (11) is valid where $k \geq 2$ is a natural number for all $x_1, \ldots, x_k \in I$, $t_1, \ldots, t_k > 0$ with $t_1 + \cdots + t_k = 1$. Now we have to prove that $P(k+1)$ is also a true statement. In fact, given any two $k+1$-tuples $x_1, \ldots, x_{k+1} \in [0, b]$, $t_1, \ldots, t_{k+1} > 0$ with $t_1 + \cdots + t_{k+1} = 1$, by parts (2) and (3) from Lemma 3.4

and by being f strongly m-convex, it follows that

$$\begin{aligned}
f\left(\sum_{i=1}^{k+1} m^{i-1}t_i x_i\right) &= f\left(t_1x_1 + m(1-t_1)\sum_{i=1}^{k} m^{i-1}\overline{t}_i\overline{x}_i\right)\\
&\le t_1 f(x_1) + m(1-t_1) f\left(\sum_{i=1}^{k} m^{i-1}\overline{t}_i\overline{x}_i\right)\\
&\qquad - cmt_1(1-t_1)\left(x_1 - \sum_{i=2}^{k+1}\frac{m^{i-2}t_i}{1-t_1}x_i\right)^2\\
&\le t_1 f(x_1) + m(1-t_1)\left[\sum_{i=1}^{k} m^{i-1}\overline{t}_i f(\overline{x}_i) - c\sum_{i=1}^{k-1}\frac{m^i\overline{t}_i\overline{a}_{i+1}}{\overline{a}_i}(\overline{x}_i-\overline{y}_i)^2\right]\\
&\qquad - c\frac{mt_1a_2}{a_1}(x_1-y_1)^2\\
&= \sum_{i=1}^{k+1} m^{i-1}t_i f(x_i) - c\sum_{i=1}^{k-1}\frac{m^{i+1}t_{i+1}a_{i+2}}{a_{i+1}}(x_{i+1}-y_{i+1})^2\\
&\qquad - c\frac{mt_1a_2}{a_1}(x_1-y_1)^2\\
&= \sum_{i=1}^{k+1} m^{i-1}t_i f(x_i) - c\sum_{i=1}^{k}\frac{m^i t_i a_{i+1}}{a_i}(x_i-y_i)^2.
\end{aligned}$$

This shows that $P(k+1)$ follows from $P(k)$. Consequently, by the principle of induction we can conclude that $P(n)$ is true for all $n \ge 2$.

Corollary 3.7 (General case) *If $f : [0,b] \longrightarrow \mathbb{R}$ is strongly m-convex with modulus c, then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} m^{i-1}t_i x_i\right) \le \frac{1}{T_n}\sum_{i=1}^{n} m^{i-1}t_i f(x_i) - c\frac{1}{T_n}\sum_{i=1}^{n-1}\frac{m^i t_i a_{i+1}}{a_i}(x_i-y_i)^2 \quad (12)$$

for all $x_1,\dots,x_n \in [0,b]$, $t_1,\dots,t_n > 0$ with $t_1+\cdots+t_n = T_n$ where a_i and y_i are defined by (9) and (10) and for all $n \ge 2$.

From [5, Theorem 4] and Theorem 3.6 (with $m = 1$) is plausible to establish the following conjecture:

Conjecture 3.8 *Given any two n-tuples $x_1,\dots,x_n \in [0,b]$, $t_1,\dots,t_n > 0$ with $t_1+\cdots+t_n = 1$, we believe that the following formula is true*

$$\sum_{i=1}^{n} t_i(x_i-\overline{x})^2 = \sum_{i=1}^{n-1}\frac{t_i a_{i+1}}{a_i}(x_i-y_i)^2$$

where $\overline{x} = t_1x_1+\cdots+t_nx_n$, $a_{i+1} = a_i - t_i$ and a_i and y_i are defined by (9) and (10) with $m = 1$.

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