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On dimension and σ-p.i.c.-functors

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Abstract

In this paper we introduce a notion of σ-p.i.c.-functor. We prove that the functor P_k of probabilities measures supported on $\leq\ k$ points is a σ - p.i.c.-functor and the logarithmic law for Lebesgue dimension takes place for P_k and paracompact p-spaces and stratifiable spaces. We consider also a problem of a preservation of the class of weakly countable-dimensional spaces by some σ-p.i.c - functors.

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1 Introduction

In [19] a notion of a projectively inductively closed functor (p.i.c.-functor) was introduced. Every normal finitary functor, in particular $\exp_{\mathrm{k}}$, is a p.i.c.-functor. But, as it follows from [19] and [20], one of the main functors of a finite degree, the functor P_k of probability measures supported on $\leq k$ points, is not a p.i.c-functor. In [21] it was proved that every p.i.c.-functor of a finite degree satisfies the logarithmic law for Lebesgue dimension dim for all paracompact $\sum$-spaces. It means that

$$\dim F_{\beta}\left(X\right) \leq k \dim X + \dim F(k)$$

for a p.i.c.-functor F of degree k and a non-empty paracompact $\sum$-space X.

In this article we introduce a notion of a σ-p.i.c.-functor. The functor P_k is a -σ-p.i.c.-functor (Theorem 3.17). The mentioned above logarithmic law takes place for any σ-p.i.c.-functor F of a finite degree and paracompact $\sum$-space X such that $F(X)$ is a normal space (Theorem 3.32). As corollaries we have that the logarithmic law for dim takes place for P_k and paracompact p-spaces (Theorem 3.44) and stratifiable spaces (Theorem 3.45). We prove also that

$P_\omega(X)$, where P_ω is the functor of probability measures with finite supports, is a weakly countable-dimensional space for any weakly countable-dimensional paracompact p-space X and stratifiable space X (Theorem 3.51). The same is true for the hyperspace functor $\exp_\omega$ of finite subsets (Theorem 3.54).

For a set A by $|A|$ we denote the cardinality of A. For a subset A of a topological space X by $Cl_{\mathrm{X}}A$ or just $\mathrm{Cl}A$ we denote the closure of A in X. A non-negative integer k we identify with the corresponding set of ordinals, i.e. $k = \{0, ..., k-1\}$, $0 = \{\emptyset\}$. All spaces are assumed to be Tychonoff, and all mappings, continuous. Any additional information on general topology and covariant functors one can find, for example, in ([10], [12], [18]).

2 Preliminary Notes

By Top we denote the category of all topological spaces and all their continuous mappings. By $Tych$ we denote the full subcategory of Top whose objects are Tychonoff spaces. A Hausdorff compact space we call a compact space or just a *compactum*. By $Comp$ we denote the full subcategory of $Tych$, whose objects are compacta. Recall several properties of a covariant functor $F : Comp \to Comp$. They say that F:

0) *preserves the empty set*, if $F(\emptyset) = \emptyset$.

1) *preserves singletons*, if $F(1) = 1$.

2) is *monomorphic*, if for any (topological) embedding $f : A \to X$, the mapping $F(f) : F(A) \to F(X)$ is also an embedding.

3) is *epimorphic*, if for any surjective mapping $f : X \to Y$, $F(f) : F(X) \to F(Y)$ is also surjective.

4) is *continuous*, if for any inverse spectrum $S = \left\{\mathrm{X}_\alpha; \pi_\beta^\alpha : \alpha \in A\right\}$of compact spaces, the limit $f : F(\lim S) \to \lim F(S)$ of the mappings $F(\pi_\alpha)$, where $\pi_\alpha : \lim S \to X_\alpha$ are the limiting projections of the spectrum S, is a homeomorphism.

5) *preserves intersections*, if for any family $\{A_\alpha : \alpha \in A\}$ of closed subsets of a compact space X, the mapping $F(i) : \cap \{F(A_\alpha) : \alpha \in A\} \to F(X)$ defined by $F(i)(a) = F(i_\alpha)(a)$,where $i_\alpha : A_\alpha \to X$ are the identity embeddings for all $\alpha \in A$, is an embedding.

6) *preserves preimages*, if for any mapping f : X $\to$ Y and an arbitrary closed set $A \subset Y$, we have $F(f^{-1}(A)) = F(f)^{-1}(F(A))$.

7) *preserves weight*, if $w(F(X)) = w(X)$ for any infinite compactum X.

Definition 2.1 . *A covariant functor $F : Comp \to Comp$ is called normal [18] if it satisfies the propertiies 0)-7).*

In what follows we shall use bigger classes than the class of normal functors. But any of them shall *preserve intersections and be monomorphic*. Let us note, that almost all properties of a normality of functors have a sense for functors acting in an arbitrary category $C \subset Top$. In particular, monomorphic and intersection-preserving functors are naturally defined in the category $Tych$. So. by a *functor* $F : Tych \to Tych$ we shall mean a covariant monomorphic functor preserving intersections.

Definition 2.2 . *A functor $F : Comp \to Comp$ is called regular if it satisfies the properties 2)-6).*

For a functor F and an element $a \in F(X)$, the *support* of a is defined as intersection of all closed sets $A \subset X$ such that $a \in F(A)$ (recall that we consider only monomorphic functors preserving intersections). This support we denote by $\operatorname{supp}_{F(X)}(a)$. When it is clear what functor and space are meant, we denote the support of a merely by *supp*(a).

A.Ch.Chigogidze [8] extended an arbitrary intersection-preserving monomorphic functor $F : Comp \to Comp$ to the category $Tych$ by setting

$$F_\beta(X) = \{a \in F(\beta X) : \operatorname{supp}(a) \subset X\}$$

for any Tychonoff space X. If $f : X \to Y$ is a continuous mapping of Tychonoff spaces and $\beta f : \beta X \to \beta Y$ is the (unique) extension of f over their Stone - Cech compactifications, then

$$F(\beta f)(F(\beta X)) \subset F_\beta(X) \tag{1.1}$$

The last inclusion is a corollary of a trivial fact

$$f(\operatorname{supp}(a)) \supset \operatorname{supp}(F(f)(a)). \tag{1.2}$$

Therefore, we can define the mapping

$$F_\beta(f) = F(\beta f)|X,$$

which makes F_β into a functor.

A.Ch.Chigogidze proved [8] that if a functor F has certain normality property, then F_β has the same property (modified when necessary). In sections 1 and 2 by a functor $F : Tych \to Tych$ we shall mean, as a rule, a functor of type F_β. For such a functor F and any compact space X the space $F(X)$ is compact.

Definition 2.3 . *A functor $F : Tych \to Tych$ is said to be compact ($\sigma-compact$) if $F(K)$ is* compact(σ − compact) *for any space $X \in Comp$.*

Remark 2.4 . *For any functor $F : Comp \to Comp$ the functor F_β is compact, but not every compact functor in $Tych$ has type F_β . As for examples one can consider the functors P_R and P_τ of Radon and τ- additive probability measures correspondingly (look at [4]).*

Going back to functors $F : Comp \to Comp$, we, evidently, have

$$a \in \ F\,(supp\,(a))\,. \tag{1.3}$$

If a functor F preserves preimages. then F *preserves supports* [18], i.e.

$$f(\text{supp}\,(\text{a}) = \text{supp}\,(F\,(f)\,(\text{a}))\,. \tag{1.4}$$

The property (1.4) can be conversed.

Proposition 2.5 *.[18].* Any functor in *Comp* preserves supports if and only if it preserves preimages.

Definition of the functor F_β and property (1.4) imply that

$$f(\text{supp}_{F_\beta\,(\text{X})}\,(a) = supp_{F_\beta(Y)}F_\beta(f)(a) \tag{1.5}$$

for any preimage preserving functor $F : Comp \to Comp$, Tychonoff spaces X and Y, continuous mapping $f : X \to Y$. and $a \in F_\beta(X)$.

Now we recall one construction given by V.N.Basmanov [5]. Let F : $Comp \to Comp$ be a functor. By $C(X.Y)$ we denote the space of all continuous mappings from X to Y with compact-open topology.

In particular, $C(k, Y)$ is naturally homeomorphic to the k-th power Y^k of the space Y; the homeomorphism takes each mapping $\xi : k \to Y$ to the point $(\xi(0), ..., \xi(k-1)) \in Y^k$.

For a functor F, a non-empty compact space X, and a positive integer k, in [5] there was defined the mapping

$$\pi_{F,X,k} : C(k, X) \times F(k) \to F(X)$$

by

$$\pi_{F,X,k}(\xi, a) = F(\xi)(a) \tag{1.6}$$

for any $\xi \in C\,(k, X)$ and $a \in F\,(k)$. If $X = \emptyset$, then $\text{dom}(\pi_{F,X,k}) = \emptyset$, and $Im\,(\pi_{F,\text{X},k}) = \emptyset$.

When it is clear what functor F and what space X are meant. we omit the subscripts F and X and write $\pi_{X,k}$ or π_k instead of $\pi_{F,X,k}$.

According to Shchepin's theorem ([18], Theorem 3.1), the mapping

$$F : C\,(Z, Y) \to C\,(F\,(Z)\,, F\,(Y))$$

is continuous for any *continuous* functor F and compacta Z and Y. This implies the following assertion.

Proposition 2.6 *.[5].* If F is a continuous functor, X is a compact space, and κ is a positive integer, then the mapping $\pi_{F,X,k}$ is continuous.

Recall a general definition. For a category C we denote by $O(C)$ and $M(C)$ the family of all objects of C and the family of all morphisms of C respectively. Let G and F be functors in some category $C \subset Top$. *A natural transformation* $i : G \to F$ is a family of mappings $i_X : G(X) \to F\left(X\right), X \in O\left(C\right)$, such that

$$F\left(f\right) \circ i_X = i_Y \circ G(f) \tag{1.7}$$

for any $f \in M\left(C\right)$.

Mappings i_X are called *components* of a transformation i. A functor G is said to be a *subfunctor* of a functor F if there is a natural transformation $i : G \to F$ such that all its components i_X are embeddings. This natural transformation i is called an *embedding* of G into F. If $i : G \to F$ is an embedding, then (1.7) implies

$$G(f) = i_Y^{-1} \circ F(f) \circ i_X \tag{1.8}$$

Remark 2.7 . *Usually subfunctors arise in a rather natural way. For example, in the category Comp the functor* $\exp^c$ *of hyperspaces of subcontinua is embedded in the functor exp of hyperspaces of closed subsets from the outset. We shall assume that it is a standard situation. This means that a functor G is a subfunctor of F if a natural transformation (embedding) $i : G \to F$ consists of identity embeddings, i.e. $G(X)$ is a subspace of $F(X)$ for any space X. In this case the condition (1.8) is equivalent to the condition*

$$F\left(f\right)\left(G\left(\mathrm{X}\right)\right) \subset G\left(Y\right) \tag{1.9}$$

Definition 2.8 . *Let F_k be a subfunctor of a functor F in Comp defined as follows. By definiton. $F_k(\emptyset) = F(\emptyset)$. For a non-empty compact space X, $F_k(X)$ is the image of the mapping $\pi_{F,\mathrm{X},k}$. For a mapping $f : \mathrm{X} \to Y$, $F_k\left(f\right)$ is the restriction of $F(f)$ to $F_k(X)$. Denote by $\bar{f} : C\left(k, \mathrm{X}\right) \to C\left(k, Y\right)$ the mapping which takes ξ to composition $f \circ \xi$. It is easy to see that*

$$\pi_{Y,k} \circ \bar{f} \times id_{F(\{k\})} = F\left(f\right) \circ \pi_{\mathrm{X},k} \tag{1.10}$$

Therefore, $F\left(f\right)\left(F_k\left(\mathrm{X}\right)\right) \subset F_k\left(Y\right)$. Hence, F_k is a functor. Clearly. F_k is a subfunctor of F with the identity embedding $F_k\left(X\right) \subset F\left(X\right)$ for an arbitrary compact space X.

A functor F is called'a *functor of degree* k (they write $\deg F = k$), if $F_k(X) = F(X)$ for any compact space X, but $F_{k-1}\left(X\right) \neq F\left(X\right))$ for some X. The next assertion (Proposition 2.9) is Shchepin's definition of the functor F_k. But using Basmanov's definition we should prove it. One can find the proof in [19].

Proposition 2.9 . For any continuous functor F and a compact space X, we have

$$F_k\left(\mathrm{X}\right) = \left\{a \in F\left(\mathrm{X}\right) : \left|\sup p\left(a\right)\right| \leq k\right\}.$$

The definition of a support and the property (1.3) imply

Proposition 2.10 . For a functor F, a compact space X. and a closed subset A of X, we have

$$F(A) = \{a \in F(\mathrm{X}) : supp(a) \subset A\}.$$

Proposition 2.11 . Let F be a functor in $Tych$, A be a closed subset of X, $a \in F(X)$, and $\sup p(a) \subset A$. Then $F(f)(a) = a$ for any mapping $f : X \to X$ such that $f|A = id_A$.

Proof. Let $i : A \to X$ be the identity embedding. Then $i = f\, o\, i$. Hence,

$$F(i) = F(f) \circ F(i). \tag{1.11}$$

Since $supp_{(a)} \subset A$ and $F(i) : F(A) \to F(X)$ is the identity embedding, we have $a = F(i)(a)$. Consequently. $F(f)(a) = F(f)\, F(i)(a) =$ (in view of (1.11))$=F(i)(a) = a$ Proposition 2.11 is proved.

Proposition 2.12 . Let F be a regular functor in $Comp$ and let G be its regular subfunctor. Then

$$G(A) = G(X) \cap F(A)$$

for any compact space X and its closed subset A.

Proof. The inclusion $\subset$ is obvious. As for the inverse inclusion, we shall prove it in several steps.

1. X is finite. Let $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\}$, $A = \{x_1, \ldots, x_m\}, x_i \neq x_j$ for $i \neq j$. Take some set $Y_0 = \{y_1, \ldots, y_{n-m}\}$ such that $Y_o \cap X = \emptyset$ Set $Y = A \cup Y_0$ and $Z = X \cup Y$ Define a mapping $f : Z \to Z$ in the following way:

$f \mid A = id,\ f(x_{m+i}) = y_i,\ f(y_i) = x_{m+i},\ i \in \{1, \ldots, n-m\}.$

Let $a \in G(X) \cap F(A)$. In accordance with Proposition 2.11. $F(f)(a) = a$. In view of (1.8) and (1.9), we have $G(f)(a) = F(f)(a) = a$. Hence, $a \in G(Y)$ by definition of f . Thus, $a \in G(X) \cap G(Y) =$ (since G preserves intersections)$= G(A)$. So, the assertion of Proposition 2.12 is proved.

2. X *is zero-dimensional.* Then there exists an inverse spectrum $S = \left\{X_a, \pi^\alpha_\beta, A\right\}$, consisting of finite spaces and epimorphisms, such that $X = \lim S$. Let $\pi_\alpha : X \to X_\alpha$ be the limiting projections of the spectrum S. Since F is a continuous functor, we can identify spaces $F(X)$ and $\lim F(S)$. Then $F(\pi_\alpha)$ are the limiting projections of the spectrum $F(S)$. Let $A_\alpha = \pi_\alpha(A)$ and $\rho^\alpha_\beta = \pi^\alpha_\beta | A_\alpha$ for any $\beta \leq \alpha \in A$. Then $A = \lim S_A$ and $\rho_\alpha = \pi_\alpha | A$, where $S_A = \left\{A_\alpha, \rho^\alpha_\beta, A\right\}$ and ρ_α are the limiting projections of the inverse spectrum S_A. Three other inverse spectra arise: $F(S_A)$. $G(S_A)$, and $G(S)$.

Then again we have $F(A) = \lim F(S_A)$, $G(A) = \lim G(S_A)$, and $G(X) = \lim G(S)$. Besides.$F(\rho_a)$, $G(\rho_a)$, and $G(\pi_\alpha)$ are the limiting projections of the spectra $F(S_A)$,$G(S_A)$, and $G(S)$ respectively. Further,

$$F(\rho_a) = F(\pi_\alpha) | F(A) \tag{1.12}$$

$$G(\rho_\alpha) = G(\pi_a) | G(A) \tag{1.13}$$

for any $\alpha \in A$. Moreover, since X_a are finite, we have

$$G(\mathrm{X}_\alpha) \cap F(A_\alpha) \subset G(A_\alpha) \tag{1.14}$$

for all $\alpha \in A$. It is easy to see, that (1.12), (1.13), and (1.14) yield

$$G(\mathrm{X}) \cap F(A) \subset G(A) . \tag{1.15}$$

So, the assertion of Proposition 2.12 for zero-dimensional spaces is proved.

3. *X is an arbitrary compactum.* There exist a zero-dimensional compactum $X_0$and an epimorphism $f : X_0 \to X$. Let $A_0 = f^{-1}(A)$, and let $a \in G(X) \cap F(A)$. Since G is an epimorphic functor, there is an element $b \in G(X_0)$ such that $G(f)(b) = a$. Then $F(f)(b) = a$ and, consequently, $b \in F(f)^{-1}(F(a))$. But F preserves preimages. Hence, $b \in F(f^{-1}(A)) = F(A_0)$. Thus, $b \in G(\mathrm{X}_0) \cap F(A_0) \subset G(A_0)$, because of (1.15). Then $a = G(f)(b) \in G(f)(G(A_0)) \subset G(A)$. Proposition 2.12 is proved.

From definition of support we get

Proposition 2.13 . Let F be a functor in *Comp*, and let G be its subfunctor. Then

$$supp_{F(\mathrm{X})}(a) \subset supp_{G(\mathrm{X})}(a)$$

for any compactum X and $a \in G(X)$.

Proposition 2.14 . Let F be a regular functor in *Comp*, G be its regular subfunctor. Then

$$supp_{F(X)}(a) = supp_{G(X)}(a)$$

for any compact space X and $a \in G(X)$.

Proof. Denote $supp_{F(X)}(a)$ by A and $supp_{G(X)}(a)$ by B. Then $a \in F(A) \cap G(B)$= (by Proposition 1.12)=$G(A)$. Thus, $supp_{G(X)}(a) \subset A$. On the other hand, $A \subset B$, because of Proposition 2.13. Proposition 2.14 is proved. Propositions 2.9 and 1.14 imply

Proposition 2.15 . Let F be a regular functor of degree$\leq k$ in *Comp*, and let G be its regular subfunctor. Then deg $G \leq k$.

From 1.8 we get immediately

Proposition 2.16 . Let F be a continuous functor in *Comp*, and let G be its continuous subfunctor. Then $G_k \subset F_k$ for any positive integer k.

Proposition 2.17 . Let F be a regular functor in *Comp*, and let G be a regular subfunctor of F. Then

$$G_\beta(X) = G(\beta X) \cap F_\beta(X)$$

for any Tychonoff space X.

Proof. Inclusion $\subset$ is trivial. Now let

$$a \in G(\beta X) \cap F_\beta(X). \tag{1.16}$$

Consequently, $supp_F(a) \subset X$. Hence, Proposition 2.14 implies that $supp_{G(a)} \subset X$. So,(1.16) yields $a \in G_\beta(X)$. Proposition 2.17 is proved.

Proposition 2.18 . Let F be a functor in *Comp*, and let G be its subfunctor. Then $G_\beta$is a subfunctor of F_β .

Proof. Let $a \in G_\beta(X)$. It means that $supp_{G(\beta X)} \subset X$. Then $supp_{F(\beta X)} \subset X$, because of Proposition 2.13. Hence, $a \in F_\beta(X)$. So, we proved that

$$G_\beta(X) \subset F_\beta(X) \tag{1.15}$$

for any Tychonoff space X. It remains to check the condition (1.9). Let $f : X \to Y$ be a mapping, and let $a \in G_\beta(X)$. We denote $supp_{G(\beta X)}(a)$ by K. Then $a \in G(K)$. Hence, $F(\beta f)(a) = G(\beta f)(a) \in G(\beta f)(K) = G(f)(K)$. Consequently, $\sup p_{G(\beta Y)}(F(\beta f)(a)) \subset f(K) \subset Y$. Thus, $F(\beta f)(a) \in G_\beta(Y)$. Proposition 2.18 is proved.

3 Main Results

3.1 On closed subfunctors and their sums

Definition 3.1 . *Let F be a functor acting in a category $C \subset Top$, and let G be a subfunctor of F . The functor G is called a closed subfunctor of the functor F if for any $X \in O(C)$ the space $G(X)$ is a closed subspace of $F(X)$.*

The next assertion is trivial.

Proposition 3.2 . In the category *Comp*, every subfunctor is closed.

From 2.8 and 3.2 we get

Corollary 3.3 . For every continuous functor $F : Comp \to Comp$ and positive integer k, the functor F_k is a closed subfunctor of F.

Going back to the category $Tych$, for a Tychonoff space X, a continuous functor $F : Comp \to Comp$, and a positive integer k, we put

$$F_k(X) = \pi_{F,\beta X,k}(C(k), X) \times F(k). \tag{2.1}$$

We also put

$$F_k(\emptyset) = F(\emptyset). \tag{2.1$_0$}$$

Now we denote the restriction of $\pi_{F,\beta,X,k}$ *to* $C(k) \times F(k)$ by $\pi_{F,X,k}$. If $g : X \to Y$ is a continuous mapping, then

$$F(\beta g)(F_k(X)) \subset F_k(Y),$$

in view of the equality (1.10) for the mapping $f = \beta g$. Therefore, setting

$$F_k(g) = F_k(\beta g)|F(X),$$

we obtain a mapping $F_k(g) : F_k(X) \to F_k(Y)$

Thus, we have defined the covariant functor

$$F_k : Tych \to Tych,$$

that extends the functor $F_k : Comp \to Comp$ to the category $Tych$. Proposition 2.9 implies the following assertion.

Proposition 3.4 *.[19].* If $F : Comp \to Comp$ is a continuous functor, then $F_k : Tych \to Tych$ is a subfunctor of the functor F_β, and

$$F_k(X) = F_\beta(X) \cap F_k(\beta X). \tag{2.2}$$

Equality (2.2) yields

Corollary 3.5 . For every continuous functor $F : Comp \to Comp$ and positive integer k, the functor $(F_k)_\beta$ is a closed subfunctor of F_β.

Remark 3.6 . Proposition 2.9 asserts that Shchepin's and Basmanov's definitions of functor F_k coincide for any continuous functor F in $Comp$. Nevertheless, we shall use both definitions depending on a situation and denote them $F_{k,s}$ and $F_{k,b}$ respectively. As for equality (2.1), we may assume that it defines functor $(F_\beta)_{k,b}$. Thus, equality (2.2) can be written as

$$(F_\beta)_{k,b} = (F_{k,b})_\beta = (F_{x,s})_\beta. \tag{2.3}$$

Proposition 3.7 . If G is a regular subfunctor of a regular-functor F in $Comp$, then G_β is a closed subfunctor of F_β.

Proof. For an arbitrary Tychonoff space X, we have$G_\beta\left(X\right) = G\left(\beta X\right) \cap \{a : supp_G\left(a\right) \subset X\}$=(according to Proposition 2.14)= $G\left(\beta X\right) \cap \{a : supp_F\left(a\right) \subset X\} \subset_{cl}$ (by Proposition 3.2) $F\left(\beta X\right) \cap \{a : supp_F\left(a\right) \subset X\} = F_\beta\left(X\right)$. Proposition 3.7 is proved.

Definition 3.8 *.[19]. A functor F is said to be finitely open if the set $F_k(k+1)$ is open in $F(k+1)$ for any positive integer k. The dual for this definition states that $F(k+1)\backslash F_k(k+1)$ is closed in $F(k+1)$.*

Remark 3.9 . As an example of a finitely open functor one can take any *finitary* functor F, i.e. a functor F such that $F(k)$ is finite for any positive integer k. In particular, the hyperspace functor exp is finitary and, consequently, finitely open.

Proposition 3.10 . If F is regular finitely open functor and G is its regular subfunctor, then G is finitely open.

Proof. Since of Proposition 2.14, $G_k(X) = G(X) \cap F_k(X)$ for any compact space X and positive integer k. Hence, $G(k+1) \setminus G_k(k+1)$ is closed in $F(k+1) \setminus F_k(k+1)$. Consequently, $G(k+1) \backslash G_k\left(k+1\right)$ is compact in view of finite openness of the functor F. Proposition 3.10 is proved.

Recall that an epimorphism $f : X \to Y$ is said to be *inductively closed* if there exists a closed subset A of X such that $f(A) = Y$ and $f|A$ is a closed mapping.

Definition 3.11 *.[20]. A continuous functor $F : Comp \to Comp$ is called projectively inductively closed (p.i.c.) if the mapping $\pi_{F,X,k}$ is inductively closed for any Tychonoff space X and positive integer k.*

Theorem 3.12 *.[20].* Every continuous finitely open functor F, that preserves empty set and preimages is a p.i.c.-functor.

Corollary 3.13 . Every finitary normal functor, in particular the functor $\exp_k$, is a p.i.c.-functor.

Let $F : Tych \to Tych$ be a functor and $F^n \subset_{cl} F,\ n \in w$ We say that F is a *union* of $F^n\left(F = \cup_{n=0}^{\infty} F^n\right), if\ F\left(X\right) = \cup_{n=0}^{\infty} F^n\left(X\right)$ for any Tychonoff space X.

Definition 3.14 . *A functor $F : Tych \to Tych$ is said to be a σ-p.i.c.-functor, if $F = \cup_{n=0}^{\infty}(F^n)_\beta$, where every F^n is a p.i.c-functor of a finite degree.*

Example 3.15 . *In view of Remark 3.9 and Theorem 3.12. the functor* $\exp_\omega = U_{n=1}^{\infty} \exp_n$ *is a σ-p.i.c.-functor. The next statement is evident.*

Proposition 3.16 . If F $=U_{n=0}^{\infty} F^n$, where each F^n is a σ-p.i.c.-functor, then F is a σ-p.i.c.-functor.

Recall that by $P : Comp \to Comp$ they denote the functor of Borel regular probability measures. This functor is normal [11]. According to Proposition 3.4 and Remark 3.5 we shall denote $(P_k)_\beta$ *by* P_k.

Theorem 3.17 . The functor P_k is a σ-p.i.c.-functor for any positive integer k.

Proof. For a compact space X and a positive integer n we set

$$P^n (X) = \left\{ \mu \in P (X) : \mu (x) \geq {}^1\!/_n \ \forall x \in supp (\mu) \right\}. \tag{2.4}$$

It is easy to verify that P^n is a subfunctor of P in $Comp$. Moreover, $P^n \subset P_n$. Further, we put $P_k^n = (P^n)_k$. It is a routine to check that

$$P_k^n \text{ is a normal functor} \tag{2.5}$$

. Now we are going to prove that

$$P_k^n \text{ is a finitely open functor.} \tag{2.6}$$

According to Definition 3.8, property (2.6) is equivalent to

$$P_k^n (m+1) \setminus P_m^n (m+1) \text{ is closed in } P_k^n (m+1) \text{ for all } m \leq k-1. \tag{2.7}$$

Let $\mu_l \in P_k^n (m+1) \setminus P_m^n (m+1)$, $l \in \omega$. Assume that the sequence $\{\mu_l\}$ converges to a measure $\mu \in P_k^n (m+1)$. Every measure $\nu \in P_k (m+1)$ is a convex combination of Dirac measures $\delta (i)$, $i = 0, \ldots, m$. *Let* $\mu_l = \sum_{i=0}^m a_i^l \delta (i)$ and $\mu = \sum_{i=0}^m a_i \delta (i)$. Since μ_l converges to μ, every sequence a_i^l converges to a_i, $i = 0, \ldots, m$. By a choice of μ_l we have $|supp(\mu_l)| = m+1$, that is supp$(\mu_l) = m+1$. Hence, $a_i^l \geq 1/n$ for all l and i. Consequently, $a_i \geq 1/n$ for all i. Then $\mu \in P_k^n (m+1) \setminus P_m^n (m+1)$. Property (2.7) is checked, P_k^n is a finitely open functor.

Theorem 3.12 and propery (2.5) imply that $(P_k^n)_\beta$ is a p.i.c-functor. In accordance with Proposition 3.7 this functor is closed in P_k. Further, a point $x \in X$ belongs to supp(μ), where $\mu \in P_k$, if and only if $\mu (x) > 0$. Consequently, P_k is a union of $(P_k^n)_\beta$. Thus P_k is a σ-p.i.c.-functor. Theorem 3.17 is proved.

Remark 3.18 . *Actually, we proved more: the functor P_k is a union of its normal finitely open subfunctors* $(P_k^n)_\beta$.

3.2 Applications to dimension

Recall some auxiliary notions and facts.

Definition 3.19 *.[2]. A network for a space X is a collection N of subsets of X such that whenever $x \in U$ with U open, there exists $F \in N$ with $x \in F \subset U$.*

A family A of subsets of X is said to be σ *-locally finite* if it is a union of countably many families A_n which are locally finite in X.

Definition 3.20 *.[17]. A topological space X is called a σ-*space*, if it has a σ–locally finite network.*

In 1969 K.Nagami [16] has invented more general class than class of σ–spaces.

Definition 3.21 *. A space X is a (strong)-$\sum$-space if there exist a σ-discrete collection N and a cover c of X by closed countably compact (compact) sets such that, whenever $C \in c$ and $C \subset U$ with open U, then $C \subset F \subset U$ for some $F \in N$.*

Clearly, from Definitions 3.20 and 3.21 we have

Proposition 3.22 *.[16].* Every perfect preimage of a σ-space is a strong $\sum$-space. In particular, every σ-space is a $\sum$-space.

K.Nagami [16] has shown that the class of strong $\sum$-spaces is strictly larger than the class of perfect preimages of σ–spaces. On the other hand, the class of perfect preimages of σ-spaces is much larger than the class of σ-spaces. For example, every compact σ-space is metrizable.

The class of paracompact p-spaces in sense of A.V.Arhangel'skii is a proper subclass of paracompact strong $\sum$-spaces.

Definition 3.23 *.[3]. A space X is called a p -space if there exists a countable family u_n such that:*

1) u_n consists of open subsets of βX;

2) $X \subset \cup u_n$ for each n;

3) $\cap_n st\,(x, u_n) \subset X$ for every $x \in X$.

Here for a family v of subsets of a space Y by $st(y, v)$ we denote the set $\cup\{V \in v : y \in V\}$

Theorem 3.24 *.[3].* The class of paracompact p-spaces coincides with the class of perfect preimages of metrizable spces.

Corollary 3.25 . Every paracompact p-space is a perfect preimage of a paracompact σ-space and, consequently, is a paracompact $\sum$-space.

Definition 3.26 *.[7]. A space X is stratifiable if there is a function G which assigns to each $n \in \omega$ and closed set $H \subset X$, an open set $G(n, H)$ containing H such that*
(1) if $H \subset K$, then $G(n, H) \subset G(n, K)$;
(2) $H = \cap_n Cl\,(G\,(n, H))$.

The class of stratifiable spaces was defined in 1961 by J.Ceder [7]. But he called these spaces by M_3-spaces. The latter form was proposed by C.R.Borges [6] in 1966. Clearly, every metrizable space is stratifiable.

Theorem 3.27 *.[14].* Stratifiable spaces are paracompact σ-spaces.

Proposition 3.22 and Theorem 3.27 imply

Corollary 3.28 . Every statifiable space is a paracompact $\sum$- space.

Theorem 3.29 *.[20].* Let F be a p.i.c.-functor of a finite degree. Then F_β preserves the class of paracompact $\sum$-spaces.

Corollary 3.13 and Theorem 3.29 yield

Corollary 3.30 . Every normal finitary functor of a finite degree, in particular the functor $\exp_k$, preserves the class of paracompact $\sum$-spaces.

Theorem 3.31 *.[21].* Let F be a p.i.c.-functor of finite degree k, and let X be a paracompact $\sum$-space. Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)$$

We shall say that a σ-p.i.c.-functor F has degree k if in a representation $F = \cup_{n=0}^{\infty} F^n$ from Definition 3.14 every functor F^n has degree $\leq k$.

Theorem 3.32 . Let $F : Tych \to Tych$ be a σ-p.i.c.-functor of finite degree k, let X be a paracompact $\sum$-space, and let F(X) be a normal space. Then

$$\dim F\,(X) \leq k \dim X + \dim F\,(k) \tag{3.1}$$

Proof. Let $F = \cup_{n=0}^{\infty} F^n$ be a representation from Definition 3.14. According to Theorem 3.31 we have

$$(3.1)^n \quad \dim F^n(X) \leq k \dim X + \dim F^n(k)$$

for each $n \in w$. But every $F^n(X)$ is a closed subspace of a normal space $F(X)$. Consequently, inequalities $(3.1)^n$ and the countable sum theorem for dim imply inequality (3.1). Theorem 3.32 is proved.

Theorem 3.32 and respectively Proposition 3.22, Corollaries 3.25 and 3.28 imply the following corollaries.

Corollary 3.33 . Let $F : Tych \to Tych$ be a σ-p.i.c.-functor of finite degree k, let X be a paracompact σ-space, and let $F(X)$ be a normal space. Then

$$\dim F(X) \leq k \dim X + \dim F(k).$$

Corollary 3.34 . Let $F : Tych \to Tych$ be a σ-p.i.c.-functor of finite degree k, let X be a paracompact p-space, and let $F(X)$ be a normal space. Then

$$\dim F(X) \leq k \dim X + \dim F(k)$$

Corollary 3.35 . Let $F : Tych \to Tych$ be a σ-p.i.c.-functor of finite degree k, let X be a stratifiable space, and let $F(X)$ be a normal space. Then

$$\dim F(X) \leq k \dim X + \dim F(k)$$

Definition 3.36 . *Let G_1, G_2 be subfunctors of a functor $F : C \to C$, where either $C = Comp$ or $C = Tych$. Their intersection $G = G_1 \cap G_2$ is defined as:* $G(X) = G_1(X) \cap G_2(X)\ \ for\, every\, X \in O(C)$,

$$G(f) = F(f)\,|G(X)\ \ for\, every\, f : X \to Y\, from\, M(C).$$

Proposition 3.37 . Let G_1, G_2 be normal subfunctors of a normal functor $F : Comp \to Comp$ of finite degree $\leq k$. Then their intersection $G = G_1 \cap G_2$ is a normal functor of degree $\leq k$.

Proof. Clearly, every subfunctor of a normal functor preserves the empty set, singletons, epimorphisms and weight. Now we prove that

$$G \text{ preserves intersections.} \tag{3.2}$$

Since we are in *Comp*, it suffices to show that G preserves intersections of two sets. Let $A_i \subset_{cl} X$, $i = 1, 2$. We have

$G(A_1 \cap A_2) = G_1(A_1 \cap A_2) \cap G_2(A_1 \cap A_2)$ =(since G_i preserve intersections)$= \cap_{i,j=1}^{2} G_j(A_i) = G(A_1) \cap G(A_2)$ by definition of $G_1 \cap G_2$.

Further,

(3.3) every monomorphic functor H of a finite degree is epimorphic.

Let us notice that here a monomorphic functor H is called a functor of finite degree $\leq k$ if for any $a \in H(X)$ there is a set $X_0 \subset X$ such that $|X_0| \leq k$ and $a \in H(X_0)$ (Shchepin's definition). Assume that $f : X \to Y$ is an epimorphism and $a \in H(Y)$. There is a set $Y_0 \subset Y$ such that $|Y_0| \leq k$ and $a \in H(Y)$. Since f is an epimorphism, there is a set $X_0 \subset X$ such that $f_0 \equiv f|X_0$ is a bijection and $f(X_0) = Y_0$. The mapping f_0 is a homeomorphism. Consequently, $H(f_0) : H(X_0) \to H(Y_0)$ is a homeomorphism too. Hence, there is an element $b \in G(X_0)$ such that $H(f)(b) = a$. By definition we have $H(f_0) = H(f)|H(X_0)$. Thus, $H(f)(b) = a$. Property (3.3) is checked.

Since $\deg F \leq k$, we have that $\deg_s F \leq k$, where deg_s is Shchepin's definition of a degree. Clearly, $deg_s H < deg_s F$ for any subfunctor H of F. Consequently, from (3.3) we get that G is a continuous functor being a mononomorphic and epimorphic functor of a finite degree (look at [18], Proposition 3.7). Hence, $\deg G \leq k$ in accordance with Proposition 2.9.

Finally,

(3.4) G preserves preimages.

In fact, let $f : X \to Y$ be a mapping and $A \subset_{cl} Y$. Then $G(f^{-1}(A)) = G_1(f^{-1}(A)) \cap G_2(f^{-1}(A))$ = (because G_i preserve preimages)= $G_1(f)^{-1}\left(G_1(A) \cap G_2(f)^{-1}\right)(G_2(A)) \supset G_1(f)^{-1}\left(G(A) \cap G_2(f)^{-1}\right)\left(G(A) \supset G(f)^{-1}\right)(A)$

On the other hand, let $a \in G(f^{-1}(A)) = G_1(f)^{-1}(G_1(A)) \cap G_2(f^) - 1(G_2(A))$. Then $G_i(f)(a) \in G_1(A) \cap G_2(A) = G(A)$. Consequently, $G(f)(a) \in G(A)$, that is $a \in G(f)^{-1}(A)$. Property (3.4) is checked. Proposition 3.37 is proved.

Proposition 3.38 . Let $G_1,\ G_2$ be normal subfunctors of a normal functor F in *Comp.* Then

$$(G_1)_\beta \cap (G_2)_\beta = (G_1 \cap G_2)_\beta$$

.

Proof. Denote $G_1 \cap G_2$ by G. According to Proposition 3.37 G is a normal functor. Let X be a Tychonoff space. Then $(G_i)_\beta(X) = G_i(\beta X) \cap F_\beta(X)$ by Proposition 2.17.

Consequently, $(G_1)_\beta(X) \cap (G_2)_\beta(X) = G_1(\beta X) \cap G_2(\beta X) \cap F_\beta(X) = G(\beta X) \cap F_\beta(X)$=(in view of Proposition 1.17)=$G_\beta(X)$. Proposition 3.38 is proved.

Theorem 3.39 . Let F be a normal subfunctor of P_k in *Comp.* Then F_β is a σ -p.i.c.-functor.

Proof. According to Remark 3.18 we have

$$(P_k)_\beta \equiv P_k = U_{n=0}^{\infty}(P_k^n)_\beta.$$

On the other hand, F_β is a closed subfunctor of $(P_k)_\beta$, because of Proposition 3.7. Consequently,

$$F_\beta = \cup_{n=0}^{\infty}\left((P_k^n)_\beta \cap F_\beta\right).$$

Hence, Proposition 3.38 yields $F_\beta = \cup_{n=0}^{\infty}(P_k^n \cap F)_\beta$.

Thus, according to Definition 3.14 it remains to show that $P_k^n \cap F$ is a p.i.c.-functor of a finite degree. Property (2.5) and Proposition 3.37 imply that $P_k^n \cap F$ is a normal functor. Since of Proposition 2.9, $\deg(P_k^n \cap F) \leq$ k. Further. P_k^n is a finitely open functor, because of property (2.6). Then $P_k^n \cap F$ is a finitely open functor by Proposition 3.10. Finally, $P_k^n \cap F$ is a p.i.c.-functor in accordance with Theorem 3.12. Theorem 3.39 is proved.

Theorems 3.32 and 3.39 yield

Theorem 3.40 . Let F be a normal subfunctor of P_k in *Comp*, let X be a paracompact $\sum$-space, and let $F_\beta\,(X)$be a normal space. Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)\,.$$

Corollaries 3.33. 3.34. 3.35, and Theorem 3.40 imply the following corollaries.

Corollary 3.41 . Let F be a normal subfunctor of P_k in *Comp*, let X be a paracompact σ-space, and let $F_\beta\,(X)$ be a normal space. Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)\,.$$

Corollary 3.42 . Let F be a normal subfunctor of P_k in *Comp*, let X be a paracompact p-space, and let $F_\beta\,(X)$ be a normal space. Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)\,.$$

Corollary 3.43 Let F be a normal subfunctor of P_k in *Comp*, let X be a stratifiable space, and let $F_\beta\,(X)$ be a normal space. Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)\,.$$

Theorem 3.44 . Let F be a normal subfunctor of P_k in *Comp*, let X be a paracompact p-space (in particular, metrizable space). Then

$$\dim F_\beta\,(X) \leq k \dim X + \dim F\,(k)\,.$$

If $F = P_k$,then

$$\dim P_k\,(X) \leq k \dim X + k - 1.$$

Proof. In view of Corollary 3.42 it suffices to show that $F_\beta(X)$ is a normal space. But it was proved in [1] that $P_\beta(X)$ is a paracompact p- space for any paracompact p-space X. Further, $(P_k)_\beta$ is a closed subfunctor of P_β by Corollary 3.5. According to Proposition 3.7 F_β is a closed subfunctor of $(P_k)_\beta$. Hence, $F_\beta(X)$ is a closed subspace of $P_\beta(X)$. Since Theorem 3.24, a closed subspace of a paracompact p-space is a paracompact p-space. Consequently, $F_\beta(X)$ is a paracompact p-space. Theorem 3.44 is proved.

Theorem 3.45 . Let F be a normal subfunctor of P_k in *Comp*, let X be a stratifiable space. Then

$$\dim F_\beta(X) \leq k \dim X + \dim F(k).$$

If $F = P_k$,then

$$\dim P_k(X) \leq k \dim X + k - 1.$$

The proof is similar to that of Theorem 3.44. The only difference is that instead of [l] one has to mention [13], where a preservation of the class of stratifiable spaces by the functor P_k was proved.

Let us recall

Definition 3.46 . *A normal space X is said to be weakly countable-dimensional if X is a union of a countable family of closed subsets X_i such that* $\dim X_i < \infty$ *for each i.*

The next two statements are trivial.

Proposition 3.47 . Every closed subspace of a weakly countable-dimensional space is a weakly countable-dimensional space.

Proposition 3.48 . Every normal space which is a union of a countable family of its closed weakly countable-dimensional subspaces is weakly countable-dimensional.

Theorem 3.49 .*[21].* Let F be a p.i.c.-functor of a finite degree transforming finite sets into finite-dimensional compacta, let X be a weakly countable-dimensional space, and let X belongs to one of the following classes:

a) $\sum$ -paracompact spaces;
b) p-paracompact spaces;
c) σ -paracompact spaces;
d) stratifiable spaces;
e) metrizable spaces.

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Corollary 3.50 *.[21].* Let F be a normal finitary functor of a finite degree, in particular, functor exp_k, let X be a weakly countable-dimensional space, and let X belongs to one of the following classes:

a) $\sum$ -paracompact spaces;
b) p-paracompact spaces;
c) σ -paracompact spaces;
d) stratifiable spaces;
e) metrizable spaces.

Then $F_\beta(X)$ is a weakly countable-dimensional space.

Recall that for a normal functor $F : Comp \to Comp$ by F_ω they denote a σ-compact subfunctor of F_β defined as: $F_\omega(X) = \cup_{k=1}^{\infty} F_k(X)$ for any Tychonoff space X .Thus, $F_\omega = \cup_{k=1}^{\infty} F_k$.

Theorem 3.51 . If X is a weakly countable-dimensional space, which is either p-paracompact (in particular, metrizable) or stratifiable, then $P_\omega(X)$is weakly countable-dimensional.

Proof. Firstly, let us show that $P_\omega(X)$ is normal. If X is p-paracompact, then $P_\omega(X)$ is an $F_\sigma-$set in $P_\beta(X)$ which is p-paracompact (see [1]). Hence, $P_\omega(X)$ is normal (even paracompact) as an $F_\sigma-$subset of a paracompact space. If X is stratifiable, then $P_\omega(X)$ is stratifiable as well (see [13]) and, consequently, normal.

The space $P_\omega(X)$ is a countable union of its closed subsets $(P_k^n)_\beta(X)$. Thus, in view of Proposition 3.48 it suffices to show that $(P_k^n)_\beta(X)$ is a weakly countable-dimensional space. But P_k^n is a finitely open functor in accordance with property (2.6). Consequently, P_k^n is a p.i.c.-functor by Theorem 3.12. Finally, applying Theorem 3.49 we get that $(P_k^n)_\beta(X)$ is a weakly countable-dimensional space. Theorem 3.51 is proved.

Proposition 3.47 and Theorem 3.51 yield

Corollary 3.52 . If X is a weakly countable-dimensional space, which is either p-paracompact (in particular, metrizable) or stratifiable, then $F(X)$ is a weakly countable-dimensional space for an arbitrary closed subfunctor $F\,of\,P_\omega$.

Corollary 3.53 If X is a weakly countable-dimensional space, which is either p-paracompact (in particular, metrizable) or stratifiable, then $P_k(X)$ is weakly countable-dimensional.

Theorem 3.54 . If X is a weakly countable-dimensional space, which is either p-paracompact (in particular, metrizable) or stratifiable, then $\exp_\omega(X)$ is weakly countable-dimensional.

Proof. If X is a paracompact p-space, then $\exp_\beta(X)$ is a paracompact p-space (see [9]). Hence, $\exp_\omega(X)$ is normal as an F_σ-set in the normal space $\exp_\beta(X)$. If X is a stratifiable space, then $\exp_\omega(X)$ is stratifiable (see [15]) and, consequently, normal. Applying Proposition 3.48 and Corollary 3.50 we complete the proof.

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References

[1] Al'-Kassas Yu. A metrizability and a paracompactness of probability measures spaces. Vestn. Mosk. Univ. Ser. Matem.. Mekh. N1 (1993), 14-17.

[2] Arhangel'skii A.V. An addition theorem for the weight of sets lying in bicompacta. Dokl. Akad. Nauk. SSSR. 126:2 (1959). 239-241.

[3] Arhangel'skii A.V. On a class of spaces containing all metric spaces and all locally bicompact spaces. Matem. Sbornik. 67:1 (1965), 55-85.

[4] T.O.Banakh. Topology of probability measure spaces. I. Matem Stud. Proc. Lwow Math. Soc. 5 (1995), 65-87.

[5] V.N.Basmanov. Covariant functors, retracts, and dimension. Dokl. Akad. Nauk SSSR 271:5 (1983), 1033-1036.

[6] Borges C.J.R. On stratifiable spaces. Pasif. J. Math. 17:1 (1966), 1-16.

[7] Ceder J.G. Some generalizations of metric spaces. Pacif. J. Math. 11 (1961), 105-125.

[8] Chigogidze A.Ch. Extension of normal functors. Vestn. Mosc. Univ. Ser. Matem., Mekh. N6 (1984), 40-42.

[9] Choban M.M.. On exponential topology. Dokl. Akad. Nauk SSSR 186:2 (1969),?

[10] Engelking R. General topology. Warszawa, PWN. 1977.

[11] Fedorchuk V.V. Covariant functors in the category of compact a, absolute retracts, and Q-manifolds. Uspekhi Matem. Nauk. 36:3 (1981), 177-195.

[12] Fedorchuk V.V. and Filippov V.V. General topology: Basic constructions. Moscow, Mosc. State Univ. 1988.

[13] Guo B.-L. and Sakai K. Spaces of measures on stratifible spaces. Kobe J. Math. 12 (1995), 161-173.

[14] Heath R.W. and Hodel R.E. Characterization of σ-spaces. Fund. Math. 77:3 (1973), 271-275.

[15] Mizokami T. and Koiwa T. On hyperspaces of compact and finite subsets. Bull. Joetsu Univ. Educ. 6 (1987), 1-14.

[16] Nagami K. Σ-spaces. Fund. Math. 65 (1969). 169-192.

[17] Okuyama A. Some generalizations of metric spaces, their metrization theorems, and product spaces. Sci. Rep. Tokyo-Kyoiku-Daigaku. Sect.-A. 9 (1968), 236-254.

[18] Shcepin E.V. Functors and uncountable powers of compact spaces. Uspekhi Matem. Nauk. 36:3 (1981). 3-62.

[19] Zhuraew T.F. On projectively quotient functors. CMUC 42:3 (2001), 561-573.

[20] Zhuraev T.F. On paracompact spaces and projectively inductively closed functors. Applied General Topology 43:1 (2002). 31-44.

[21] Zhuraev T.F. Geometrical and topological properties some a covariant functors Vestn.Kr.CU. (2010). 10:9. 38-40.

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