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On g α r - Connectedness and g α r - Compactness in Topological Spaces

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Abstract

In this paper, the authors introduce a new type of connected spaces called generalized α regular-connected spaces (briefly $g\alpha r$-connected spaces) in topological spaces. The notion of generalized α regular-compact spaces is also introduced (briefly $g\alpha r$-compact spaces) in topological spaces. Some characterizations and several properties concerning $g\alpha r$-connected spaces and $g\alpha r$-compact spaces are obtained.

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1 Introduction

Topological spaces are mathematical structures that allow the formal definitions of concepts such as connectedness, compactness, interior and closure. In 1974, Das [4] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. In 1990, Ganster [7] defined and investigated semi-Lindelof spaces. Since then, Hanna and Dorsett

[10], Ganster and Mohammad S. Sarsak [8] investigated the properties of semi-compact spaces. The notion of connectedness and compactness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Ganster and Steiner [9] introduced and studied the properties of gb-closed sets in topological spaces. Benchalli et al [2] introduced gb - compactness and gb - connectedness in topological spaces. Dontchev and Ganster[5] analyzed sg - compact space. Later, Shibani [13] introduced and analyzed rg - compactnes and rg - connectedness. Crossely et al [3] introduced semi - closure. Vadivel et al [14] studied $rg\alpha$ - interior and $rg\alpha$ - closure sets in topological spaces. The aim of this paper is to introduce the concept of $g\alpha r$-connected and $g\alpha r$-compactness in topological spaces.

2 Preliminary Notes

Definition 2.1. *A subset A of a topological space (X, τ), is called sg closed, if $scl(A) \subseteq U$. The complement of sg closed set is said to be sg open set . The family of all sg open sets (respectively semi generalised closed sets) of (X, τ) is denoted by $SG - O(X, \tau)$[respectively $SG - CL(X, \tau)$].*

Definition 2.2. *A subset A of a topological space (X, τ), is called generalized α regular-closed set [11] (briefly $g\alpha r$-closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X. The complement of $g\alpha r$-closed set is called $g\alpha r$-open. The family of all $g\alpha r$-open [respectively $g\alpha r$-closed] sets of (X, τ) is denoted by $g\alpha r - O(X, \tau)$ [respectively $g\alpha r - CL(X, \tau)$].*

Definition 2.3. *A subset A of a topological space (X, τ) is called b-open set[1] if $A \subseteq cl(int(A)) \cup int(cl(A))$. The complement of b-open set is b-closed sets. The family of all b-open sets (respectively b-closed sets) of (X, τ) is denoted by $bO(X, \tau)$ (respectively $bCL(X, \tau)$)*

Definition 2.4. *The $g\alpha r$-closure of a set A, denoted by $g\alpha r - Cl(A)$[12] is the intersection of all $g\alpha r$-closed sets containing A.*

Definition 2.5. *The $g\alpha r$-interior of a set A, denoted by $g\alpha r - int(A)$[12] is the union of all $g\alpha r$-open sets containing A.*

Definition 2.6. *A topological space X is said to be gb-connected [2] if X cannot be expressed as a disjoint of two non-empty gb-open sets in X. A sub set of X is gb-connected if it is gb-connected as a subspace.*

Definition 2.7. *A subset A of a topological space (X, τ) is called generalized α regular-closed set[11] (briefly $g\alpha r$-closed set) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*

3 Main Results $g\alpha r$-Connectedness

Definition 3.1. *A topological space X is said to be $g\alpha r$-connected if X cannot be expressed as a disjoint of two non - empty $g\alpha r$-open sets in X. A subset of X is $g\alpha r$-connected if it is $g\alpha r$-connected as a subspace.*

Example 3.2. *Let $X = \{a, b, c\}$ and let $\tau = \{X, \varphi, \{a\}, \{c\}, \{a, c\}\}$. It is $g\alpha r$-connected.*

Theorem 3.3. *For a topological space X, the following are equivalent.*

(i) *X is $g\alpha r$-connected.*

(ii) *X and φ are the only subsets of X which are both $g\alpha r$-open and $g\alpha r$-closed.*

(iii) *Each $g\alpha r$-continuous map of X into a discrete space Y with at least two points is constant map.*

Proof. (i) $\Rightarrow$ (ii) : Suppose X is $g\alpha r$ - connected. Let S be a proper subset which is both $g\alpha r$ - open and $g\alpha r$ - closed in X. Its complement $X - S$ is also $g\alpha r$ - open and $g\alpha r$ - closed. $X = S \cup (X - S)$, a disjoint union of two non empty $g\alpha r$ - open sets which is contradicts (i). Therefore $S = \varphi$ or X.
(ii) $\Rightarrow$ (i) : Suppose that $X = A \cup B$ where A and B are disjoint non empty $g\alpha r$ - open subsets of X. Then A is both $g\alpha r$ - open and $g\alpha r$ - closed. By assumption $A = \varphi$ or X. Therefore X is $g\alpha r$ - connected.
(ii) $\Rightarrow$ (iii) : Let $f : X \to Y$ be a $g\alpha r$ - continuous map. X is covered by $g\alpha r$ - open and $g\alpha r$ - closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption $f^{-1}(y) = \varphi$ or X for each $y \in Y$. If $f^{-1}(y) = \varphi$ for all $y \in (Y)$, then f fails to be a map. Then there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.
(iii) $\Rightarrow$ (ii) : Let S be both $g\alpha r$ - open and $g\alpha r$ - closed in X. Suppose $S \neq \varphi$. Let $f : X \to Y$ be a $g\alpha r$ - continuous function defined by $f(S) = \{y\}$ and $f(X - S) = \{w\}$ for some distinct points y and w in Y. By (iii) f is a constant function. Therefore $S = X$. □

Theorem 3.4. *Every $g\alpha r$ - connected space is connected.*

Proof. Let X be $g\alpha r$ - connected. Suppose X is not connected. Then there exists a proper non empty subset B of X which is both open and closed in X. Since every closed set is $g\alpha r$ - closed, B is a proper non empty subset of X which is both $g\alpha r$ - open and $g\alpha r$ - closed in X. Using by Theorem 3.3, X is not $g\alpha r$ - connected. This proves the theorem. □

The converse of the above theorem need not be true as shown in the following example.

Example 3.5. *Let $X = \{a, b, c\}$ and let $\tau = \{X, \varphi, \{b\}, \{c\}, \{a, c\}\}$. X is connected but not $g\alpha r$ - connected. Since $\{b\}, \{a, c\}$ are disjoint $g\alpha r$ - open sets and $X = \{b\} \cup \{a, c\}$.*

Theorem 3.6. *If $f : X \to Y$ is a $g\alpha r$ - continuous onto and X is $g\alpha r$ - connected, then Y is connected.*

Proof. Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non - empty open set in Y. Since f is $g\alpha r$ - continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty $g\alpha r$ - open sets in X. This contradicts the fact that X is $g\alpha r$ - connected. Hence Y is connected. □

Theorem 3.7. *If $f : X \to Y$ is a $g\alpha r$ - irresolut and X is $g\alpha r$ - connected, then Y is $g\alpha r$ - connected.*

Proof. Suppose that Y is not $g\alpha r$ connected. Let $Y = A \cup B$ where A and B are disjoint non - empty $g\alpha r$ open set in Y. Since f is $g\alpha r$ - irresolut and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non - empty $g\alpha r$ - open sets in X. This contradicts the fact that X is $g\alpha r$ - connected. Hence Y is $g\alpha r$ - connected. □

Definition 3.8. *A topological space X is said to be $T_{g\alpha r}$ - space if every $g\alpha r$ - closed set of X is closed subset of X.*

Theorem 3.9. *Suppose that X is $T_{g\alpha r}$ - space then X is connected if and only if it is $g\alpha r$ - connected.*

Proof. Suppose that X is connected. Then X cannot be expressed as disjoint union of two non - empty proper subsets of X. Suppose X is not a $g\alpha r$ - connected space. Let A and B be any two $g\alpha r$ - open subsets of X such that $X = A \cup B$, where $A \cap B = \varphi$ and $A \subset X, B \subset X$. Since X is $T_{g\alpha r}$ - space and A, B are $g\alpha r$ - open. A, B are open subsets of X, which contradicts that X is connected. Therefore X is $g\alpha r$ - connected.
Conversely, every open set is $g\alpha r$ - open. Therefore every $g\alpha r$ - connected space is connected. □

Theorem 3.10. *If the $g\alpha r$ - open sets C and D form a separation of X and if Y is $g\alpha r$ - connected subspace of X, then Y lies entirely within C or D.*

Proof. Since C and D are both $g\alpha r$ - open in X, the sets $C \cap Y$ and $D \cap Y$ are $g\alpha r$ - open in Y. These two sets are disjoint and their union is Y. If they were both non - empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely C or D. □

Theorem 3.11. *Let A be a $g\alpha r$ - connected subspace of X. If $A \subset B \subset g\alpha r - cl(A)$ then B is also $g\alpha r$ - connected.*

Proof. Let A be $g\alpha r$ - connected and let $A \subset B \subset g\alpha r - cl(A)$. Suppose that $B = C\ cupD$ is a separation of B by $g\alpha r$ - open sets. By using Theorem 3.10, A must lie entirely in C or D. Suppose that $A \subset C$, then $g\alpha r - cl(A) \subset g\alpha r - cl(B)$. Since $g\alpha r - cl(C)$ and D are disjoint, B cannot intersect D. This contradicts the fact that C is non empty subset of B. So $D = \varphi$ which implies B is $g\alpha r$ - connected. □

Theorem 3.12. *A contra $g\alpha r$ - continuous image of an $g\alpha r$ - connected space is connected.*

Proof. Let $f : X \to Y$ is a contra $g\alpha r$ - continuous function from $g\alpha r$ - connected space X on to a space Y. Assume that Y is disconnected. Then $Y = A \cup B$, where A and B are non empty clopen sets in Y with $A \cap B = \varphi$. Since f is contra $g\alpha r$ - continous, we have $f^{-1}(A)$ and $f^{-1}(B)$ are non empty $g\alpha r$ - open sets in X with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varphi) = \varphi$. This shows that X is not $g\alpha r$ - connected, which is a contradiction. This proves the theorem. □

4 Main Results $g\alpha r$ - Compactness

Definition 4.1. *A collection $\{A_\alpha : \alpha \in \Lambda\}$ of $g\alpha r$ -open sets in a topological space X is called a $g\alpha r$ - open cover of a subset B of X if $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ holds.*

Definition 4.2. *A topological space X is $g\alpha r$ - compact if every $g\alpha r$ - open cover of X has a finite sub - cover.*

Definition 4.3. *A subset B of a topological space X is said to be $g\alpha r$ - compact relative to X, if for every collection $\{A_\alpha : \alpha \in \Lambda\}$ of $g\alpha r$ - open subsets of X such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda\}$ there exists a finite subset Λ_0 of Λ such that $B \subset \bigcup \{A_\alpha : \alpha \in \Lambda_0\}$.*

Definition 4.4. *A subset B of a topological space X is said to be $g\alpha r$ - compact if B is $g\alpha r$ - compact as a subspace of X.*

Theorem 4.5. *Every $g\alpha r$ - closed subset of $g\alpha r$ - compact space is $g\alpha r$ - compact relative to X.*

Proof. Let A be $g\alpha r$ - closed subset of a $g\alpha r$ - compact space X. Then A^c is $g\alpha r$ - open in X. Let $M = \{G_\alpha : \alpha \in \Lambda\}$ be a cover of A by $g\alpha r$ - open sets in X. Then $M^* = M \cup A^c$ is a $g\alpha r$ - open cover of X. Since X is $g\alpha r$ - compact, M^* is reducible to a finite sub cover of X, say $X = G_{\alpha 1} \cup$

$G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m} \cup A^c$, $G_{\alpha k} \in M$. But A and A^c are disjoint. Hence $A \subset G_{\alpha 1} \cup G_{\alpha 2} \cup G_{\alpha 3} \cup \ldots \cup G_{\alpha m} G_{\alpha k} \in M$, this implies that any $g\alpha r$ open cover M of A contains a finite sub - cover. Therefore A is gb - compact relative to X. That is, every $g\alpha r$ - closed subset of a $g\alpha r$ - compact space X is $g\alpha r$ - compact. □

Definition 4.6. *A function $f : X \to Y$ is said to be $g\alpha r$ - continuous if $f^{-1}(V)$ is $g\alpha r$ - closed in X for every closed set V of Y.*

Theorem 4.7. *A $g\alpha r$ - continuous image of a $g\alpha r$ - compact space is compact.*

Proof. Let $f : X \to Y$ be a $g\alpha r$ - continuous map from a $g\alpha r$ - compact space X onto a topological space Y. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \Lambda\}$ is a $g\alpha r$ - open cover of X. Since X is $g\alpha r$ - compact, it has a finite sub - cover say $\{f^{-1}(A_1), f^{-1} : i \in \Lambda(A_2), \ldots, f^{-1}(A_n)\}$. Since f is onto $\{A_1, A_2, \ldots, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact. □

Definition 4.8. *A function $f : X \to Y$ is said to be $g\alpha r$ - irresolute if $f^{-1}(V)$ is $g\alpha r$ - closed in X for every $g\alpha r$ - closed set V of Y.*

Theorem 4.9. *If a map $f : X \to Y$ is $g\alpha r$ - irresolute and a subset B of X is $g\alpha r$ - compact relative to X, then the image $f(B)$ is $g\alpha r$ - compact relative to Y.*

Proof. Let$\{A_\alpha : \alpha \in \Lambda\}$ be any collection of $g\alpha r$ - open subsets of Y such that $f(B) \subset \bigcup\{A_\alpha : \alpha \in \Lambda\} \subset$. Then $B \subset \bigcup\{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$. Since by hypothesis B is $g\alpha r$ - compact relative to X, there exists a finite subset $\Lambda_0 \in \Lambda$ such that $B \subset \bigcup\{f^{-1}(A_\alpha) : \alpha \in \Lambda_0\}$. Therefore we have $f(B) \bigcup \subset \{(A_\alpha) : \alpha \in \Lambda_0\}$, it shows that $f(B)$ is $g\alpha r$ - compact relative to Y. □

Theorem 4.10. *A space X is $g\alpha r$ - compact if and only if each family of $g\alpha r$ - closed subsets of X with the finite intersection property has a non - empty intersection.*

Proof. Given a collection A of subsets of X, let $C = \{X - A : A \in A\}$ be the collection of their complements. Then the following statements hold.

(a) A is a collection of $g\alpha r$ - open sets if and only if C is a collection of $g\alpha r$ - closed sets.

(b) The collection A covers X if and only if the intersection $\bigcap_{c \in C} C$ of all the elements of C is empty.

(c) The finite sub collection $\{A_1, A_2, \ldots A_n\}$ of A covers X if and only if the intersection of the corresponding elements $C_i = X - A_i$ of C is empty.The statement (a) is trivial, while the (b) and (c) follow from De Morgan's law. $X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J}(X - A_\alpha)$. The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement. The statement X is $g\alpha r$ - compact is equivalent to : Given any collection A of $g\alpha r$ - open subsets of X, if A covers X, then some finite sub collection of A covers X. This statement is equivalent to its contra positive, which is the following.
Given any collection A of $g\alpha r$ - open sets, if no finite sub - collection of A of covers X, then A does not cover X. Let C be as earlier, the collection equivalent to the following:

Given any collection C of $g\alpha r$ - closed sets, if every finite intersection of elements of C is not - empty, then the intersection of all the elements of C is non - empty. This is just the condition of our theorem.

□

Definition 4.11. *A space X is said to be $g\alpha r$ - Lindelof space if every cover of X by $g\alpha r$ - open sets contains a countable sub cover.*

Theorem 4.12. *Let $f : X \to Y$ be a $g\alpha r$ - continuous surjection and X be $g\alpha r$ - Lindelof, then Y is Lindelof Space.*

Proof. Let $f : X \to Y$ be a $g\alpha r$ - continuous surjection and X be $g\alpha r$ - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by $g\alpha r$ - open sets. Since X is $g\alpha r$ - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is Lindelof space. □

Theorem 4.13. *Let $f : X \to Y$ be a $g\alpha r$ - irresolute surjection and X be $g\alpha r$ - Lindelof, then Y is $g\alpha r$ - Lindelof Space.*

Proof. Let $f : X \to Y$ be a $g\alpha r$ - irresolute surjection and X be $g\alpha r$ - Lindelof. Let $\{V_\alpha\}$ be an open cover for Y. Then $\{f^{-1}(V_\alpha)\}$ is a cover of X by $g\alpha r$ - open sets. Since X is $g\alpha r$ - Lindelof, $\{f^{-1}(V_\alpha)\}$ contains a countable sub cover, namely $\{f^{-1}(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable subcover for Y. Thus Y is $g\alpha r$ - Lindelof space. □

Theorem 4.14. *If $f : X \to Y$ is a $g\alpha r$ - open function and Y is $g\alpha r$ -Lindelof space, then X is Lindelof space.*

Proof. Let $\{V_\alpha\}$ be an open cover for X. Then $\{f(V_\alpha)\}$ is a cover of Y by $g\alpha r$ - open sets. Since Y is $g\alpha r$ Lindelof, $\{f(V_\alpha)\}$ contains a countable sub cover, namely $\{f(V_{\alpha n})\}$. Then $\{V_{\alpha n}\}$ is a countable sub cover for X. Thus X is Lindelof space. □

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