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On Some Applications of Lie Theoretic Approach to Basic Analogue of Meijers G-Function

S. K. Sharma
School of Physical Sciences,
ITM University,
Gwalior - 474 001, INDIA
E-mail : sksharma_itm@rediffmail.com

Renu Jain
School of Mathematics and Allied Sciences,
Jiwaji University,
Gwalior - 474 001, INDIA
E- mail : renujain3@rediffmail.com

Abstract. Having defined the q-recurrence relation of basic analogue of Meijer's G-function by using technique of q-calculus and a basis function of several variables for the said function ,we have been construct the various q-difference operators and their Lie algebra. In this paper, the operators used to characterize the generating functions.

Keywords and phrases: Meijer's G-function, q-difference operators, Lie algebra

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1. Introduction

We begin our study with the basic analogue of G-function in terms of Mellin Barnes type contour integral, see Saxena, et. al. [8], in the following manner :

$$G_{A,B}^{m,n}\left[z;q\left|{}^{a_1,\ldots\ldots a_A}_{b_1,\ldots\ldots. b_B}\right.\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-s}\right)\prod_{j=1}^{n} G\left(q^{1-a_j+s}\right)\pi z^s}{\prod_{j=m+1}^{B} G\left(q^{1-b_j+s}\right)\prod_{j=n+1}^{A} G\left(q^{a_j-s}\right)G\left(q^{1-s}\right)\sin \pi s} ds \tag{1.1}$$

Where

$$G(q^{\alpha}) = \left\{\prod_{n=0}^{\infty}\left(1-q^{\alpha+n}\right)\right\}^{-1} = \frac{1}{(q^{\alpha};q)_{\infty}} \tag{1.2}$$

and $0 \le m \le B;\ 0 \le n \le A$.

The contour C is a line parallel to Re(ωs) = 0, with indentations, if necessary, in such a manner that all the poles of $G\left(q^{b_j-s}\right);\ 1 \le j \le m$ are to the right and those of $G\left(q^{1-a_j+s}\right);\ 1 \le j \le n$ to the left of C. The integral converges if $\mathrm{Re}\left[s\log|z| - \log\sin \pi s\right] < 0$ for large value of $|s|$ on the contour C, that is, if $\left|\arg(z)-\omega_2\omega_1^{-1}\log|z|\right|<\pi$, where $|q|<1$, $\log q = -\omega = -(\omega_1+i\omega_2)$, ω, ω_1, ω_2 are definite quantities, ω_1 and ω_2 being real.

For basic analogue of G-function, we use the notation $G_q(.)$, defined as follows:

$$G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1,\ldots\ldots a_A\\ b_1,\ldots\ldots. b_B\end{matrix}\right.\right]=G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_A\\ b_B\end{matrix}\right.\right]=$$

$$(1-q)^{\sum_{j=1}^{n}a_j-\sum_{j=1}^{m}b_j-A-1+m+n}\{G(q)\}^{A+B-2(m+n-1)}\times\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G(q^{b_j-s})\prod_{j=1}^{n}G(q^{1-a_j+s})\pi z^s(1-q)^{s(B-A)}}{\prod_{j=m+1}^{B}G(q^{1-b_j+s})\prod_{j=n+1}^{A}G(q^{a_j-s})G(q^s)G(q^{1-s})\sin\pi s}ds \quad (1.3)$$

and then making use of identity (see Askey [1])

$$\Gamma_q(x)=\frac{G(q^x)}{(1-q)^{x-1}.G(q)};\ |q|<1 \quad (1.4)$$

We find that the R.H.S. of (1.3) reduced to

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi z^s}{\prod_{j=m+1}^{B}\Gamma_q(1-b_j+s)\prod_{j=n+1}^{A}\Gamma_q(a_j-s)\Gamma_q(s)\,\Gamma_q(1-s)\sin\pi s}ds \quad (1.5)$$

Now, taking the limit of both sides, as $q\to 1-$ and making use of result $\lim_{q\to 1-}\Gamma_q(a)=\Gamma(a)$ it yields

$$\lim_{q\to 1-}G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1,\ldots\ldots a_A\\ b_1,\ldots\ldots. b_B\end{matrix}\right.\right]=G{}^{m,n}_{A,B}\left[z\left|\begin{matrix}a_1,\ldots\ldots a_A\\ b_1,\ldots\ldots. b_B\end{matrix}\right.\right] \quad (1.6)$$

We have established certain recurrence relations associated with the basic analogue of Meijer's G-function (see for details Sharma, et.al.[12])

$$\left[zD_{z,q}-\left(\frac{q^{a_1-1}-1}{q-1}\right)\right]G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots\ldots b_B\end{matrix}\right.\right]=q^{a_1-1}G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1-1,a_2,..,a_A\\ b_1,\ldots\ldots\ldots,b_B\end{matrix}\right.\right] \quad (1.7)$$

where $n\geq 1$

$$\left[zD_{z,q}-\left(\frac{q^{a_A-1}-1}{q-1}\right)\right]G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots\ldots b_B\end{matrix}\right.\right]=-G_q{}^{m,n}_{A,B}\left[qz;q\left|\begin{matrix}a_1,..,a_{A-1},a_A-1\\ b_1,\ldots\ldots\ldots,b_B\end{matrix}\right.\right] \quad (1.8)$$

where $n<A$

$$\left[zD_{z,q}-\left(\frac{q^{b_1}-1}{q-1}\right)\right]G_q{}^{m,n}_{A,B}\left[z;q\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots\ldots b_B\end{matrix}\right.\right]=-G_q{}^{m,n}_{A,B}\left[qz;q\left|\begin{matrix}a_1,\quad\ldots,\quad a_A\\ b_1+1,.b_2\ldots,b_B\end{matrix}\right.\right] \quad (1.9)$$

where $m \geq 1$

$$\left[zD_{z,q} - \left(\frac{q^{b_B} - 1}{q - 1} \right) \right] G_{q\,A,B}^{\;m,n} \left[x \middle| \begin{matrix} a_1, \ldots, a_A \\ b_1, \ldots, b_B \end{matrix} \right] = q^{b_B} G_{q\,A,B}^{\;m,n} \left[z, q \middle| \begin{matrix} a_1, \ \ldots, \ \ a_A \\ b_1, .. b_{B-1}, b_B + 1 \end{matrix} \right] \tag{1.10}$$

where $m < B$

and

$$zG_{q\,A,B}^{\;m,n} \left[z; q \middle| \begin{matrix} a_1, .., a_A \\ b_1, .., b_B \end{matrix} \right] = \frac{1}{q} G_{q\,A,B}^{\;m,n} \left[qz; q \middle| \begin{matrix} a_1 + 1, \ldots, a_A + 1 \\ b_1 + 1, \ldots, b_B + 1 \end{matrix} \right] \tag{1.11}$$

With the help of relations (1.7) to (1.11) and the method of Miller, W.(see [5]) a Lie algebra associated with the $G_q(.)$ functions namely . A basis function containing A+B variables $t_1, t_2, \ldots\ldots t_A, u_1, u_2, \ldots\ldots u_B$ have defined as

$$F_{A,B}^{m,n} \left[z; q \middle| \begin{matrix} t_A \\ u_B \end{matrix} \middle| \begin{matrix} a_A \\ b_B \end{matrix} \right] = G_{q\,A,B}^{\;m,n} \left[z; q \middle| \begin{matrix} a_A \\ b_B \end{matrix} \right] t_1^{a_1 - 1} \ldots\ldots\ldots t_A^{a_A - 1} \, u_1^{b_1} \ldots\ldots u_B^{b_B} \tag{1.12}$$

and q-partial differential operators

$$T_j = t_j \partial_{t_j, q},$$

$$L_j = t_j^{-1} \left(z\partial_{z,q} - t_j \partial_{t_j, q} \right), \ 1 \leq j \leq A, \tag{1.13}$$

$$U_k = u_k \partial_{u_k, q},$$

$$R_k \equiv u_k \left[z\partial_{z,q} - u_k \partial_{u_k, q} \right], \ 1 \leq k \leq B,$$

and $V = zt_1 \ldots, t_A u_1 \ldots u_B$

These 2(A+B)+1 operators generate a Lie algebra $G_{A,B}$ with commutation relations, defined by Manocha, [4].

$$qT_j L_j - L_j T_j = -L_j$$

$$U_k R_k - qR_k U_k = R_k \tag{1.14}$$

$$T_j V - qVT_j = V ; \ 1 \leq j \leq A,$$

$$U_k V - qVU_k = V ; \ 1 \leq k \leq B$$

All other commutators of two generators of $G_{A,B}$ are zero.

The action of the operators (1.13) on the basis functions (1.12) are given by Sharma,et.al.[12]

$$T_j F_{A,B}^{m,n} = \left(\frac{1 - q^{a_j - 1}}{1 - q} \right) F_{A,B}^{m,n}, \ 1 \leq j \leq A \tag{1.15}$$

$$U_k F_{A,B}^{m,n} = \left(\frac{1 - q^k}{1 - q} \right) F_{A,B}^{m,n} \ \ , 1 \leq k \leq B \tag{1.16}$$

$$VF_{A,B}^{m,n} = \frac{1}{q} F_{A,B}^{m,n}\left[qz;q \Big|_{u_B}^{t_A} \Big|_{b_B+1}^{a_A+1} \right] \quad (1.17)$$

$$L_j F_{A,B}^{m,n} = q^{a_j-1} F_{A,B}^{m,n}\left[z;q \Big|_{u_B}^{t_A} \Big|_{b_1, \quad ...,b_B}^{a_1,...,a_j-1,...,a_A} \right]; \quad \text{if } 1 \le j \le n \quad (1.18)$$

$$L_j F_{A,B}^{m,n} = -F_{A,B}^{m,n}\left[qz;q \Big|_{u_B}^{t_A} \Big|_{b_1, \quad ...,b_B}^{a_1,...,a_j-1,...,a_A} \right]; \quad \text{if } n+1 \le j \le A \quad (1.19)$$

$$R_k F_{A,B}^{m,n} = -F_{A,B}^{m,n}\left[qz;q \Big|_{u_B}^{t_A} \Big|_{b_1,...,b_k+1,...,b_B}^{a_1,..., \quad a_A} \right]; \quad \text{if } 1 \le k \le m \quad (1.20)$$

$$R_k F_{A,B}^{m,n} = q^{b_k} F_{A,B}^{m,n}\left[z;q \Big|_{u_B}^{t_A} \Big|_{b_1,...,b_k+1,...,b_B}^{a_1,..., \quad a_A} \right]; \quad \text{if } m+1 \le k \le B \quad (1.21)$$

In the present paper, we shall obeys to characterize the generating functions for the basic analogue of Meijer's G- functions. The detail account of basic analogue of Meijer's G-function with their applications can be found in the research papers due to Gasper and Rahman [3] ,Purohit, et.al.[6],Rajkovic, et. al.[7], Saxena, et.al.[9], Sharma, et.al.[10,11] and Yadav, et.al.[14,15],

2. Generating Functions with $G_q(.)$-Function.

We now apply Weisner's method [13] to characterize generating functions for $G_q(.)$ functions as simultaneous eigenvectors of A+B independent operators constructed from the generators of $G_{A,B}$. The basis functions $F_{A,B}^{m,n}$ have such a description

$$T_j F_{A,B}^{m,n} = \left(\frac{1-q^{a_j-1}}{1-q} \right) F_{A,B}^{m,n}, \quad 1 \le j \le A;$$

$$U_k F_{A,B}^{m,n} = \left(\frac{1-q^{k}}{1-q} \right) F_{A,B}^{m,n}, \quad 1 \le k \le B$$

We derive the multiplication theorem for $G_q(.)$ function corresponding to operator L_A

We consider the expression, for $|\beta| < \infty$

$$E_q\{\beta(1-q)L_A\}\, F_{A,B}^{m,n}$$

$$= \sum_{h=0}^{\infty} \frac{\beta^h (1-q)^h q^{h(h-1)}}{(q:q)_h} (L_A)^h F_{A,B}^{m,n}\left[z;q \Big|_{u_B}^{t_A} \Big|_{b_1,...,b_B}^{a_1,...,a_A} \right] \quad (2.1)$$

In view of the equation (1.19) yields

$$E_q\{\beta(1-q)L_A\}\, F_{A,B}^{m,n}$$

$$=\sum_{h=0}^{\infty}\frac{\beta^{h}(1-q)^{h}\,q^{h(h-1)}}{(q:q)_{h}}(-1)^{h}F_{A,B}^{m,n}\left[q^{h}z;q\left|\begin{matrix}t_A\\u_B\end{matrix}\right|\begin{matrix}a_1,...,a_{A-1},a_A-h\\b_1,...,b_B\end{matrix}\right]$$

I

$$=\sum_{h=0}^{\infty}\frac{\beta^{h}(1-q)^{h}\,q^{h(h-1)}}{(q:q)_{h}}(-1)^{h}G_{q\,A,B}^{\;m,n}\left[q^{h}z;q\left|\begin{matrix}t_A\\u_B\end{matrix}\right|\begin{matrix}a_1,...,a_{A-1,}a_A-h\\b_1,...,b_B\end{matrix}\right]$$

$$t_1^{a_1-1}...t_2^{a_2-1}...t_{A-1}^{a_{A-1}}\,t_A^{(a_A-h)-1}.u_1^{b_1}...u_B^{b_B} \tag{2.2}$$

Next, we expand

$$\left(1-\frac{\beta q^{a_A-1}}{t_A}\right)^{a_A-1}F_{A,B}^{m,n}\left[\frac{z}{\left(1-\frac{\beta q^{a_A-1}}{t_A}\right)};q\left|\begin{matrix}t_A\\u_B\end{matrix}\right|\begin{matrix}a_1,...,a_A\\b_1,...,b_B\end{matrix}\right]$$

In view of definition (1.12) and (1.1) the above expression becomes

$$\frac{1}{2\pi i}\int_c\frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi z^s\left(1-\frac{\beta q^{a_A-1}}{t_A}\right)^{-(1-a_A+s)}}{\prod_{j=m+1}^{B}\Gamma_q(1-b_j+s)\prod_{j=n+1}^{A}\Gamma_q(a_j-s)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}ds\,.t_1^{a_1-1}...t_A^{a_A-1}.u_1^{b_1}...u_B^{b_B}$$

On making use of q-binomial theorem, we obtain

$$\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi z^s\left\{\sum_{h=0}^{\infty}\frac{(q^{1-a_A+s};q)_h}{(q:q)_h}\left(\frac{\beta q^{a_A-1}}{t_A}\right)^h\right\}}{\prod_{j=m+1}^{B}\Gamma_q(1-b_j+s)\prod_{j=n+1}^{A}\Gamma_q(a_j-s)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}ds$$

$$.\,t_1^{a_1-1}...t_A^{a_A-1}.u_1^{b_1}...u_B^{b_B}$$

Interchanging the order of integration and summation and after certain simplifications, we obtain

$$\sum_{h=0}^{\infty}\frac{(\beta q^{a_A-1})^h(1-q)^h}{(q;q)_h}\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\frac{(q^{1-a_A+s};q)_h}{(1-q)^h}\pi z^s}{\prod_{j=m+1}^{B}\Gamma_q(1-b_j+s)\prod_{j=n+1}^{A}\Gamma_q(a_j-s)\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}ds$$

$$\times\,t_1^{a_1-1}.t_2^{a_2-1}...t_{A-1}^{a_{A-1}}.t_A^{(a_A-h)-1}.u_1^{b_1}...u_B^{b_B}$$

$$=\sum_{h=0}^{\infty}\frac{\beta^h(-1)^h(1-q)^h q^{\frac{h(h-1)}{2}}}{(q;q)_h}\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}\Gamma_q(b_j-s)\prod_{j=1}^{n}\Gamma_q(1-a_j+s)\pi(q^hz)^s}{\prod_{j=m+1}^{B}\Gamma_q(1-b_j+s)\prod_{j=n+1}^{A-1}\Gamma_q(a_j-s)\Gamma_q\{(a_A-h)-s\}\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}ds$$

$$\times t_1^{a_1-1}.t_2^{a_2-1}...t_{A-1}^{a_{A-1}}.t_A^{(a_A-h)-1}.u_1^{b_1}...u_B^{b_B}$$

$$=\sum_{h=0}^{\infty}\frac{\beta^h(-1)^h(1-q)^h q^{\frac{h(h-1)}{2}}}{(q;q)_h} G_{q\,A,B}^{m,n}\left[q^h z;q\left|{}^{a_1,\ldots,a_{A-1},a_A-h}_{b_1,\ldots,b_B}\right.\right] t_1^{a_1-1}.t_2^{a_2-1}\ldots t_{A-1}^{a_{A-1}}.t_A^{(a_A-h)-1}.u_1^{b_1}\ldots u_B^{b_B}$$

$$\Rightarrow\left(1-\frac{\beta q^{a_A-1}}{t_A}\right)^{a_A-1} G_{q\,A,B}^{m,n}\left[\frac{z}{\left(1-\frac{\beta q^{a_A-1}}{t_A}\right)};q\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots,b_B\end{matrix}\right.\right] t_1^{a_1-1}\ldots t_A^{a_A-1}.u_1^{b_1}\ldots u_B^{b_B}$$

$$=\sum_{h=0}^{\infty}\frac{\left(\frac{-\beta q^{a_A-1}}{t_A}\right)^h(1-q)^h q^{\frac{h(h-1)}{2}}}{(q;q)_h\left(q^{a_A-1}\right)^h} G_{q\,A,B}^{m,n}\left[q^h z;q\left|{}^{a_1,\ldots,a_{A-1},a_A-h}_{b_1,\ldots,b_B}\right.\right] t_1^{a_1-1}.t_2^{a_2-1}\ldots t_A^{a_A-1}.u_1^{b_1}\ldots u_B^{b_B}$$

Indeed, factoring out the quantity $t_1^{a_1-1}\ldots t_A^{a_A-1}.u_1^{b_1}\ldots u_B^{b_B}$ and employing substitution, $\left(\frac{1-\beta q^{a_A-1}}{t_A}\right)^{-1}=\omega$,we obtain the generating relation

$$G_{q\,A,B}^{m,n}\left[\omega z;q\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots,b_B\end{matrix}\right.\right]$$

$$=\omega^{(a_{A1}-1)}\sum_{h=0}^{\infty}\frac{(\frac{1}{\omega}-1)^h(1-q)^h q^{\frac{h(h-1)}{2}}}{(q;q)_h\left(q^{a_A-1}\right)^h} G_{q\,A,B}^{m,n}\left[q^h z;q\left|{}^{a_1,a_2,\ldots,a_{A-1}\,a_A-h}_{b_1,\ldots,\quad b_B}\right.\right] \tag{2.3}$$

on taking the limit of both sides, as q→1- and making use of result

$\lim_{q\to 1-}\frac{(q:q)_h}{(1-q)^h}=h!$ Then, the identity (2.3) becomes

$$G_{A,B}^{m,n}\left[\omega z\left|\begin{matrix}a_1,\ldots,a_A\\ b_1,\ldots,b_B\end{matrix}\right.\right]=\omega^{a_A-1}\sum_{h=0}^{\infty}\frac{(\omega^{-1}-1)^h}{h!}G_{A,B}^{m,n}\left[z\left|{}^{a_1,a_2,\ldots a_{A-1},a_A-h}_{b_1,\ldots,\quad b_B}\right.\right]$$

which is a multiplication theorem for the G-function by Erdelyi et. al. [2].

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