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On a chaotic weighted shift $z^p D^{p+1}$ of order p in generalized Fock-Bargmann spaces

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Abstract

This article is devoted to the study of the chaotic properties of some specific bakward shift unbounded operators $H_p = z^p D^{p+1}; p = 0, 1, ...$ realized as differential operators in generalized Fock- Bargmann spaces where D is the adjoint of the operator of multiplication by the independent variable z on generalized Fock-Bargmann spaces.

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1 Introduction

A continuous operator T on a Banach space X is said to be hypercyclic if the following condition is met:
There exists an element $\phi \in X$ that its orbit $Orb(T, \phi) = \{\phi, T\phi, T^2\phi, ...\}$ is dense in X and is said to be chaotic in the sense of Devaney **[2, 11]** if the following conditions is met:
1) T is hypercyclic.
2) The set $\{\phi \in X; \exists \quad n \in I\!N$ such that $T^n\phi = \phi\}$ of periodic points of operator T is dense in X.
It is well known that linear operators in finite-dimensional linear spaces can't be chaotic but the nonlinear operator may be. Only in infinite-dimensional

linear spaces can linear operators have chaotic properties.These last properties are based on the phenomenon of hypercyclicity or the phenomen of non-wandercity.
The study of the phenomenon of hypercyclicity originates in the papers by Birkoff [**6**] and Maclane [**19**] that show, respectively, that the operators of translation and differentiation, acting on the space of entire functions are hypercyclic.
The theories of hypercyclic operators and chaotic operators have been intensively developed for bounded linear operator, we refer to [**6, 9, 10**] and references therein and for a bounded operator, Ansari asserts in [**1**] that powers of a hypercyclic bounded operator are also hypercyclic

For an unbounded operator, Salas exhibit in [**22**] an unbounded hypercyclic operator whose square is not hypercyclic. The result of Salas show that one must be careful in the formal manipulation of operators with restricted domains. For such operators it is often more convenient to work with vectors rather than with operators themselves.
Now, let T be an unbounded operator on a separable infinite dimensional Banach space X.

We define the following sets:

$$D(T) = \{\phi \in X; T\phi \in X\} \tag{1.1}$$

$$D(T^\infty) = \bigcap_{n=0}^{\infty} D(T^n) \tag{1.2}$$

The notion of chaos for unbounded operators was defined in [**5**] by Bés et al as follows:

Definition 1.1 *A linear unbounded densely defined operator* $(T, D(T))$ *on a Banach space* X *is called chaotic if the following conditions are met:*
1) T^n *is closed for all positive integers* n..
2) there exists an element $\phi \in D(T^\infty)$ *whose orbit* $Orb(T,\phi) = \{\phi, T\phi, T^2\phi, ...\}$ *is dense in* X
3) the set $\{\phi \in X; \exists \quad m \in \mathbb{N}$ *such that* $T^m\phi = \phi\}$ *of periodic points of operator* T *is dense in* X.

Recently these theories are begin developed on some concrete examples of unbounded linear operators, see [**4,7,12**]. In [**12**] it has been shown that the operators $H_p = z^p \frac{d^{p+1}}{dz^{p+1}}; p = 0, 1, ...$ are chaotic in the sense of Definition 1.1

on the classic Fock-Bargmann space [**3**] :

$$F_2 = \{\phi : \mathbb{C} \longrightarrow \mathbb{C}\, entire; \int_{\mathbb{C}} | \phi(z) |^2 e^{-|z|^2} dxdy < \infty\} \quad (1.3)$$

where $z = x + iy$.

In the present work, we consider generalized Fock-Bargmann spaces (the spaces of entire functions with $e^{-|z|^\beta}$ measure; $\beta > 0$) and we shall prove that the operators $H_p = z^p D^{p+1}; p = 0, 1, ...$ in these spaces are chaotic where D is the adjoint operator of the operator of multiplication by the independent variable z on these spaces. D belongs to class Gelfond-Leontiev operators of generalized differentiation [**8**]

This paper is organized as follows : In section 2 we give some elementary properties of generalized Fock-Bargmann spaces and the action of $H_p = z^p D^{p+1}; p = 0, 1, ...$ on these spaces. In section 3 we recall some sufficient conditions on hypercyclicity of unbounded operator given by Bès-Chan-Seubert theorem [**5**]. As our operator H_p is a unilateral weighted backward shift with an explicit weight, we use the results of Bès et al to proof the chaoticity of H_p in generalized Fock-Bargmann spaces (we can also use the results of Bermudez et al [**4**] to proof the chaoticity of our operator H_p).

2 Action of $z^p D^{p+1}$ of order p on generalized Fock-Bargmann spaces

We define the generalized Fock-Bargmann space by :

$$F_\beta = \{\phi : \mathbb{C} \longrightarrow \mathbb{C}\, entire; \int_{\mathbb{C}} | \phi(z) |^2 e^{-|z|^\beta} d\mu(z) < \infty\} \quad (2.1)$$

where $\beta > 0$ is an arbitrary constant, $d\mu(z) = \frac{\beta}{2\pi\Gamma(\frac{2}{\beta})} dxdy$ and $z = x + iy$.
Note that F_2 coincides with the classic Fock-Bargmann space.
F_β is a Hilbert space with an inner product

$$< \phi, \psi >= \frac{\beta}{2\pi\Gamma(\frac{2}{\beta})} \int_{\mathbb{C}} \phi(z)\overline{\psi(z)} e^{-|z|^\beta} dxdy \quad (2.2)$$

and the associated norm is denoted by $||\ .\ ||$.

Let $m_0 = 0$, $m_n = \frac{\Gamma(\frac{2}{\beta}(n+1))}{\Gamma(\frac{2n}{\beta})}$ $n = 1, 2, ...$ and $[m_n]! = m_1.m_2......m_n$ then it may be shown that the functions

$$e_0(z) = 1 \text{ and } e_n(z) = \frac{z^n}{\sqrt{[m_n]!}}; n = 1, 2, ... \tag{2.3}$$

form a complet orthonormal set in F_β.

Define the principal vectors $e_\lambda \in F_\beta$ (for every $\lambda \in \mathbb{C}$) as complex valued functions
$e_\lambda(z) = e(z, \lambda) = 1 + \sum_{n=1}^{\infty} e_n(z)\overline{e_n(\lambda)}$ of λ and z in $\mathbb{C}$
If $\phi(z) = \sum_{n}^{\infty} a_n e_n(z)$ then $< \phi, e_\lambda >= \phi(\lambda)$ (the reproducing property) because $\int_{\mathbb{C}} \sum_{n}^{\infty} a_n e_n(z)\overline{(1 + \sum_{n=1}^{\infty} e_n(z)e_n(\lambda))} e^{-|z|^\beta} d\mu(z) = a_0 + \sum_{n=1}^{\infty} a_n e_n(\lambda) \mid\mid e_n \mid\mid = \phi(\lambda)$
or, in other words

$$\phi(z) = \int_{\mathbb{C}} \phi(\lambda)\overline{e_\lambda(z)} e^{-|\lambda|^\beta} d\mu(\lambda) \text{ for all } \phi \in F_\beta \tag{2.4}$$

so that $e_\lambda(z)$ is called a reproducing kernel for F_β

Note that the reproducing kernel $e_\lambda(z)$ is uniquely determined by the Hilbert space F_β and the evaluation linear functional $\phi \in F_\beta \to \phi(z) \in \mathbb{C}$ is a bounded linear functional on F_β.

So applying (2.4) to the function e_z at λ; we get $e_z(\lambda) =< e_z; e_\lambda >$ for z; $\lambda \in \mathbb{C}$ and by the above relations, for $z \in \mathbb{C}$ we obtain

$$\mid\mid e_z \mid\mid = \sqrt{< e_z, e_z >} = \sqrt{e(z, z)}.$$

A systematic study of these generalized Fock-Bargmann spaces can be founded in **[16]** where Irac-Astaud and Rideau have constructed an deformed harmonic algebra (DHOA) on F_β and in **[17]** where Knirsch and Schneider have invesigated the continuity and Schattenvon Neumann p-class membership of Hankel operators with anti-holomorphic symbols on these spaces with $\beta \in \mathbb{N}$.
Note that the generalized Fock-Bargmann spaces F_β are different from the generalized Bargmann spaces E_m $m = 0, 1, ...$ defined in **[13]**.It would be interesting to characterize the orthogonal space of F_β in $L_2(\mathbb{C}, e^{-|z|^\beta} d\mu(z))$ for $\beta \neq 2$.
Now on the generalized Fock-Bargmann representation F_β, we denote the operator of multiplication by the independent variable z on F_β by :

$$M\phi(z) = z\phi(z) \text{ with domain } \mathbb{D}(M) = \{\phi \in F_\beta; z\phi \in F_\beta\} \tag{2.5}$$

The operator M acts on $e_n(z)$ as following:

$$Me_n(z) = \frac{\sqrt{\Gamma(\frac{2}{\beta}(n+2))}}{\sqrt{\Gamma(\frac{2}{\beta}(n+1))}} e_{n+1}(z) \tag{2.6}$$

Then its adjoint is generalized differentiation given by :

$$De_n(z) = \frac{\sqrt{\Gamma(\frac{2}{\beta}(n+1))}}{\sqrt{\Gamma(\frac{2}{\beta}n)}} e_{n-1}(z) \tag{2.7}$$

and for $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ we have $D1 = 0$ and $D\phi(z) = \frac{1}{z}\sum_{n=0}^{\infty} a_n m_n z^n$ where $m_n = \frac{\Gamma(\frac{2}{\beta}(n+1))}{\Gamma(\frac{2}{\beta}n)}$ with domain:

$$I\!D(D) = \{\phi \in F_\beta; D\phi \in F_\beta\} \tag{2.8}$$

Note that if $\beta = 2$ the generalized differentiation operator D is:

$$D\phi(z) = \frac{d}{dz}\phi(z) \tag{2.9}$$

Now we define a family of weighted shifts H_p acting on F_β as following

$$H_p = M^p D^{p+1} \text{ with domain } I\!D(H_p) = \{\phi \in F_\beta; H_p\phi \in F_\beta\} \tag{2.10}$$

Then we get

$$H_p^* e_n(z) = M^{p+1}D^p e_n(z) = \sqrt{m_{n+1}}\prod_{j=1}^{p}[m_{n-j+1}]e_{n+1}(z).$$

i.e. H_p^* is weighted shift with weight $\omega_n = \sqrt{m_{n+1}}\prod_{j=1}^{p}[m_{n-j+1}]$ for $n = 1, ...$ and as we have denoted $[m_n]! = m_1.m_2.......m_n$ then $\omega_n = \sqrt{m_{n+1}}\frac{[m_n]!}{[m_{n-p}]!}$ for $n = 1, ...$

Remark 2.1 *(i) If $\beta \neq 2$ and $p = 0$ then the operator $H_0 = D$ is particular case of Gelfond-Leontiev operator of generalized differentiation* [**8**] *on F_β and coincides with the usual differentiation on F_2.*
(ii) For $\beta = 2$, It is known in [**12**] *that :*
(a) the operator H_p with its domain $I\!D(H_p)$ is an operator chaotic on the classic Fock-Bargmann space.
(b) $H_0\phi_\lambda(z) = \lambda\phi_\lambda(z) \quad \forall \quad \lambda \in \mathbb{C}$, where $\phi_\lambda(z) = \sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}e_n(z)$ and
$\| \phi_\lambda \|^2 = e^{|\lambda|^2}$
(c) The function $e^{-|\lambda|^2}\phi_\lambda(z)$ is called a coherent normalized quantum optics

(see **[18]** *and* **[21]***)*
(d) For $p = 1$, it is known that $H_1 + H_1^*$ is a not selfadjoint operator and it is chaotic on the classic Fock-Bargmann space **[7]** *and this operator play an essential role in Reggeon field theory (see* **[14]** *and* **[15]***)*

Before to show that the operator $H_p = z^p D^{p+1}$ with its domain $I\!D(H_p)$ is an operator chaotic on the generalized Fock-Bargmann space F_β, we begin by

Lemma 2.2 *(i) Let $p \in I\!N$ and $e_p(z) = \dfrac{z^p}{\sqrt{[m_p]!}}$ then $D^m e_p(z) = 0$, $\forall \quad m \geq p+1$*
(ii) Let $\omega_n = \sqrt{m_{n+1}\frac{[m_n]!}{[m_{n-p}]!}}$ with $n \geq p$ and if we denote by $\gamma_{p,n} = \omega_p.\omega_{p+1}........\omega_{n-1}$ for $n \geq p+1$ and $\gamma_{p,p} = 1$ then the function $G_\lambda(z) = e_p(z) + \sum\limits_{n=p+1}^{\infty} \dfrac{\lambda^{n-p}}{\gamma_{p,n}} e_n(z)$ is eigenfunction of $z^p D^{p+1}$ associated to λ for all $\lambda \in \mathbb{C}$ i.e. $z^p D^{p+1} G_\lambda(z) = \lambda G_\lambda(z) \quad \forall \lambda \in \mathbb{C}$
(iii) Let $\tilde{\phi}_\lambda(z) = 1 + \sum\limits_{n=1}^{\infty} \dfrac{\lambda^n}{\sqrt{[m_n]!}} e_n(z)$ then $D\tilde{\phi}_\lambda(z) = \lambda\tilde{\phi}_\lambda(z)$ for all $\lambda \in \mathbb{C}$ and we shall called it the generalized coherent state on F_β.

Proof

i) For $p = 0$ we have $De_0(z) = 0$ then $D^m e_0(z) = 0 \quad \forall \quad m \geq 1$.
For $p \geq 1$ we have $De_p(z) = \sqrt{m_p} e_{p-1}(z)$ then $D^p e_p(z) = \sqrt{[m_p]!} e_0(z)$ and $D^{p+1} e_p(z) = \sqrt{[m_p]!} De_0(z) = 0$, in particular $D^m e_p(z) = 0 \quad \forall \quad m \geq p+1$.
ii) Let $G_\lambda(z) = e_p(z) + \sum\limits_{n=p+1}^{\infty} \dfrac{\lambda^{n-p}}{\gamma_{p,n}} e_n(z)$ with $\gamma_{p,n} = \omega_p.\omega_{p+1}........\omega_{n-1}$ for $n \geq p+1$ and $\gamma_{p,p} = 1$.
Then

$$z^p D^{p+1} G_\lambda(z) = 0 + \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\gamma_{p,n}} z^p D^{p+1} e_n(z) = \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\gamma_{p,n-2}} e_{n-1}(z)$$

$$= \lambda \sum_{n=p}^{\infty} \frac{\lambda^{n-p}}{\gamma_{p,n-1}} e_n(z) = \lambda[e_p(z) + \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\gamma_{p,n-1}} e_n(z)]$$

iii) For $p = 0$ we have $G_\lambda(z) = \tilde{\phi}_\lambda(z)$ then $D\tilde{\phi}_\lambda(z) = \lambda\tilde{\phi}_\lambda(z)$ for all $\lambda \in \mathbb{C}$.
Note that for $\lambda = 1$ the function $\tilde{\phi}(z) = 1 + \sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{[m_n]!}} e_n(z)$ is a periodic point of the operator D, i.e $D\tilde{\phi}(z) = \tilde{\phi}(z)$.

Let us now recall some asymptotic properties of analytic functions. They are characterized by their growth and the density of their zeros.

Let $M(R)$ be the maximum modulus of an analytic function $f(z)$ for $|z| = R$. Its growth is described by the order ρ and the type σ, which are defined as follows:

$$- \rho = \overline{lim} \quad \frac{lnlnM(R)}{lnR}; \quad | z |= R \to \infty \tag{2.11}$$

$$- \sigma = \overline{lim} \quad \frac{lnM(R)}{R^\rho}; \quad | z |= R \to \infty \tag{2.12}$$

These definitions imply that $M(R) \sim e^{\sigma R^\rho}$ as R goes to infinity (here the $\sim$ indicates that $M(R)$ is log-asymptotic to $e^{\sigma R^\rho}$)

The relation (2.4) yields an important estimate for the functions in F_β :

$$| \phi(z) | \leq || \phi || \ e^{\frac{1}{2}|z|^\beta} \qquad (2.13).$$

or in other words, F_β is inclued in the set of analytic functions in the complex plane with order $\rho = \beta$ and of type $\sigma = \frac{1}{2}$

We shall now establish some properties on the sequence m_n and on the generalized coherent state $\tilde{\phi}_\lambda(z)$

Lemma 2.3 *(i) Let* $m_0 = 0, m_n = \frac{\Gamma(\frac{2}{\beta}(n+1))}{\Gamma(\frac{2n}{\beta})}; n = 1, 2, ...$
then $m_n \sim (\frac{2}{\beta})^{\frac{2}{\beta}} n^{\frac{2}{\beta}}, n \to +\infty$
(ii) The order of $\tilde{\phi}_\lambda(z)$ *is* $\rho = \frac{\beta}{2}$ *and its type is* $\sigma = 1$

Proof

(i) It is well know that $\Gamma(x) \simeq \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}$ then

$$m_n \simeq \frac{\sqrt{2\pi}[\frac{2}{\beta}(n+1)]^{\frac{2}{\beta}(n+1)-\frac{1}{2}} e^{-\frac{2}{\beta}(n+1)}}{\sqrt{2\pi}[\frac{2}{\beta}n]^{\frac{2}{\beta}n-\frac{1}{2}} e^{-\frac{2}{\beta}n}} \simeq [\frac{\frac{2}{\beta}(n+1)}{\frac{2}{\beta}n}]^{\frac{2}{\beta}n-\frac{1}{2}} [\frac{2}{\beta}(n+1)]^{\frac{2}{\beta}} e^{-\frac{2}{\beta}}$$

$$\simeq [(1+\frac{1}{n})^n]^{\frac{2}{\beta}}][1+\frac{1}{n}]^{\frac{-1}{2}}[\frac{2}{\beta}]^{\frac{2}{\beta}}[n+1]^{\frac{2}{\beta}} e^{-\frac{2}{\beta}} \simeq (\frac{2}{\beta})^{\frac{2}{\beta}} n^{\frac{2}{\beta}}$$

Also note that the property i) can be verified by using the relation

$\frac{\Gamma(z+a}{\Gamma(z+b)} \simeq z^{a-b}$ when $\mid z \mid\to \infty$ uniformly for $\mid argz \mid\leq \pi-\delta$

$(\delta fixed; 0 < \delta < \pi)$ for all $a \in \mathbb{C}$ and $b \in \mathbb{C}$

Consequently we have:

$$Lim \frac{m_n}{(\frac{2}{\beta})^{\frac{2}{\beta}} n^{\frac{2}{\beta}}} = 1, n \to +\infty \qquad (2.14).$$

(ii) A necessary and sufficient condition that $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ should be an integral function of finite order ρ (see Titchmarsh in **[23]**, p.253 or Vourdas in **[24]**, p. 4870) is that

$$\rho = \overline{Lim} \quad \frac{nl_n(n)}{l_n(\frac{1}{|a_n|})}; \qquad n \to \infty \qquad (2.15).$$

and its type σ is determined by the formula

$$(\sigma e \rho)^{\frac{1}{\rho}} = \overline{Lim} \quad n^{\frac{1}{\rho}} \mid a_n \mid^{\frac{1}{n}}; \qquad n \to \infty \qquad (2.16).$$

To apply these results at $\tilde{\phi}_\lambda(z)$ to found its order $\rho = \frac{\beta}{2}$ and its type $\sigma = 1$, we begin by noting that by virtue of property (2.14) we have

$\forall \epsilon > 0, \exists N > 0$ such that $((\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon)n^{\frac{2}{\beta}} \leq\mid m_n \mid\leq (\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon)n^{\frac{2}{\beta}} \quad \forall n \geq N.$

Then $\quad \forall n \geq N$ we have $\mid m_1.m_2......m_N[(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid\leq$

$\mid [m_n]! \mid\leq\mid m_1.m_2......m_N[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid$

and

$$\frac{\mid \lambda^n \mid}{\mid m_1.m_2......m_N[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid} \leq \frac{\mid \lambda^n \mid}{\mid [m_n]! \mid} \leq \frac{\mid \lambda^n \mid}{\mid m_1.m_2......m_N[(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid}$$

Now, we consider the functions :

$$1) \ \tilde{\phi}_1(z) = 1 + \sum_{n=1}^{N} \frac{\lambda^n z^n}{[m_n]!} + \sum_{n=N+1}^{\infty} \frac{\lambda^n z^n}{[m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}}}$$

then by virtue of property (2.15) its order is given by:

$$\rho_1 = \overline{lim} \frac{nl_n(n)}{l_n \mid [m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid} \quad n \to \infty$$

and

$$2)\tilde{\phi}_2(z) = 1 + \sum_{n=1}^{N} \frac{\lambda^n z^n}{[m_n]!} + \sum_{n=N+1}^{\infty} \frac{\lambda^n z^n}{[m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}}}$$

then by virtue of property (2.15) its order is given by:

$$\rho_2 = \overline{lim} \frac{n l_n(n)}{l_n \mid [m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid} \quad n \to \infty$$

The asymptotic development of Gamma function is given by Stirling formula as follows :

$$\Gamma(x) \simeq \sqrt{\frac{2\pi}{x}} (\frac{x}{e})^x [1 + \frac{1}{12x} + ...+] \tag{2.17}$$

By virtue of this property (2.17) we get $\Gamma(n+1) \simeq [\frac{n}{e}]^n \sqrt{2\pi n}$ and

$$l_n[(n!)^\alpha] \simeq \alpha n l_n[\frac{n}{e}(2\pi n)^{\frac{1}{2n}}] \tag{2.18}$$

$\alpha = \frac{2}{\beta}$ which operates in the explicit calculation of ρ_1 or ρ_2

For other asymptotic expansions for the Gamma function, we can see the recent work of Nemes in **[20]**.

Now as

$$\rho_1 = \overline{lim} \frac{n l_n(n)}{l_n \mid [m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{(n!)^{\frac{2}{\beta}}}{(N!)^{\frac{2}{\beta}}} \mid} \quad n \to \infty$$

$$= \overline{lim} \frac{n l_n(n)}{l_n(\frac{[m_N]!}{(N!)^{\frac{2}{\beta}}}) + (n-N) l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + \frac{2}{\beta} l_n(n!)} \quad n \to \infty$$

$$= \overline{lim} \frac{n l_n(n)}{l_n(\frac{[m_N]!}{(N!)^{\frac{2}{\beta}}}) - N l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + n l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + \frac{2}{\beta} l_n(n!)} \quad n \to \infty$$

then by virtue of the property (2.18) we deduce that

$$\rho_1 = \overline{lim} \frac{n l_n(n)}{l_n(\frac{[m_N]!}{(N!)^{\frac{2}{\beta}}}) - N l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + n l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + n \frac{2}{\beta} l_n(\frac{n}{e} \sqrt[2n]{2\pi n})} n \to \infty$$

$$= \overline{lim} \frac{n l_n(n)}{l_n(\frac{[m_N]!}{(N!)^{\frac{2}{\beta}}}) - N l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + n l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + n \frac{2}{\beta}[l_n(\frac{n}{e}) + l_n \sqrt[2n]{2\pi n}]} \quad n \to \infty$$

$$= \overline{lim} \frac{l_n(n)}{\frac{1}{n}[l_n(\frac{[m_N]!}{(N!)^{\frac{2}{\beta}}}) - N l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]] + l_n[(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon] + \frac{2}{\beta} l_n(n) - \frac{2}{\beta} + \frac{2}{\beta} l_n(\sqrt[2n]{2\pi n})} \quad n \to \infty$$

$$= \frac{\beta}{2}$$

Then the order of $\tilde{\phi}_1(z)$ is $\rho_1 = \frac{\beta}{2}$

By taking $[(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon]$ in above calculation, we deduce that the order of $\tilde{\phi}_2(z)$ is $\rho_2 = \frac{\beta}{2}$ and as $\rho_1 \leq \rho \leq \rho_2$ we get the order of $\tilde{\phi}_\lambda(z)$,$\rho = \frac{\beta}{2}$

To obtain the type σ of $\tilde{\phi}_\lambda(z)$, we apply the property (2.16) for the function $\tilde{\phi}_1(z)$ of order $\rho_2 = \frac{\beta}{2}$ then its type σ_1 is given by:

$$(\sigma_1 e \frac{\beta}{2})^{\frac{2}{\beta}} = \overline{Lim} \frac{n^{\frac{2}{\beta}}}{\sqrt[n]{[m_N]![(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon]^{n-N} \frac{n!^{\frac{2}{\beta}}}{N!^{\frac{2}{\beta}}}}} \text{ , } n \to \infty$$

$$= \overline{lim} \frac{n^{\frac{2}{\beta}}}{((\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon) \sqrt[n]{\frac{[m_N]!}{((\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon)^N N!^{\frac{2}{\beta}}} n!^{\frac{2}{\beta}}}} \text{ , } n \to \infty$$

Let $\gamma = \sqrt[n]{\frac{[m_N]!}{((\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon)^N N!^{\frac{2}{\beta}}}}$ then $(\sigma_1 e \frac{\beta}{2})^{\frac{2}{\beta}} = \overline{Lim} \frac{n^{\frac{2}{\beta}}}{((\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon) \gamma \sqrt[n]{n!^{\frac{2}{\beta}}}}$, $n \to \infty$

As $n!^{\frac{2}{\beta}} \simeq [\frac{n}{e}(2\pi n)^{\frac{1}{2n}}]^{\frac{2n}{\beta}}$, then $\sqrt[n]{n!^{\frac{2}{\beta}}} \simeq n^{\frac{2}{\beta}} (\frac{1}{e})^{\frac{2}{\beta}} (\sqrt[2n]{2\pi n})^{\frac{2}{\beta}} \simeq \frac{e^{\frac{2}{\beta}}}{n^{\frac{2}{\beta}} (2\pi n)^{\frac{1}{n\beta}}}$

Then $\sigma_1^{\frac{2}{\beta}} = [\frac{\frac{2}{\beta}}{\frac{2}{\beta}^{\frac{2}{\beta}} + \epsilon}]^{\frac{2}{\beta}} \overline{lim} \frac{1}{\gamma (2\pi n)^{\frac{1}{n\beta}}}$, $n \to \infty$. In particular we have

$\sigma_1 \leq \frac{\frac{2}{\beta}}{\sqrt[\frac{2}{\beta}]{(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon}}$ and $\sigma \geq \frac{\frac{2}{\beta}}{\sqrt[\frac{2}{\beta}]{(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon}}$.

Now by using the calculation of type σ_2 we deduce that

$$\frac{\frac{2}{\beta}}{\sqrt[\frac{2}{\beta}]{(\frac{2}{\beta})^{\frac{2}{\beta}} + \epsilon}} \leq \sigma \leq \frac{\frac{2}{\beta}}{\sqrt[\frac{2}{\beta}]{(\frac{2}{\beta})^{\frac{2}{\beta}} - \epsilon}} \ \forall \quad \epsilon > 0$$

Consequently we have $\sigma = \frac{\frac{2}{\beta}}{\sqrt[\frac{2}{\beta}]{(\frac{2}{\beta})^{\frac{2}{\beta}}}} = 1$.

3 Chaoticity of the operator $H_p = z^p D^{p+1}$ on generalized Fock-Bargmann space F_β

The operator $H_p = z^p D^{p+1}$ is an unbounded operator on generalized Fock-Bargmann space F_β. Consider the subset $\mathbb{P} \subset F_\beta$ consisting of all polynomials which is dense in F_β and it is included in the domain of H_p. Thus it is densely defined.
Now we recall a sufficient condition for the hypercyclicity of an unbounded operator given by $Bès - Chan - Seubert$ theorem

Theorem 3.1 *(Bès-Chan-Seubert* **[5]** *)Let X be a separable infinite dimensional Banach and let T be a densely defined linear operator on X. Then T is hypercyclic if*
i) T^m is closed operator for all positive integers m.
ii) There exist a dense subset Y of the domain $D(T)$ of T and a (possibly nonlinear and discontinuous) mapping $S : Y \longrightarrow Y$ so that $TS = I_{|Y}$ ($I_{|Y}$ is identity on Y) and $T^n, S^n \longrightarrow 0$ pointwise on Y as $n \longrightarrow \infty$.

Let us now formulate and prove the main result of the paper.

Theorem 3.2 *Let F_β be the generalized Fock-Bargmann space with orthonormal basis $e_n(z) = \frac{z^n}{\sqrt{[m_n]!}}$ and $H_p = z^p D^{p+1}$ with domain*

$$\mathbb{D}(H_p) = \{\phi \in F_\beta; H_p\phi \in F_\beta\}$$

Then H_p is chaotic operator in F_β.

We present the proof of chaoticity of H_p (i.e. H_p satisfies the conditions (1) and (2) of Definition (1.1)) under lemmas form.

Lemma 3.3 *Let $H_p e_n = \omega_{n-1} e_{n-1}$ where $e_n(z) = \frac{z^n}{\sqrt{[m_n]!}}$ and $\omega_n = \sqrt{m_{n+1}} \frac{[m_n]!}{[m_{(n-p)}]!}$ for $n \geq p \geq 0$, then for each positive integer m, the operator $(H_p)^m$, with domain $D((H_p)^m) = \{\phi \in F_\beta; (H_p)^m\phi \in F_\beta\}$, is a closed operator*

Proof
As $(H_p)^m$ is closed if and only if the graph $G((H_p)^m)$ is closed linear manifold

of $F_\beta \times F_\beta$ then let $(\phi_n, (H_p)^m \phi_n)$ be a sequence in $G((H_p)^m)$ which converges to (ϕ, ψ) in $F_\beta \times F_\beta$.

As ϕ_n converges to ϕ in F_β then $z^p D^{p+1}\phi_n$ converges to $z^p D^{p+1}\phi$ pointwise on $\mathbb{C}$ and $(H_p)^m)\phi_n$ converges to $(H_p)^m)\phi$ pointwise on $\mathbb{C}$.

As $(H_p)^m\phi_n$ converges to ψ we deduce that $(H_p)^m)\phi = \psi$ and $\phi \in \mathbb{D}((H_p)^m)$ hence the $G((H_p)^m)$ is closed.

Remark 3.4 *The above proof of the closeness of the operator $(H_p)^m$ is analogous to proof of lemma (2.3) from* [**5**] *or lemma (2.5) from* [**12**]

Lemma 3.5 *Let $H_p = z^p D^{p+1}$ with domain $\mathbb{D}(H_p) = \{\phi \in F_\beta; H_p\phi \in F_\beta\}$where $H_p e_n = \omega_{n-1}e_{n-1}$, $e_n(z) = \frac{z^n}{\sqrt{[m_n]!}}$ and $\omega_n = \sqrt{m_{n+1}\frac{[m_n]!}{[m_{(n-p)}]!}}$ for $n \geq p \geq 0$*
Then H_p is hypercyclic.

Proof

Let $Y = \{\phi_k(z) = \sum_{n=p}^{k} a_n e_n(z)\}$ This space is dense in F_β.
Let $S_p : Y \longrightarrow Y$ and $S_p e_n = \frac{1}{\omega_n} e_{n+1}; n \geq p \geq 0$. Then $H_p S_p \phi_k(z) = \phi_k(z)$,i.e $H_p S_p = I_{|Y}$.

By virtue of the property i) of lemma (1.1) $[H_p]^k e_n = 0$ for all $k > n \geq p$ then we deduce that any element of Y can be annihilated by a finite power k_n of H_p since as $[\prod_{j=n}^{k_n+n} \omega_j]^{-1} \longrightarrow 0; k_n \longrightarrow \infty$ we have

$S_p^{k_n} e_n = [\prod_{j=n}^{k_n+n} \omega_j]^{-1} e_{k+n} \longrightarrow 0$ in F_β.

Now the hyperciclycity of H_p follows from the theorem of *Bès* and al. recalled above.

Lemma 3.6 *Let $H_p = z^p D^{p+1}$ with domain $\mathbb{D}(H_p) = \{\phi \in F_\beta; H_p\phi \in F_\beta\}$ where $H_p e_n = \omega_{n-1}e_{n-1}$, $e_n(z) = \frac{z^n}{\sqrt{[m_n]!}}$ and $\omega_n = \sqrt{n+1}\frac{[m_n]!}{[m_{(n-p)}]!}$ for $n \geq p \geq 0$*
Then there exist $k > 0$ and $g \in \mathbb{D}(H_p^k)$ such that $H_p^k g(z) = g(z)$.

Proof

Let $\lambda \in \mathbb{C}$ and

$$G_\lambda(z) = e_p(z) + \sum_{n=p+1}^{\infty} \frac{\lambda^{n-p}}{\omega_p \omega_{p+1} \ldots\ldots \omega_{n-1}} . e_n(z) \tag{3.1}$$

then by virtue of lemma (2.1) G_λ is an generalized eigenvector for H_p corresponding to eigenvalue λ so it is therefore a periodic point of H_p where λ is a root of unity.

We will check that G_λ is in the domain of H_p. In fact let $r > 0$ and $\mid \lambda \mid < r$. As $m_n \to \infty; n \to \infty$ then

$$lim \prod_{j=p}^{n-1} \omega_j = \infty \ ; \ n \longrightarrow \infty \tag{3.2}$$

and there exist $n_0 > 0$ and $q < 1$ such that

$$\frac{r}{(\omega_p \omega_{p+1} \ldots\ldots \omega_{n-1})^{\frac{1}{n}}} \leq q \text{ for } n \geq n_0 \tag{3.3}$$

Since for $\mid \lambda \mid < r$ we have

$$\frac{\mid \lambda \mid^{(n-p)}}{(\omega_p \omega_{p+1} \ldots\ldots \omega_{n-1})^2} \leq q^{2n}; n \geq n_0 \tag{3.4}$$

and G_λ is in generalized Fock-Bargmann space.

Now as

$$< G_\lambda, e_p >= 1 \tag{3.5}$$

and

$$< G_\lambda, e_{n+1} > = \frac{\lambda^{n-p+1}}{\omega_p \omega_{p+1} \ldots\ldots \omega_n} \tag{3.6}$$

we get

$$|< G_\lambda, e_{n+1} >|^2 = \frac{\lambda^{2(n-p+1)}}{(\omega_p \omega_{p+1} \ldots\ldots \omega_n)^2} \tag{3.7}$$

and

$$|< G_\lambda, e_{n+1} >|^2 \, (\omega_n)^2 = \frac{\lambda^{2(n-p+1)}}{(\omega_p \omega_{p+1} ...... \omega_{n-1})^2} \leq q^{2n} \mid \lambda \mid^2 \text{ for } n \geq n_0 \qquad (3.8)$$

we result that

$$\sum_{n=p}^{\infty} |< G_\lambda, e_{n+1} >|^2 \, (\omega_n)^2 < \infty \qquad (3.9)$$

i.e. $G_\lambda \in I\!D(H_p)$.

Let us prove that operator H_p satisfies the condition (2) of the definition (1.1)

Lemma 3.7 *The set of periodic points of H_p is dense in F_β.*

Proof

let $\lambda_{k,m} = e^{\frac{2ik\pi}{m}}, m \in I\!N, k = 0, 1, ..., m-1$ is a root of unity and
$G = Span\{G_{\lambda_{k,m}}(z)\}$
By virtue of the property (2.4), we deduce that the system $G_{\lambda_{k,m}}$ is complete in F_β and the linear span G of this system is dense in F_β.
Or for a direct proof, we assume that there exist a nonzero vector $g \in F_\beta$ which is orthogonal to G.

Let

$$g(z) = \sum_{n=p}^{\infty} b_n e_n(z) \qquad (3.10)$$

such that

$$< g, G_\lambda >= 0 \text{ for each } G_\lambda \in G \qquad (3.11).$$

and

$$\phi(\lambda) =< g, g_\lambda > \text{ for } \mid \lambda \mid < 1 \text{ and } \phi(\lambda) = 0 \text{ for } \mid \lambda \mid = 1 \qquad (3.12)$$

$\phi(\lambda)$ is continuous function on the closed units disc that is holomorphic on the interior.$\phi(\lambda)$ vaniches at each root of unity, hence on the entire unit circle hence $\phi(\lambda)$ vaniches for all $\lambda \mid \leq 1$.

We deduce that $b_n = 0$ for $n \geq p$ then G is dense in F_β.

Remark 3.8 *The lemmas (3.3), (3.5) , (3.6) and (3.7) show the chaoticity of H_p.*

We conclude that main results of this work can be considered as a generalization of the result in **[12]** on operator H_p acting in classic Fock-Bargmann space F_2.

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