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On Denseness in Asymmetric Metric Spaces

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Abstract. First, we introduce forward (backward) sets in an asymmetric metric space. Then, we prove some theorem and results; As an important result we prove that if X is forward and backward compact asymmetric metric space and $Y \subset X$ both forward and backward dense in X, Then $\tau_+ = \tau_-$

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1.Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$.

In the realms of applied mathematics and materials science we defind many recent application of asymmetric metric spaces; for example, in rate- independent models for plasticity [2], shape- memory alloys [4], and models for material failure [5]. There are other application of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton- Jacobi equations [3] in mind. The study of asymmetric metrics apparently goes back to Wilson [6]. Following his terminology, asymmetric metrics are often called quasi-metrics. Author in [1] has discussed completely on asymmetric metric spaces.In this work we introduce the concept of "denseness" in asymmetric metric spaces. We recall the elementary definitions and other results that will be needed in the sequel.

Definition 1. 1. A function $d: X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if:

(1) For every $x, y \in X, d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,

(2) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.2 [1,example 3.5]. Let $\propto > 0$. Then d: $\mathbb{R} \times \mathbb{R} \to \mathbb{R} \geq 0$ defind by

$$d(x, y) = \begin{cases} e^{y} - e^{x} & y \geq x, \\ e^{-y} - e^{-x} & y < x, \end{cases}$$

Is an asymmetric metric.

Definition 1.3. The forward topology τ_{+} induced by d is the topology generated by the forward open balls $B^{+}(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$ for $x \in X, \varepsilon > 0$.

Likewise, the backward topology τ_{-} induced by d is the topology generated by the backward open balls $B^{-}(x, \varepsilon) = \{y \in X: d(y, x) < \varepsilon\}$ for $x \in X, \varepsilon > 0$.

Definition 1.4. A sequence $\{x_k\}_{k \in \mathbb{N}}$ forward converges to $x_0 \in X$ if and only if

$\lim_{k\to\infty} d(x_0, x_k) = 0$, respectively $\lim_{k\to\infty} d(x_k, x_0) = 0$. Then we write $x_k \xrightarrow{f} x_0, x_k \xrightarrow{b} x_0$ respecttively.

Example 1. 5. Let $(\mathbb{R}, d)$ be an asymmetric space, where d is as in example 1. 2. It is easy to show that the sequence $\left\{x + \frac{1}{n}\right\}_{n \in \mathbb{N}}$ $(x \in \mathbb{R})$ is both forward and backward converges to x.

Definition 1.6. Suppose (X, d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f: X \to Y$ be a function. We say f is forward continuous, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^{+}(x, \delta)$ implies $f(y) \in B^{+}(f(x), \varepsilon)$, respectively $f(y) \in B^{-}(f(x), \varepsilon)$. However, note that uniform forward continuity and uniform backward continuity are the same.

Definition 1.7. A set $S \subset X$ is forward compact if every open cover of S in the forward topology has a finite subcover. We say that S is forward relatively compact if $\bar{S}$ is forward compact, where $\bar{S}$ denotes the closure in the forward topolpgy. We say S is forward sequentially compact if every sequence has a forward convergent subsequence with limit in S. Finally, S $\subset$ X is forward complete if every forward Cauchy sequence is forward convergent.Note that there is a corresponding backward definition in each case, which is obtained by replacing ‘forward’ with ‘backward’ in each definition. For further, see [1,3].

2. Main Results

In this section, we define the concept of forward (backward) dense sets in asymmetric metric spaces. Then we obtain some results. Throughout this section (X, d) denotes an asymmetric metric space.

Definition 2. 1. A subset Y of X is called forward dense when for each $\varepsilon > 0$ and

$x \in X, B^+(x,\varepsilon) \cap Y \neq \emptyset$. Smillary, one can define backward dense.

Example 2. 2. Let $\propto > 0$ be a constant, $X = \mathbb{R}, d: X \times X \to \mathbb{R}_0^+$ defined by

$$d(x,y) = \begin{cases} y - x & y \geq x, \\ \propto (x-y) & y < x. \end{cases}$$

Then (X,d) is an asymmetric metric space. We claim that $Y = Q$ is forward dense in X. For this, for given $\varepsilon > 0$ and $x \in X$, we have $B^+(x,\varepsilon) = \{y \in \mathbb{R}: d(x,y) < \varepsilon\}$. There are two case:

Case 1: $d(x,y) = y - x < \varepsilon$. Then $y < x + \varepsilon$. So $B^+(x,\varepsilon) = (-\infty, x+\varepsilon)$,

Clearly, $B^+(x,\varepsilon) \cap Y \neq \emptyset$.

Case 2: $d(x,y) = \propto (x-y) < \varepsilon$. Then $y > x - \frac{\varepsilon}{\alpha}$. Hence $B^+(x,\varepsilon) = \left(x - \frac{\varepsilon}{\alpha}, +\infty\right)$,

Which again $B^+(x,\varepsilon) \cap Y \neq \emptyset$. Therefore, Y is forward dense in X.

Theorem 2. 3. Let $f: (X, d_X) \to (Z, d_Z)$ be a forward continuous and onto map. Assume that $Y \subset X$ be forward dense in X. Then $f(Y)$ is forward dense in Z.

Proof. Fixed $\varepsilon > 0$ and $z \in Z$. There exists $x \in X$ so that $f(x) = z$. We show that

$B^+(z,\varepsilon) \cap f(Y) \neq \emptyset$. Since f is forward continuous, so there exists $\delta > 0$ such that $d_X(x,y) < \delta$ implies

$$d_Z(f(x), f(y)) < \varepsilon \qquad 2.3.1$$

Also, we have

$$B^+(x,\delta) \cap Y \neq \emptyset$$

Since Y is forward dense in X. Let $y^* \in B^+(x,\delta) \cap Y$. Thus $d_X(x,y^*) < \delta$ and $y^* \in Y$. Now, by 2. 3. 1, we have $d_Z(f(x), f(y^*)) < \varepsilon$, which means $f(y^*) \in B^+(f(x),\varepsilon) = B^+(z,\varepsilon)$. Clearly $f(y^*) \in f(Y)$.

Lemma 2. 4. Let $Y \subset X$ be forward dense in X. Then for each $x \in X$, there exists a sequence such as $(y_n) \subset Y$ so that $y_n \xrightarrow{f} x$ as $n \to \infty$.

Proof. Set $\varepsilon = \frac{1}{n}$ $(n \in \mathbb{N})$. Then for each n, there exists $y_n \in B^+(x,\varepsilon) \cap Y$. Since Y is forward dense in X. this implies $d(x,y_n) < \frac{1}{n}$ for each n. Now, if n is sufficiently large, then $y_n \xrightarrow{f} x$ as $n \to \infty$.

We know from [6], compactness implies the uniqueness of limits of sequence. By this fact and lemma 2. 4, we obtain the next theorem.

Theorem 2. 5. Let X be forward (backward) compact and Y is forward (backward) dense in X. Then $B^+(x,\varepsilon) = \{x\}(B^-(x,\varepsilon) = \{x\})$ for each $\varepsilon > 0$ and $x \in X$.

Proof. For given $\varepsilon > 0$ and $x \in X$, let $y \in B^+\left(x, \frac{\varepsilon}{2}\right)$. Then $(x,y) < \frac{\varepsilon}{2}$. On the other hand, $y \in X$ and Y is forward dense in X. So by lemma 2. 4, there exists a sequence, say (y_n), such that $y_n \xrightarrow{f} y$ as $n \to \infty$. Therefore, we may choose $N \in \mathbb{N}$ so that $d(y, y_n) < \frac{\varepsilon}{2}$, for all $n \geq \mathbb{N}$. Now, we obtain $d(x, y_n) \leq d(x,y) + d(y, y_n) < \varepsilon$, for all $n \geq \mathbb{N}$. This means $y_n \xrightarrow{f} x$ as $n \to \infty$. Since $\varepsilon > 0$ and $x \in X$ where arbitrary, so the proof is completed.

Remark 2. 6. If we remove the condition "forward compactness" in theorem 2. 5, then the assertion need not be hold. For this, set $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}_0^+$ by

$$d(x,y) = \begin{cases} y - x & y \geq x, \\ \propto (x - y) & y < x, \end{cases}$$

Where $\propto > 0$ is a constant. We know that $\mathbb{R}$ is not forward compact. Also Y is forward dense in X, by example 2. 2; whereas $B^+(x,\varepsilon) \neq \{x\}$.

Corollary 2. 7. Let X be forward and backward compact and $Y \subset X$ both of forward and backward dense in X, Then $\tau_+ = \tau_-$.

Proof. By theorem 2. 5, for each $\varepsilon > 0$ and $x \in X$. We have $B^-(x,\varepsilon) = B^+(x,\varepsilon) = \{x\}$.

Therefore $\tau_+ = \tau_-$. Finally, we prove an interesting result.

Theorem 2. 8. Let X be forward and backward compact, $Y_1 \subset X$ backward dense and $Y_2 \subset X$ forward dense in X such that $Y_1 \cap Y_2 = \emptyset$ then $X = Y_1 \cup Y_2$.

Proof. Suppose $x \in X \setminus (Y_1 \cup Y_2)$. Then, by lemma 2. 4, there exists a sequence, say $(y_n) \subset Y_2$ such that $y_n \xrightarrow{f} x$ as $n \to \infty$. Also there exists a sequence, say $(Z_n) \subset Y_1$, so that $Z_n \xrightarrow{b} x$ as $n \to \infty$. We obtain $(y_n) \in B^+(x,\varepsilon)$ and $(Z_n) \in B^-(x,\varepsilon)$. But $B^-(x,\varepsilon) = B^+(x,\varepsilon) = \{x\}$ by corollary 2. 7. Therefore, there exists $N \in \mathbb{N}$ such that $y_n = Z_n$ for all $n \geq \mathbb{N}$. This implies $Y_1 \cap Y_2 = \emptyset$ which is contraction. Hence, $X = Y_1 \cup Y_2$.

Remark 2. 9. The condition "compactness", in theorem 2. 8, is not necessary.

Consider$X = \mathbb{R}$, $Y_1 = Q$ and $Y_2 = Q^c$ in example 2. 2. Clearly, $Y_1 \cap Y_2 = \emptyset$. Also, $\mathbb{R} = Q \cup Q^c$. But $\mathbb{R}$ is not compact (forward/backward).

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