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Note on an open question regarding an integral inequality

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Abstract

In the paper "Improved Answers To An Open Problem Concerning An Integral Inequality" published in Mathematica Aeterna, Vol. 2, 2012, no. 4, 321 - 324, an open question was posed. In this short paper, we give the solution of the mentioned paper.

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1 Introduction

The following problem was proposed in the paper [2]:

Problem 1.1. *Assume constant* $\gamma > 0$. *Let* $f(x) \geq 0$ *be a continuous function on* $[0,1]$ *satisfying the inequality*

$$\int_t^1 f^{\gamma}(x)\,dx \geq \int_t^1 x^{\gamma} dx, \forall t \in [0,1]. \tag{1.1}$$

Does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\alpha} f^{\beta}(x)\,dx \tag{1.2}$$

hold for $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta < \gamma$? .

In the present paper, we give the solution of the problem presented in [2].

2 Main Results

We firstly introduce the following lemmas, which are useful in our main results.

Lemma 2.1. *(The Aritmitic- Geometric Mean Inequality)Let* $X_1, X_2, ..., X_n > 0, \alpha_1,\alpha_2, ...\alpha_n \geq 0$ *and* $\sum_{i=1}^{n} \alpha_i = 1$ *then*

$$\sum_{i=1}^{n} \alpha_i X_i \geq \prod_{i=1}^{n} X_i^{\alpha_i} \tag{1.3}$$

In [3], the following Lemma was proved

Lemma 2.2. *Let* $f(x)$ *be continuous and not identically zero on* $[a, b]$*, differentiable in* (a, b)*, with* $f(a) = 0$*, and let* α, β *be positive real numbers such that* $\alpha > \beta > 1$*. If*

$$\left[f^{\frac{\alpha-\beta}{\beta-1}}(x)\right]' \underset{\leq}{\geq} \frac{(\alpha-\beta)\beta^{\frac{1}{\beta-1}}}{\beta-1} \tag{2.1}$$

for all $x \in (a, b)$*, then*

$$\int_a^b [f(x)]^\alpha \, dx \underset{\leq}{\geq} \left[\int_a^b f(x) \, dx\right]^\beta \tag{2.2}$$

Now we give answer to the posed problem.

Theorem 2.3. *Assume constant* $\gamma > 0$*.Let* $f(x)$ *be continuous and not identically zero on* $[0, 1]$*, differentiable in* $(0, 1)$*, with* $f(0) = 0$*, satisfying*

$$\int_t^1 f^\gamma(x) \, dx \geq \int_t^1 x^\gamma dx, \forall t \in [0, 1], \tag{2.3}$$

$$\left[f^{\frac{\gamma-2(\alpha+\beta)}{\alpha+\beta}}(x)\right]' \leq 2\frac{\gamma-2(\alpha+\beta)}{\alpha+\beta}, \tag{2.4}$$

$$1 \leq \int_0^1 f^{\alpha+\beta}(x) \, dx \leq \frac{\gamma}{\alpha+\beta} - 1. \tag{2.5}$$

Then

$$\int_0^1 f^{\alpha+\beta}(x) \, dx \geq \int_0^1 x^\alpha f^\beta(x) \, dx, \tag{2.6}$$

holds for every real number $\alpha \geq 0$ and $\beta \geq 0$, such that $\frac{\gamma}{\alpha+\beta} > 2$.

Proof. For every $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$, by The Aritmitic- Geometric Mean Inequality, we get

$$\frac{\alpha}{\gamma} x^{\gamma} + \frac{\beta}{\gamma} f^{\gamma}(x) + \frac{\gamma-(\alpha+\beta)}{\gamma} \geq x^{\alpha} f^{\beta}(x) \; . \tag{2.7}$$

Integrating both sides of the inequality (2.7), we further have

$$\frac{\alpha}{\gamma}\int_0^1 x^{\gamma} dx + \frac{\beta}{\gamma}\int_0^1 f^{\gamma}(x)dx + \frac{\gamma-(\alpha+\beta)}{\gamma} \geq \int_0^1 x^{\alpha} f^{\beta}(x)dx \; . \tag{2.8}$$

Using (2.3) , we obtain

$$\frac{\alpha+\beta}{\gamma}\int_0^1 f^{\gamma}(x)dx + \frac{\gamma-(\alpha+\beta)}{\gamma} \geq \int_0^1 x^{\alpha} f^{\beta}(x)dx \; , \tag{2.9}$$

By using Theorem (2.3), taking into account that $\frac{\gamma}{\alpha+\beta} > 2 > 1$ and from (2.4), we have

$$\int_0^1 f^{\gamma}(x)dx = \int_0^1 \left[f^{\alpha+\beta}\right]^{\frac{\gamma}{\alpha+\beta}}(x)dx \leq \left[\int_0^1 f^{\alpha+\beta}(x)dx\right]^2 , \tag{2.10}$$

from (2.9) and (2.10) , we have

$$\frac{\alpha+\beta}{\gamma}\left[\int_0^1 f^{\alpha+\beta}(x)dx\right]^2 + \frac{\gamma-(\alpha+\beta)}{\gamma} \geq \int_0^1 x^{\alpha} f^{\beta}(x)dx \; , \tag{2.11}$$

by (2.5), we deduce that

$$\left[\int_0^1 f^{\alpha+\beta}(x)dx - 1\right]\left[\int_0^1 f^{\alpha+\beta}(x)dx - \frac{\gamma}{\alpha+\beta} + 1\right] \leq 0, \tag{2.12}$$

that is

$$\left[\int_0^1 f^{\alpha+\beta}(x)dx\right]^2 - \frac{\gamma}{\alpha+\beta}\int_0^1 f^{\alpha+\beta}(x)dx + \frac{\gamma}{\alpha+\beta} - 1 \leq 0, \tag{2.13}$$

then

$$\left[\int_0^1 f^{\alpha+\beta}(x)dx\right]^2 + \frac{\gamma}{\alpha+\beta} - 1 \leq \frac{\gamma}{\alpha+\beta}\int_0^1 f^{\alpha+\beta}(x)dx, \tag{2.14}$$

the inequality (2.14) can be written as

$$\frac{\alpha+\beta}{\gamma}\left[\int_0^1 f^{\alpha+\beta}(x)dx\right]^2+\frac{\gamma-(\alpha+\beta)}{\gamma}\leq\int_0^1 f^{\alpha+\beta}(x)dx, \tag{2.15}$$

Summing up the two inequalities (2.11) and (2.15), we get the result. □

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