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# On Some Lebesgue and Hardy Vector Classes

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**Abstract**

Vector classes $L_p(X)$ and $H_p(X)$ are considered, where $X$ is a Banach space. These classes are the generalizations of similar Lebesgue and Hardy classes in scalar case. Two different definitions for Hardy class are given, and their equivalence is proved. Riemann boundary value problems in different formulations are considered. Under certain conditions, their correct solvability is proved. Subspace bases in $L_p(X)$ are also considered. An abstract analogue of the "1/4-Kadets" theorem is obtained.

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# 1 Introduction

Classical Hardy classes are playing an important role in many areas of mathematics, including harmonic analysis. The theory of these classes is well developed, and valuable monographs like [1-3], etc have been dedicated to it. Their abstract generalizations have been studied by many authors (see, for example, [4-11]). In general, these works either consider the case of Hilbert space or introduce various Hardy class-related definitions, studying their equivalence and the conjugate spaces of them. It should be noted that, in the context of the factorization of operators, the abstract Hardy classes have been defined (see the monographs [4, 10] and references therein), some of their properties have been studied and some abstract analogues of boundary value problems of the theory of analytic functions have been considered. The obtained results were mainly used in factorization of some classes of operators and operator pencils. Riemann boundary value problems in Hardy classes in scalar case have been extensively studied in [12]. An abstract analogue of this problem was considered in a Banach algebra in the context of the factorization of operators (see, for example, [10]). The direct generalizations of these problems in finite-dimensional case have also been well studied, their theory being developed in [13]. In this paper, we consider infinite dimensional case of above classes. Vector classes $H_p(X)$ are defined in two ways: as a subspace of the Lebesgue vector space $L_p(X)$ and as a space of vector-valued analytic functions. The equivalence of these two definitions is proved. We state the natural analogues of classical Riemann problems in Hardy classes and, under certain restrictions, prove their correct solvability. The obtained results are applied to issues of double and perturbed bases in $L_p(X)$. Abstract analogues of the Riemann problem in various settings are considered. The obtained results are used to study the basicity of systems of subspaces in $L_p(X)$, where $X$ is a separable Banach space. An abstract analogue of the "1/4-Kadets theorem" is obtained. It should be noted that in the scalar case similar results were obtained in [14-17].

## 2 $L_p^{\pm}(X)$ classes

We will use the standard notation. $N$ will be the set of all positive integers, while $Z_+ = \{0\} \bigcup N$; $Z$ will denote the set of all integers; $C$ will be the set of all complex numbers. $\Leftrightarrow$ will mean "if and only if", and $\exists!$ will mean "there exists a unique". $B$-space will mean "a Banach space". We will use a concept of subspace basis. Let $X$ be a $B$-space. System of subspaces $\left\{X_n^+; X_k^-\right\}_{n,k\in N} \subset X$ is said to be a basis for $X$ if $\forall x \in X$, $\exists! \{x_n^{\pm}\}_{n\in N}$, $x_n^{\pm} \in X_n^{\pm}$:

$$x = \sum_{n=1}^{\infty} x_n^+ + \sum_{n=1}^{\infty} x_n^-.$$

Let $X$ be a separable $B$-space. Denote by $L_p(X)$ a class of measurable (no matter strongly or weakly because $X$ is separable) functions on $\vartheta : [0, 2\pi] \to X$ such that

$$\|\vartheta\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} \|\vartheta(t)\|_X^p \, dt < +\infty, \; 1 \le p < +\infty,$$

where $\|\cdot\|_X$ is a norm in $X$. With such a norm, $L_p(X)$ turns into a separable $B$-space. Functions from $L_p(X)$ that coincide with each other a.e. (with respect to the Lebesgue measure) are considered identical. Let the sequence $\{\vartheta_n\}_{n\in N} \subset L_p(X)$ converge to $\vartheta \in L_p(X)$ in $L_p(X)$, i.e.

$$\|\vartheta_n - \vartheta\|_p^p \to 0,\ n \to \infty.$$

Then we can choose a subsequence $\{\vartheta_{n_k}\}_{k\in N}$ such that

$$\sum_{k=1}^{\infty} \int_0^{2\pi} \|\vartheta_{n_k}(t) - \vartheta(t)\|_X^p \, dt < +\infty .$$

It follows from the theorem of Beppo Levi that the series $\sum_{k=1}^{\infty} \|\vartheta_{n_k}(t) - \vartheta(t)\|_X^p$ converges a.e., and, as a result, $\|\vartheta_{n_k}(t) - \vartheta(t)\|_X^p \to 0$, $k \to \infty$, a.e. on $[0, 2\pi]$. Consequently, every convergent sequence in $L_p(X)$ has a subsequence which is also a.e. convergent. Denote by $L_p^{(k)}(X)$ the subspace of $L_p(X)$ formed by functions of the form $e^{ikt}a$, $a \in X$, where $k \in Z$ is an integer. Let $q : \frac{1}{p} + \frac{1}{q} = 1$ be the conjugate of the number $p$. Similarly we define the space $L_q(X)$ and the family of subspaces $\left\{L_q^{(k)}(X)\right\}_{k\in Z}$. It is known (see, for example [9, 14]) that $L_q(X^*)$ can be identified with $(L_p(X))^*$, and an arbitrary element $\vartheta^* \in (L_p(X))^*$ is realized by the element $\vartheta \in L_q(X^*)$ through the expression

$$\vartheta^*(f) = \int_0^{2\pi} \vartheta^*(t) f(t) \, dt,\ \forall f \in L_p(X),$$

where $\vartheta^*(f) =: \langle f, \vartheta^* \rangle$ is the value of functional $\vartheta^*$ on the element $f$. Let us show that

$$L_p(X) = \sum_{k=-\infty}^{+\infty} \dot{+} L_p^{(k)}(X),$$

i.e. every $\vartheta \in L_p(X)$ has the expansion

$$\vartheta(t) = \sum_{k=-\infty}^{+\infty} e^{ikt}\vartheta_k,$$

where $\vartheta_k \in X$, $\forall k \in Z$. We first show that the system $\left\{L_p^{(k)}(X)\right\}_{k\in Z}$ is complete in $L_p(X)$. Let the element $\vartheta^* \in (L_p(X))^*$ cancel all the $L_p^{(k)}(X)$'s out, $k \in Z$, i.e.

$$\vartheta^*(f_k) = 0,\ \forall f_k \in L_p^{(k)}(X),\ \forall k \in Z.$$

Let $\vartheta \in L_q(X^*)$ be realized by $\vartheta^*$ and $f_k$ have the form $f_k(t) = e^{ikt}f$, $f \in X$. Thus, we have

$$\int_0^{2\pi} e^{ikt}(f, \vartheta(t)) \, dt = 0,\ \forall f \in X,\ \forall k \in Z.$$

It hence follows that $(f, \vartheta(t)) = 0$ a.e. on $[0, 2\pi]$, i.e. $(f, \vartheta(t)) = 0$, $\forall t \in I \backslash E_f$, where $I \equiv [0, 2\pi]$ and $mesE_f = 0$. Let $\tilde{X} \subset X$ be a countable, everywhere dense set in $X$. Assume $E = \bigcup\limits_{f \in \tilde{H}} E_f$. It is clear that $mesE = 0$. From $(f, \vartheta(t)) = 0$, $\forall f \in \tilde{X}$, $\forall t \in I \backslash E$ it follows that $\vartheta(t) = 0$, $\forall t \in I \backslash E$. Completeness of system $\left\{L_p^{(k)}(X)\right\}_{k \in Z}$ in $L_p(X)$ is now proved. Take $\forall f \in L_p(X)$. Obviously

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \in X, \forall k \in Z.$$

Consider $P_k : L_p(X) \to L_p^{(k)}(X)$, $k \in Z$: $P_k f = e^{ikt} f_k$. It is not difficult to see that

$$P_i P_j = \delta_{ij} P_j, \forall i, j \in Z,$$

where $\delta_{ij}$ is the Kronecker symbol. Applying Holder's inequality, we get that $P_k$ is continuous. Consider the partial sums

$$S_n = \sum_{k=-n}^{n} P_k, n \in Z_+ = Z \backslash \{-N\}. \tag{1}$$

Using the expression for $P_k$ in (1), absolutely similar to the classical case we obtain

$$S_n f = \frac{1}{2\pi} \int_0^{2\pi} D_n(\tau - t) f(\tau) d\tau, \forall n \in N,$$

where $D_n(t)$ is the Dirichlet kernel

$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}, n \in N.$$

We have

$$\| [S_n f] (t) \|_X \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\|f(\tau + t)\|_X}{2 \sin \frac{\tau}{2}} d\tau, \forall n \in N.$$

Hence we immediately obtain

$$\| [S_n f](\cdot) \|_p^p \leq \| H [\|f\|_X](\cdot) \|_p^p, \tag{2}$$

where $H$ is the Hilbert transform

$$[Hf](t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\tau + t)}{2 \sin \frac{\tau}{2}} d\tau.$$

Throughout this paper we consider $f$ as a function periodically extended (with the period $2\pi$) to the whole of axis $R$. Since the Hilbert transform acts boundedly in $L_p(0, 2\pi)$, $1 < p < +\infty$, from (2) we obtain

$$\| [S_n f](\cdot) \|_p \leq c_p \|f(\cdot)\|_p, \forall n \in N,$$

where $c_p$ is a constant depending only on $p$. As a result, we get that the family of projectors $\{S_n\}_{n\in N}$ is uniformly bounded in $L_p(X)$. Then, as is known (see, for example, [19]), system $\{P_n\}_{n\in Z}$ forms a basis for $L_p(X)$. Consequently, $\forall f \in L_p(X)$ has an expansion in $L_p(X)$:

$$f(t) = \sum_{n=-\infty}^{+\infty} (P_n f)(t) = \sum_{n=-\infty}^{+\infty} e^{int} f_n.$$

It is absolutely obvious that such an expansion is unique. Thus, we have

**Theorem 2.1** *The family of projectors $\{P_n\}_{n\in Z}$ forms a strong basis for $L_p(X)$, i.e.*

$$I = \sum_{n=-\infty}^{+\infty} P_n \Leftrightarrow L_p(X) = \sum_{n=-\infty}^{+\infty} \dot{+} L_p^{(n)}(X).$$

Assume

$$L_p^+(X) \equiv \{f \in L_p(X) : P_n f = 0, \forall n < 0\}.$$

$L_p^+(X)$ is a subspace of $L_p(X)$. Take $\forall f \in L_p^+(X)$ :

$$f(t) = \sum_{n=0}^{+\infty} e^{int} f_n. \tag{3}$$

Let $\omega \equiv \{z \in C : |z| < 1\}$ and $\partial\omega \equiv \{z \in C : |z| = 1\}$. By $P_r(\cdot)$ we denote the Poisson kernel

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

By virtue of the relation

$$\left(re^{it}\right)^k = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, e^{iks} ds,\ k \geq 0,\ 0 \leq r < 1,$$

we obtain from (3) that

$$\sum_{k=0}^{\infty} \left(re^{it}\right)^k f_k = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, f(s)\, ds.$$

Suppose

$$F\left(re^{it}\right) = \sum_{k=0}^{\infty} \left(re^{it}\right)^k f_k.$$

So

$$F\left(re^{it}\right)^k = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, f(s)\, ds.$$

The latter formula implies the validity of the following generalized Fatou theorem [4]:

***Theorem F.*** *$F\left(re^{it}\right)$ tends strongly to $f\left(t\right)$ in $X$ if $z=re^{it}$ tends non-tangentially to $e^{it}$ staying in the unit circle for every point $t$ for which*

$$\frac{1}{2s}\int_{t-s}^{t+s}f\left(\tau\right)d\tau\to f\left(t\right)\;\; strongly\ as\ s\to 0,$$

*i.e. almost everywhere.*

Similarly, using the relation

$$z^{k}=\frac{1}{2\pi i}\int_{\partial\omega}\frac{\tau^{k}d\tau}{\tau-z},\, z\in\omega,\, k\geq 0,$$

we get the formula

$$F\left(z\right)=\frac{1}{2\pi i}\int_{\partial\omega}\frac{f\left(\tau\right)d\tau}{\tau-z}, \tag{4}$$

where $f\left(e^{it}\right)\equiv f\left(t\right)$. It is clear that (4) presents an analytic function in $C\backslash\partial\omega$. Take $\forall\vartheta\in X^{*}$. We have

$$\vartheta\left(F\left(z\right)\right)=\frac{1}{2\pi i}\int_{\partial\omega}\frac{\vartheta\left(f\left(\tau\right)\right)}{\tau-z}d\tau.$$

Using classical Sokhotski-Plemelj formulas ( see e.g. [12] ), we obtain that

$$\vartheta\left(F^{\pm}\left(\tau\right)\right)=\pm\frac{1}{2}\vartheta\left(f\left(\tau\right)\right)+S\left[\vartheta\left(f\left(\tau\right)\right)\right]\;\; for\ almost\ every\ \tau\in\partial\omega, \tag{5}$$

where $S$ is a singular operator

$$\left[Sg\right]\left(\xi\right)=\frac{1}{2\pi i}\int_{\partial\omega}\frac{g\left(\tau\right)}{\tau-\xi}d\tau,\; \xi\in\partial\omega.$$

Relation (5) holds for every $\tau\in\partial\omega\backslash E_{\vartheta}$ with $mesE_{\vartheta}=0$. Let $Y\subset X^{*}$ be a countable, everywhere dense set in $X^{*}$ . Assume $E=\bigcup\limits_{\vartheta\in\tilde{X}}E_{\vartheta}$. It is clear that $mesE=0$. Consequently

$$\vartheta\left(F^{\pm}\left(\tau\right)\right)=\pm\frac{1}{2}\vartheta\left(f\left(\tau\right)\right)+S\left[\vartheta\left(f\left(\tau\right)\right)\right],\forall\tau\in\partial\omega\backslash E, \tag{6}$$

$\forall\vartheta\in Y$. As $Y$ is everywhere dense in $X^{*}$, it is evident that (6) holds for $\forall\vartheta\in X^{*}$. Therefore we obtain the following Sokhotski-Plemelj formula

$$F^{\pm}\left(\tau\right)=\pm\frac{1}{2}f\left(\tau\right)+\left[Sf\right]\,\left(\tau\right)\;\; for\ almost\ every\ \tau\in\partial\omega. \tag{7}$$

Since the shift

$$\Phi(t) = \left( \int_{-\pi}^{\pi} \| f(\theta - t) - f(\theta) \|_X^p \, d\theta \right)^{\frac{1}{p}},$$

is continuous as $t \to 0$ (see, for example, [9]), then, absolutely similar to the classical case, we can prove that the relation

$$\int_{-\pi}^{\pi} \left\| F\left(re^{it}\right) - f(t) \right\|_X^p dt \to 0, \tag{8}$$

holds as $r \to 1 - 0$. From

$$\left\| F\left(re^{it}\right) \right\|_X \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t - s) \| f(s) \|_X \, ds,$$

it follows directly that the inequality

$$\| P_r(f) \|_p \leq M \| f \|_p,$$

is valid, where $M$ is a constant independent of $f$ and $r$, and $P_r(\,\cdot\,)$ is the Poisson operator

$$[P_r(f)](t) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t - s) f(s) \, ds.$$

Similar considerations are also valid for the Cauchy operator

$$[\mathscr{K}_r(f)](t) \equiv \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\tau) \, d\tau}{\tau - re^{it}}.$$

Thus, the following theorem is true.

**Theorem 2.2** *Let $X$ be a separable $B$-space with the separable conjugate $X^*$ and $f \in L_p(X)$, $1 < p < +\infty$. Then the Sokhotski-Plemelj formula (7) is true for the function $F(z)$ defined by the Cauchy type integral (4). Moreover, if $f \in L_p^+(X)$, then*

$$\lim_{r \to 1-0} \| [T_r(f)](t) - f(t) \|_p = 0, \| T_r(f) \|_p \leq M \| f \|_p,$$

*where either $T_r = P_r$ is the Poisson operator or $T_r = \mathscr{K}_r$ is the Cauchy operator, $r : 0 \leq r < 1$, and $M$ is a constant independent of $f$.*

Similarly we define a class of $L_p^-(X)$:

$$L_p^-(X) \equiv \{ f \in L_p(X) : P_n f = 0, \forall n \geq 0 \}.$$

Consequently, $\forall g\in L_p^-(X)$ has the expansion

$$g(t)=\sum_{n=-\infty}^{-1}g_ne^{int}=\sum_{n=1}^{\infty}g_{-n}e^{-int},$$

where

$$g_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)\,e^{-int}dt,\ \forall n\le -1.$$

It is absolutely clear that $\hat g\in L_p^+(X)$, where $\hat g(t)\equiv g(-t)$. The function from $H_p^+(X)$ corresponding to $\hat g$ will be denoted by $\hat G(z)$. It is clear that $\hat G(0)=0$. Let

$$H_p^{+,0}(X)\equiv\left\{F\in H_p^+(X):\,F(0)=0\right\},$$

and

$$L_p^{+,0}(X)\equiv H_p^{+,0}(X)\,/_{\partial\omega}.$$

Denote

$$\hat L_p^{+,0}(X)\equiv\left\{g:\hat g(t)=g(-t)\in L_p^{+,0}(X)\right\}.$$

It is not difficult to see that $L_p(X)$ has a direct expansion

$$L_p(X)=L_p^+(X)\dot{+}\hat L_p^{+,0}(X).$$

Let $g\in\hat L_p^{+,0}$. We have

$$\hat G\left(re^{it}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)\,\hat g(\theta)\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(-\theta-t)\,\hat g(-\theta)\,d\theta=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta+t)\,g(\theta)\,d\theta.$$

It is clear that $\hat G\in H_p^{+,0}(X)$. So we obtain the following Cauchy type integral

$$\hat G(z)=\frac{1}{2\pi i}\int_{\partial\omega}\frac{\hat g(\tau)\,d\tau}{\tau-z}=\frac{1}{2\pi i}\int_{\partial\omega}\frac{g(\bar\tau)\,d\tau}{\tau-z}.$$

# 3 Equivalent definition for classes $H_p^{\pm}(X)$

Denote by $\tilde H_p^+(X)$ the class of analytical $X$-valued functions $f$ in $\omega$ with the norm

$$\|f\|_{\tilde H_p}\equiv\sup_{0<r<1}\int_{-\pi}^{\pi}\left\|f\left(re^{it}\right)\right\|^p dt<+\infty. \tag{9}$$

From Theorem 2.2 it follows directly that $H_p^+(X) \subset \tilde{H}_p^+(X)$. Take $\forall F \in \tilde{H}_p^+(X)$. Let

$$F(z) = \sum_{n=0}^{\infty} f_n z^n, \ z \in \omega.$$

It is absolutely clear that $F_\vartheta(z) = \vartheta(F(\vartheta)) \in H_p^+$, $\forall\vartheta \in X^*$. Assume that $X^*$ is separable. It is known that $F_\vartheta(z)$ has non-tangential boundary values $F_\vartheta^+(z)$ a.e. on $\partial\omega$, i.e. $\exists e_\vartheta \subset \partial\omega : mes\, e_\vartheta = 0$, and $F_\vartheta(z) \to F_\vartheta^+(\tau)$ as $z \overset{\triangleright}{\to} \tau \in \partial\omega \backslash e_\vartheta$ non-tangentially. Let $Y \subset X^*$ be a countable, everywhere dense set in $X^*$. Assume $e = \bigcup_{\vartheta\in Y} e_\vartheta$. It is clear that $mes\, e = 0$ and $F_\vartheta(z) \to F_\vartheta^+(\tau)$ as $z \overset{\triangleright}{\to} \tau$ (the diacritic symbol $\triangleright$ means non-tangential convergence) $\forall\tau \in \partial\omega\backslash e, \forall\vartheta \in Y$. It follows directly that $F_\vartheta(z) \to F_\vartheta^+(\tau)$, $z \overset{\triangleright}{\to} \tau$, $\forall\tau \in \partial\omega\backslash e$, $\forall\vartheta \in X^*$. It is known that $F_\vartheta^+ \in L_p(\partial\omega)$. Let $T_F(\vartheta) = \lim_{r\to 1} F_\vartheta(re^{it})$, $\forall\vartheta \in X^*$. By Fatou's lemma, from (9) we have

$$\left(\int_{\partial\omega} |[T_F(\vartheta)](\tau)|^p |d\tau|\right)^{\frac{1}{p}} \le \sup_r \left(\int_{-\pi}^{\pi} \left|F_\vartheta\left(re^{it}\right)\right|^p dt\right)^{\frac{1}{p}} \le \|\vartheta\| \ \|F\|_{\tilde{H}_p}.$$

Take $\forall\tau = e^{it} \in \partial\omega\backslash e$ and fix it. Thus, $\vartheta(F_r(\tau)) \to F_\vartheta^+(\tau)$, $r \to 1$, $\forall\vartheta \in X^*$, i.e. the family $\{F_r(\tau)\}$ converges weakly in $X$ as $r \to 1-0$, where $F_r(\tau) = F(r\tau)$. The weak completeness of $X$ implies $\exists f(\tau) \in X : F_r(\tau) \to f(\tau)$ weakly in $X$ as $r \to 1-0$, i.e. $\vartheta(F_r(\tau)) \to \vartheta(f(\tau))$, $r \to 1-0$, $\forall\vartheta \in X^*$, $\forall\tau \in \partial\omega\backslash e$. Hence, $F_\vartheta^+(\tau) = \vartheta(f(\tau))$, $\forall\vartheta \in X^*, \forall\tau \in \partial\omega\backslash e$. It is absolutely clear that $\vartheta(f(\vartheta)) \in H_p^+/_{\partial\omega} = L_p^+$, $\forall\vartheta \in X^*$. Consider

$$G\left(re^{it}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(t)\, dt,$$

where $f(t) \equiv f(e^{it})$. We have

$$\vartheta\left(G\left(re^{it}\right)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, \vartheta(f(t))\, dt.$$

It is clear that $\vartheta(G(z)) \overset{\triangleright}{\to} \vartheta(f(\tau))$ for almost every $\tau \in \partial\omega$. From the uniqueness theorem we obtain

$$\vartheta(F(z)) \equiv \vartheta(G(z)), \forall\vartheta \in X^*,$$

and, consequently, $F(z) \equiv G(z)$. Similarly, for some $f \in L_p(X)$ we get

$$F(z) = \frac{1}{2\pi i}\int_{\partial\omega} \frac{f(\tau)\, d\tau}{\tau - z}.$$

It is clear

$$[\vartheta(f)]_n = \int_{-\pi}^{\pi} \vartheta\left(f\left(e^{it}\right)\right) e^{-int} dt = 0,\, \forall n < 0, \forall \vartheta \in X^*.$$

Hence

$$\int_{-\pi}^{\pi} f\left(e^{it}\right) e^{-int} dt = 0,\, \forall n < 0,$$

and, as a result, $f \in L_p^+(X)$, which means $F \in H_p^+(X)$. So we get the validity of

**Theorem 3.1** *Let $X$ be a separable $B$-space with the separable conjugate $X^*$. Then the above defined spaces $H_p^+(X)$ and $\tilde{H}_p^+(X)$ coincide.*

Using the results of previous section, from this theorem we immediately obtain the following

**Statement 3.2** *Let the $B$-space $X$ satisfy the conditions of Theorem 3.1. Then, every element $F \in \tilde{H}_p^+(X)$, $1 < p < +\infty$, has non-tangential boundary values $f(\tau) \in L_p^+(X)$ a.e. on $\partial\omega$. Moreover, the Poisson*

$$F\left(re^{it}\right) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-s) f(s)\, ds,$$

*and the Cauchy formulas*

$$F(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\tau)\, d\tau}{\tau - z},\, z \in \omega,$$

*are true for it.*

Similarly we define the class $H_p^-(X)$ of functions that are analytical outside $\omega$ and vanish at infinity. The norm in $H_p^-(X)$ is defined as

$$\|f\|_{H_p^-} = \sup_{1<r<+\infty} \left( \int_{-\pi}^{\pi} \left\| f\left(re^{it}\right) \right\|_X^p dt \right)^{\frac{1}{p}}.$$

Absolutely similar to the previous case, we can show that every function $f \in H_p^-(X)$ has non-tangential boundary values $f^- \in L_p(X)$ from the outside of $\omega$ with

$$\int_{-\pi}^{\pi} f^-\left(e^{it}\right) e^{-int} dt = 0,\, \forall n \geq 0.$$

Consequently, $f^-$ has an expansion

$$f^-(t) = \sum_{n=1}^{\infty} f_{-n} e^{-int}.$$

It follows directly that the spaces $H_p^{+,0}(X)$ and $H_p^-(X)$ are isometrically isomorphic. So we get the validity of

**Theorem 3.3** *Let $X$ be a separable $B$-space with the separable conjugate $X^*$. Then the above defined spaces $H_p^{+,0}(X)$ and $H_p^-(X)$ are isometrically isomorphic.*

Similar to the previous case, we get the validity of

**Statement 3.4** *Let the $B$-space $X$ satisfy the conditions of Theorem 3.3. Then every element $F \in H_p^-(X)$, $1 < p < +\infty$, has non-tangential boundary values $f \in L_p^-(X)$ a.e. on $\partial\omega$ and the Cauchy formula*

$$F(z) = [\mathscr{K}_r f]\left(e^{it}\right), \; z = re^{it}, \quad r > 1,$$

*is true for it.*

# 4 Riemann boundary value problems with a scalar coefficient

Let $G : \partial\omega \to C$ be some scalar-valued function satisfying the following conditions: 1) $G^{\pm 1} \in L_\infty(\partial\omega)$; 2) $\arg G$ is a piecewise Holder function on $\partial\omega$. By ${}_mH_p^-(X)$ we denote the class of analytic functions in $C \backslash \overline{\omega}$, with their orders $m_0 \le m$ at infinity, such that the regular parts of their Laurent series expansions in the neighborhood of an infinitely remote point belong to $H_p^-(X)$. Consider the following Riemann problem

$$F_1^+(\tau) + G(\tau) F_2^-(\tau) = g(\tau), \; \tau \in \partial\omega. \tag{10}$$

By solution of problem (10) we mean a pair of analytic functions $(F_1; F_2) \in H_p^+(X) \times_m H_p^-(X)$ whose non-tangential boundary values on $\partial\omega$ a.e. satisfy the relation (10). Take $\forall\vartheta \in X^*$ and assume $F_{\vartheta,k} = \vartheta(F_k)$, $g_\vartheta = \vartheta(g)$. We have

$$F_{\vartheta,1}^+(\tau) + G(\tau) F_{\vartheta,2}^-(\tau) = g_\vartheta(\tau), \; \tau \in \partial\omega. \tag{11}$$

It is clear that $(F_{\vartheta,1}; F_{\vartheta,2}) \in H_p^+ \times_m H_p^-$. The theory of problems (11) in the Hardy classes has been sufficiently well studied (see, for example, [12]). Let the index æ of problem (11) be equal to zero. Then, as is known, this problem has a unique solution in classes $H_p^+ \times H_p^-$ which can be represented in the form of the Cauchy type integral $F_{\vartheta,1}(z) = K_\vartheta(z)$, for $|z| < 1$; $F_{\vartheta,2}(z) = K_\vartheta(z)$, for $|z| > 1$, where

$$K_\vartheta(z) = \frac{Y(z)}{2\pi} \int_{-\pi}^{\pi} \frac{g_\vartheta\left(e^{it}\right) dt}{Y^+\left(e^{it}\right)\left(1 - ze^{-it}\right)},$$

and $Y(z)$ is a canonical solution of the corresponding homogeneous problem, which depends only on the coefficient $G$. Using the arbitrariness of $\vartheta$, we obtain that the solution of problem (10) is expressed by the Cauchy type integral

$$F(z)=\frac{Y(z)}{2\pi}\int_{-\pi}^{\pi}\frac{g(e^{it})\,dt}{Y^{+}(e^{it})(1-ze^{-it})},$$

where $F_1(z)=F(z)\,,\ |z|<1;\ F_2(z)=F(z)\,,\ |z|>1$. It is not difficult to see that in this case the problem (10) has a unique solution from $H_p^{+}(X)\times H_p^{-}(X)$. The general case is studied in a similar way. As a result, we arrive at the following conclusion.

**Theorem 4.1** *Let the $B$-space $X$ satisfy the conditions of Theorem 3.3 and let the coefficient $G$ of problem* (10) *satisfy the conditions 1), 2). Then the problem* (10) *is solvable in classes $H_p^{+}(X)\times_m H_p^{-}(X)$ if and only if the scalar problem*

$$F^{+}(\tau)+G(\tau)F^{-}(\tau)=\vartheta(g(\tau))\,,\ \tau\in\partial\omega,$$

*is solvable in classes $H_p^{+}\times_m H_p^{-}$, $\forall\vartheta\in X^{*}$.*

Under conditions 1) and 2), the index $æ_{\vartheta}$ of scalar problem is finite, and, consequently, the problem is Noetherian (see, for example, [12]). In particular, for $æ_G=0$ the scalar problem is a Fredholm problem. As a result, we obtain the following

**Corollary 4.2** *Let all the conditions of Theorem 4.1 be fulfilled and $æ_G=0$. Then the problem* (10) *has a unique solution in $H_p^{+}(X)\times H_p^{-}(X)$ for $\forall g\in L_p(X)\,,\ 1<p<+\infty$.*

# 5 Conjugation problem with operator coefficient

By $\mathscr{L}_p$ we will denote the Banach space of bounded operators from $L_p(X)$ to $L_p(X)$, i.e. $\mathscr{L}_p\equiv L(L_p(X);L_p(X))$. Let $A,B\in\mathscr{L}_p(X)$ be some operators. Take $g\in L_p(X)$ and consider the equation

$$A(\tau)F^{+}(\tau)+B(\tau)F^{-}(\tau)=g(\tau)\ \ for\ almost\ every\,\tau\in\partial\omega, \qquad (12)$$

where $F^{\pm}\in L_p^{\pm}(X)$. In other words, we search for the pair $(F^{+};F^{-})\in L_p^{+}(X)\times L_p^{-}(X)$, which satisfies the relation (12). Before proceeding to the solution of problem (12), we make some preliminary considerations. From the proof of Theorem 2.1 it follows directly that the relations

$$L_p^{+}(X)=\sum_{k=0}^{\infty}\dot{+}L_p^{(k)}(X)\,,L_p^{-}(X)=\sum_{k=1}^{\infty}\dot{+}L_p^{(-k)}(X)\,,$$

hold, and, moreover, the direct sum decomposition

$$L_p(X) = L_p^+(X) \dot{+} L_p^-(X), \tag{13}$$

is valid. We denote by $P^{\pm}$ the projectors, generated by the decomposition (13). It is known that $P^{\pm}$ are continuous (see, e.g., [20]). Consequently, $\exists m > 0$ :

$$\left\|P^+ f\right\|_p + \left\|P^- f\right\|_p \le m \|f\|_p, \forall f \in L_p(X). \tag{14}$$

$\inf\{m's\ satisfying (14)\}$ will be denoted by $\theta_X\left(L_p^+; L_p^-\right)$. Take $\forall F \in L_p(X)$. Define the operator $T : L_p(X) \to L_p(X)$ as follows

$$TF = AF^+ + BF^-,$$

where $F^{\pm} = P^{\pm} F$. As a result, equation (12) takes the form $TF = g$. Assume $\Delta T = I - T$, where $I \in \mathscr{L}_p$ is an identity operator . Thus, we obtain the equation

$$(I - \Delta T) F = g, g \in L_p(X). \tag{15}$$

It is absolutely clear that the equations (12) and (15) are equivalent. It is known that if $\|\Delta T\| < 1$, then $I - \Delta T = T$ is invertible. We have

$$\Delta TF = F - TF = F^+ - AF^+ + F^- - BF^- = (I - A) F^+ + (I - B) F^-.$$

Therefore

$$\|\Delta TF\|_p \le \|I - A\| \left\|F^+\right\|_p + \|I - B\| \left\|F^-\right\|_p \le$$
$$\le \eta(A; B)\, \theta_X\left(L_p^+; L_p^-\right) \|F\|,$$

where $\eta(A; B) = \max\{\|I - A\|; \|I - B\|\}$. It is clear that if $\eta(A; B) < \theta_X^{-1}\left(L_p^+; L_p^-\right)$, then $\|\Delta T\| < 1$. We will call $\theta_X\left(L_p^+; L_p^-\right)$ the direct norm of decomposition (13) and denote it simply by $\theta_X^{+,-}$ :

$$\theta_X^{+,-} = \theta_X\left(L_p^+; L_p^-\right).$$

So the following theorem is true

**Theorem 5.1** *Let the B-space $X$ satisfy the conditions of Theorem 3.3, and the operators A,B satisfy the inequality*

$$\eta(A; B) < \left(\theta_X^{+,-}\right)^{-1}. \tag{16}$$

*Then the equation* (12) *has a unique solution for* $\forall g \in L_p(X)$, $1 < p < +\infty$. *Moreover,* $\exists M > 0$ :

$$\left\|F_g^{\pm}\right\|_p \le M \|g\|_p, \forall g \in L_p(X),$$

*where $F_g^{\pm}$ are the solutions of equation* (12) *corresponding to $g$.*

In fact, the latter assertion of the above theorem follows directly from the expression

$$F_g^{\pm} = P^{\pm}F = P^{\pm}\left(T^{-1}g\right). \tag{17}$$

Note that the operators $A$ and $B$, due to the inequality (16), are automorphisms in $L_p(X)$. Because it is not difficult to see that $\theta_X\left(L_p^+; L_p^-\right) \geq 1$, and, consequently, $\eta(A; B) < 1$. The rest follows from the classical facts. Let all the conditions of Theorem 5.1 be fulfilled. Then the problem (12) is uniquely solvable. Let $F_g^{\pm}$ be the corresponding solutions with $F_g^{\pm} \in L_p^{\pm}(X)$. On the basis of the above considerations, these solutions have the expansions

$$F^+ = \sum_{n=0}^{\infty} f_n^+ e^{int}, F^- = \sum_{n=0}^{\infty} f_n^- e^{-int}.$$

Taking into account these expansions in (12), we obtain that every $g \in L_p(X)$ can be expanded in terms of the system $\left\{AL_p^{(n)}(X); BL_p^{(-k)}(X)\right\}_{n\geq 0,\, k\geq 1}$. The fact that the operators $A$ and $B$ are automorphisms in $L_p(X)$ implies the uniqueness of such an expansion. Thus, the following theorem is valid.

**Theorem 5.2** *Let all the conditions of Theorem 5.1 be fulfilled. Then the system of subspaces*

$$\left\{AL_p^{(n)}(X); BL_p^{(-k)}(X)\right\}_{n\geq 0,\, k\geq 1},$$

*forms a basis for* $L_p(X)$.

As an example, consider the case $A = Ie^{i\alpha t}$, $B = Ie^{-i\alpha t}$, where $I \in L_p(X)$ is an identity operator, and $\alpha \in R$ is some parameter. Let us estimate $\eta(A; B)$. Let the operators $A$ and $B$ in $\mathscr{L}_p(X)$ be generated by the operator functions $A(t)$ and $B(t)$, respectively, i.e. $A(t), B(t) \in L(X), \forall t \in [-\pi, \pi]$. We have $(I - A)f = f(t) - A(t)f(t)$, $\forall f \in L_p(X)$. Consequently

$$\|I - A\|_{\mathscr{L}_p} = \sup_{\|f\|_{L_p(X)}=1} \|(I - A)\|_{L_p(X)} =$$

$$= \sup_{\|f\|_{L_p(X)}=1} \left(\int_{-\pi}^{\pi} \|f(t) - A(t)f(t)\|_X^p\, dt\right)^{\frac{1}{p}} =$$

$$= \sup_{\|f\|_{L_p(X)}=1} \left(\int_{-\pi}^{\pi} \|(I_X - A(t))f(t)\|_X^p\, dt\right)^{\frac{1}{p}} \leq$$

$$\leq \sup_{\|f\|_{L_p(X)}=1} \left(\int_{-\pi}^{\pi} \|I_X - A(t)\|^p \|f(t)\|_X^p\, dt\right)^{\frac{1}{p}}.$$

Let

$$\|I - A\| = \sup_{t \in (-\pi,\pi)} vrai \, \|I_X - A(t)\| .$$

Consequently

$$\|I - A\|_{\mathscr{L}_p} \leq \|I - A\|_{\infty} .$$

Similarly we establish

$$\|I - B\|_{\mathscr{L}_p} \leq \|I - B\|_{\infty} .$$

Let

$$\eta_{\infty}(A; B) = \max\left\{\|I - A\|_{\infty} ; \|I - B\|_{\infty}\right\} .$$

It is absolutely obvious that $\eta(A; B) \leq \eta_{\infty}(A; B)$. Therefore, from Theorem 5.2 we directly obtain the following

**Corollary 5.3** *Let the $B$-space $X$ satisfy all the conditions of Theorem 3.3 and the inequality $\eta_{\infty}(A; B) < \left(\theta_X^{+,-}\right)^{-1}$ hold. Then the system*

$$\left\{AL_p^{(n)}(X) ; BL_p^{(-k)}(X)\right\}_{n \geq 0,\, k \geq 1} ,$$

*forms a basis for $L_p(X)$.*

Consider the case when $X$ is an $H$-space with the scalar product $(\,\cdot\,;\,\cdot\,)$. In this case, $L_2(X)$ is also an $H$-space with the scalar product

$$(f; g)_{L_2(X)} = \int_{-\pi}^{\pi} (f(t) ; g(t))\, dt, \, \forall f, g \in L_2(X) .$$

It is not difficult to see that the spaces $L_2^+(X)$ and $L_2^-(X)$ are orthogonal, and, as a result, $\theta_X^{+,-} = \sqrt{2}$. So we get

**Corollary 5.4** *Let $X$ be a separable $H$-space and the operators $A$,$B$ satisfy the condition $\eta_{\infty}(A; B) < \frac{1}{\sqrt{2}}$. Then the system*

$$\left\{AL_p^{(n)}(X) ; BL_p^{(-k)}(X)\right\}_{n \geq 0,\, k \geq 1} ,$$

*forms a basis for $L_2(X)$.*

Now let's consider the case $A(t) = I_X e^{i\alpha(t)}$, $B(t) = I_X e^{-i\alpha(t)}$, where $\alpha \in L_{\infty}$ is some function. Let $F \in L_2(X)$, where $X$ is a separable $H$-space. We have

$$\Delta TF = (I - A) F^+ + (I - B) F^- = \left(I - I_X e^{i\alpha(t)}\right) F^+ +$$

$$+ \left(I - I_X e^{-i\alpha(t)}\right) F^- = \left(1 - e^{i\alpha(t)}\right) F^+ + \left(1 - e^{-i\alpha(t)}\right) F^- .$$

Let $\vartheta \in X$ be an arbitrary element. From the previous expression we get

$$(\Delta TF;\vartheta) = \left(1 - e^{i\alpha(t)}\right)\left(F^{+};\vartheta\right) + \left(1 - e^{-i\alpha(t)}\right)\left(F^{-};\vartheta\right).$$

It is absolutely obvious that $(F^{\pm};\vartheta) \in L_p^{\pm}$. For simplicity, we introduce the following notation

$$F_\vartheta^{\pm}(t) \equiv \left(F^{\pm}(t);\vartheta\right), F_\vartheta(t) = F_\vartheta^{+}(t) + F_\vartheta^{-}(t), u_{\pm} = ReF_\vartheta^{\pm},\ v_{\pm} = ImF_\vartheta^{\pm}.$$

We have

$$\left|F_\vartheta^{\pm}\right|^2 = |u_{\pm}|^2 + |v_{\pm}|^2, |F_\vartheta|^2 = |u_{+} + u_{-}|^2 + |v_{+} + v_{-}|^2.$$

We will follow the method used in [21]. Thus

$$|(\Delta TF;\vartheta)|^2 =$$

$$= \left| \left(2\sin^2\frac{\alpha}{2} - i\sin\alpha\right)(u_{+} + iv_{+}) + \left(2\sin^2\frac{\alpha}{2} + i\sin\alpha\right)(u_{-} + iv_{-}) \right|^2 =$$

$$= 4\sin^4\frac{\alpha}{2}(u_{+} + u_{-})^2 + 4\sin^2\frac{\alpha}{2}\sin\alpha(u_{+} + u_{-})(v_{+} - v_{-}) + \sin^2\alpha(v_{+} - v_{-})^2 +$$

$$+4\sin^4\frac{\alpha}{2}(v_{+} + v_{-})^2 + 4\sin^2\frac{\alpha}{2}\sin\alpha(v_{+} + v_{-})(u_{-} - u_{+}) + \sin^2\alpha(u_{-} - u_{+})^2 =$$

$$\left(4\sin^4\frac{\alpha}{2} + \sin^2\alpha\right)\left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right) + \left(4\sin^4\frac{\alpha}{2} - \sin^2\alpha\right)\times$$

$$\times(2u_{+}u_{-} + 2v_{+}v_{-}) + 4\sin^2\frac{\alpha}{2}\sin\alpha(2v_{+}u_{-} - 2u_{+}v_{-}).$$

In view of relations

$$2u_{+}u_{-} = (u_{+} + u_{-})^2 - \left(u_{+}^2 + u_{-}^2\right), 2v_{+}v_{-} = (v_{+} + v_{-})^2 - \left(v_{+}^2 + v_{-}^2\right),$$

$$2v_{+}u_{-} \le v_{+}^2 + u_{-}^2, -2u_{+}v_{-} \le u_{+}^2 + v_{-}^2,$$

we have

$$|(\Delta TF;\vartheta)|^2 \le \left(4\sin^4\frac{\alpha}{2} + \sin^2\alpha\right)\left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right) + \left(4\sin^4\frac{\alpha}{2} - \sin^2\alpha\right)\cdot$$

$$\left[|F_\vartheta|^2 - \left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right)\right] + 4\sin^2\frac{\alpha}{2}\sin\alpha\left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right) = 4\sin^4\frac{\alpha}{2}\cdot$$

$$|F_\vartheta|^2 + \sin^2\alpha\left[2\left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right) - |F_\vartheta|^2\right] + 4\sin^2\frac{\alpha}{2}\sin\alpha\left(\left|F_\vartheta^{+}\right|^2 + \left|F_\vartheta^{-}\right|^2\right).$$

Consequently

$$|(\Delta TF;\vartheta)|^2 + \sin^2\alpha\,|F_\vartheta|^2 \le 4\sin^4\frac{\alpha}{2}\,|F_\vartheta|^2 +$$

$$+\left(2\sin^2\alpha+4\sin^2\frac{\alpha}{2}\sin|\alpha|\right)\left(\left|F_\vartheta^+\right|^2+\left|F_\vartheta^-\right|^2\right).$$

Hence

$$\sup_{\|\vartheta\|_X=1}\left|(\Delta T(t)\,F(t)\,;\vartheta)\right|^2+\sin^2\alpha\sup_{\|\vartheta\|_X=1}\left|(F(t)\,;\vartheta)\right|^2\leq$$

$$4\sin^4\frac{\alpha}{2}\sup_{\|\vartheta\|_X=1}\left|(F(t)\,;\vartheta)\right|^2+\left(2\sin^2\alpha+4\sin^2\frac{\alpha}{2}\sin|\alpha|\right)\times$$

$$\left(\sup_{\|\vartheta\|_X=1}\left|\left(F^+(t)\,;\vartheta\right)\right|^2+\sup_{\|\vartheta\|_X=1}\left|\left(F^-(t)\,;\vartheta\right)\right|^2\right).$$

Thus

$$\|\Delta T(t)\,F(t)\|^2+\sin^2\alpha\,\|F(t)\|^2\leq 4\sin^4\frac{\alpha}{2}\,\|F(t)\|^2+$$

$$+\left(2\sin^2\alpha+4\sin^2\frac{\alpha}{2}\sin|\alpha|\right)\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right).$$

The latter means that

$$\|\Delta T(t)\,F(t)\|^2\leq 4\sin^4\frac{\alpha}{2}\,\|F(t)\|^2+4\sin^2\frac{\alpha}{2}\sin|\alpha|\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right)+$$

$$+\sin^2\alpha\left[2\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right)-\|F(t)\|^2\right].$$

As

$$2\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right)-\|F(t)\|^2\geq 0,$$

we have

$$\|\Delta T(t)\,F(t)\|^2\leq 4\sin^4\frac{\|\alpha\|_\infty}{2}\,\|F(t)\|^2+$$

$$+4\sin^2\frac{\|\alpha\|_\infty}{2}\sin\|\alpha\|_\infty\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right)$$

$$+\sin^2\|\alpha\|_\infty\left[2\left(\left\|F^+(t)\right\|^2+\left\|F^-(t)\right\|^2\right)-\|F(t)\|^2\right].$$

By integrating, we obtain

$$\|\Delta TF\|_{\mathscr{L}_2}^2\leq\left(4\sin^4\frac{\|\alpha\|_\infty}{2}-\sin^2\|\alpha\|_\infty\right)\|F\|_{\mathscr{L}_2}^2+$$

$$+\left(2\sin^2\|\alpha\|_\infty\right)+4\sin^2\frac{\|\alpha\|_\infty}{2}\sin\|\alpha\|_\infty\left(\left\|F^+\right\|_{\mathscr{L}_2}^2+\left\|F^-\right\|_{\mathscr{L}_2}^2\right).$$

As

$$\|F\|_{\mathscr{L}_2}^2=\left\|F^+\right\|_{\mathscr{L}_2}^2+\left\|F^-\right\|_{\mathscr{L}_2}^2,$$

we have

$$\|\Delta TF\|_{\mathscr{L}_2}^2\leq\left(4\sin^4\frac{\|\alpha\|_\infty}{2}+4\sin^2\frac{\|\alpha\|_\infty}{2}\sin\|\alpha\|_\infty+\sin^2\|\alpha\|_\infty\right)\|F\|_{\mathscr{L}_2}^2,$$

$$\|\Delta TF\|_{\mathscr{L}_2} \leq \left(2\sin^2\frac{\|\alpha\|_\infty}{2} + \sin\|\alpha\|_\infty\right)\|F\|_{\mathscr{L}_2}.$$

As a result

$$\|\Delta T\| \leq 1 - \cos\|\alpha\|_\infty + \sin\|\alpha\|_\infty .$$

It is absolutely clear that if $\|\alpha\|_\infty < \frac{\pi}{4}$, then $\|\Delta T\| \leq 1$. So we get the validity of

**Corollary 5.5** *Let $X$ be a separable $H$-space and $\|\alpha\|_\infty < \frac{\pi}{4}$. Then the system*

$$\left\{e^{i\alpha(t)} L_p^{(n)}(X); e^{-i\alpha(t)} L_p^{(-k)}(X)\right\}_{n\geq 0,\, k\geq 1},$$

*forms a basis for $L_p(X)$.*

Note that in the scalar case, i.e. in case when $X \equiv C$ is the complex plane, this result was first established in [21]. This result is an abstract analogue of the well-known "1/4-Kadets" theorem on the Riesz basicity of perturbed system of exponents (see e.g. [22; 23]). Consider another case when $A(t) \equiv e^{iT(t)}$, $B(t) \equiv e^{-iT(t)}$ with $T(t) \in L(X)$, $\forall t \in [-\pi, \pi]$. Suppose $\delta_T = \sup\limits_{t\in(-\pi,\pi)} vrai\,\|T(t)\| < +\infty$. We have

$$\left\|I - e^{\pm iT(t)}\right\| = \left\|\sum_{k=1}^{\infty}\frac{(\pm i)^k T^k(t)}{k!}\right\| \leq \sum_{k=1}^{\infty}\frac{\delta_T^k}{k!} = e^{\delta_T} - 1.$$

Let $F \in L_p(X)$. Consider

$$\left\|\left(I - e^{\pm iT(t)}\right)F(t)\right\|_{L_p(X)}^p = \int_{-\pi}^{\pi}\left\|\left(I - e^{\pm iT(t)}\right)F(t)\right\|_X^p dt \leq$$

$$\leq \int_{-\pi}^{\pi}\left\|I - e^{\pm iT(t)}\right\|^p \|F(t)\|_X^p dt \leq \left(e^{\delta_T} - 1\right)^p \|F\|_{L_p(X)}^p .$$

Hence

$$\left\|I - e^{\pm iT(\cdot)}\right\|_{\mathscr{L}_p} \leq e^{\delta_T} - 1.$$

Thus, if $\delta_T < \ln 2$, then

$$\left\|I - e^{\pm iT(\cdot)}\right\|_{\mathscr{L}_p} < 1.$$

Then from Theorem 5.2 we obtain

**Corollary 5.6** *Let the $B$-space $X$ satisfy all the conditions of Theorem 5.1 and $\sup\limits_{t\in(-\pi,\pi)} vrai\,\|I_X - T(t)\| < \ln 2$.*

*Then the system*

$$\left\{e^{iT(t)} L_p^{(n)}(X); e^{-iT(t)} L_p^{(-k)}(X)\right\}_{n\geq 0,\, k\geq 1},$$

*forms a basis for $L_p(X)$.*

# 6 Operator boundary value problem

Let $X$ be a separable $B$-space. Consider the operator boundary value problem

$$A(\tau) F^{+}(\tau) + B(\tau) F^{-}(\tau) = g(\tau), \ \tau \in \partial\omega, \tag{18}$$

where $A; B; g \in \mathscr{L}_p$. By solution of the problem (18) we mean a pair of operator functions $(F^{+}(z); F^{-}(z))$ such that $(F^{+}(z)x; F^{-}(z)x) \in H_p^{+}(X) \times H_p^{-}(X)$ for $\forall x \in X$ and their non-tangential boundary values on $\partial\omega$ a.e. satisfy the relation (18). So let's assume that all the conditions of Theorem 5.1 are fulfilled. Let $g(\tau) \in L(X)$ for almost every $\tau \in (-\pi, \pi)$ and

$$\int_{-\pi}^{\pi} \|g(\tau)\|^p \, d\tau < +\infty. \tag{19}$$

We denote the space of such operators by $L_p(L(X))$. Take $\forall x \in X$ and consider the boundary value problem

$$A(\tau) F_x^{+}(\tau) + B(\tau) F_x^{-}(\tau) = g(\tau) x, \ \tau \in \partial\omega. \tag{20}$$

From (19) it directly follows that $g_x(\cdot) \in L_p(X)$, where $F_x^{\pm}(\cdot) = F^{\pm}(\cdot)x$, $g_x(\cdot) = g(\cdot)x$. We will solve the boundary value problem (20) in classes $H_p^{+}(X) \times H_p^{-}(X)$. As it follows from Theorem 5.1, the operator $T : H_p^{+}(X) \times H_p^{-}(X) \to L_p(X)$ defined by

$$TF = A(\tau) F^{+}(\tau) + B(\tau) F^{-}(\tau),$$

with $F = F^{+} + F^{-}$, $F^{\pm} \in H_p^{\pm}(X)$ is invertible. It is absolutely clear that $F_x^{\pm}(\cdot) = P^{\pm} T^{-1}(g(\cdot)x)$, where $P^{\pm} : L_p(X) \to H_p^{\pm}(X)$ are the corresponding projectors. We have

$$\left( \int_{-\pi}^{\pi} \left\| F_x^{\pm}(\tau) \right\|_X^p d\tau \right)^{\frac{1}{p}} \leq c_1 \left( \int_{-\pi}^{\pi} \left\| T^{-1}(g(\tau)x) \right\|_X^p d\tau \right)^{\frac{1}{p}} \leq$$

$$\leq c_2 \left( \int_{-\pi}^{\pi} \|g(\tau)x\|_X^p d\tau \right)^{\frac{1}{p}} \leq c_2 \left( \int_{-\pi}^{\pi} \|g(\tau)\|^p d\tau \right)^{\frac{1}{p}} \|x\|,$$

where $c_k$, $k = 1, 2$; are the constants independent of $g$ and $x$. Thus

$$\left\| F_x^{\pm}(\cdot) \right\|_{L_p(X)} \leq c_2 \|g(\cdot)\|_{L_p(L(X))} \|x\|. \tag{21}$$

By $\mathscr{P}$ we denote the restriction operator on $\partial\omega$, i.e. $\mathscr{P}F = f/_{\partial\omega}$, $F \in H_p^{\pm}(X)$. It is clear that $\mathscr{P}$ performs an isomorphism $\mathscr{P} : H_p^{\pm}(X) \leftrightarrow L_p^{\pm}(X)$. Let $F_x^{\pm}(\cdot) = \mathscr{P}^{-1} F_x^{\pm}(\cdot)$. It is obvious that $F_x^{\pm}(z) \in H_p^{\pm}(X)$ linearly depends on $x \in X$. Denote by $F_g^{\pm}(z)$ the operator mapping the element $x$

to the function $F_x^{\pm}(z)$, i.e. $\left[F_g^{\pm}(z)\right](x) = F_x^{\pm}(z)$. We have $\left[F_g^{\pm}(\cdot)\right](x) = \mathscr{P}^{-1}P^{\pm}T^{-1}(g(\cdot)x)$. In fact, it is clear that $K_{\vartheta}(z) = \vartheta(F(z))$. The rest obviously follows from the Sokhotski-Plemelj formula, because $K_{\vartheta}^{\pm}(\tau) = \vartheta(F^{\pm}(\tau))$. Applying Sokhotski-Plemelj formula to $F(z)$ and taking into account the separability of $X^*$, we find that the boundary values $F_1$ and $F_2$ satisfy the relation (10) for almost every $\tau \in \partial\omega$. Thus, we obtain that the relation

$$A(\tau)\left[F_g^{+}(\tau)\right](x) + B(\tau)\left[F_g^{-}(\tau)\right](x) = g(\tau)\,x, \tag{22}$$

holds for every $x \in X$ and for almost every $\tau \in \partial\omega$. In other words, $\exists e_x \subset \partial\omega$, $mes\, e_x = 0$, and the relation (22) is fulfilled for $\forall\tau \in \partial\omega\backslash e_x$. Let $\tilde{X} \subset X$ be a countable, everywhere dense set in $X$. Then it is clear that $mes\, e = 0$, where $e = \bigcup_{x\in\tilde{X}} e_x$. Consequently, the equality (22) holds for $\forall\tau \in \partial\omega\backslash e$ and $\forall x \in \tilde{X}$. Hence, (22) holds for $\forall x \in X$ and $\forall\tau \in \partial\omega\backslash e$. As a result, we have

$$A(\tau)\,F_g^{+}(\tau) + B(\tau)\,F_g^{-}(\tau) = g(\tau)\,,\, \forall\tau \in \partial\omega\backslash e,$$

i.e. $F_g^{\pm}(z)$ are the sought operator functions. The uniqueness of solution follows from the fact that an arbitrary solution of problem (18) must satisfy the relation (22), and, as it follows from Theorem 5.1, such a solution is unique. As a result, we get the validity of the following

**Theorem 6.1** *Let all the conditions of Theorem 5.1 be fulfilled with respect to the $B$-space $X$ and the operators $A$; $B$. Then the operator boundary value problem* (18) *has a unique solution for an arbitrary operator* $g \in L_p(L(X))\,,\, 1 < p < +\infty$.

# 7 Conclusion

Now let's briefly overview the issues studied in this paper. 1) We gave two different definitions for the vector Hardy class and proved their equivalence. Analogous class in the exterior of the unit circle is also defined; 2) We considered the abstract Riemann boundary value problem in the Hardy classes with scalar coefficients and studied its Noetherness under small perturbations; 3) We considered the abstract Riemann boundary value problem in the Hardy classes with operator coefficients and studied its correct solvability; 4) We used the obtained results to study the basicity of systems of subspaces in ; 5) We obtained an abstract analogue of the "1/4-Kadets" theorem for the bases from subspaces.

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