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Multiple positive solutions of a m-point p-Laplacian boundary value problem involving derivative on time scales[♭]

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Abstract

This paper is concerned with the existence of positive solutions to the p-Laplacian dynamic equation $\left(\phi_p(u^{\triangle\nabla}(t))\right)^{\nabla} + h(t)f(t, u(t), u^{\triangle}(t)) = 0,\ t \in [0,T]_{\mathbb{T}}$, subject to boundary conditions $u(0) - B_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)) = 0,\ u^{\triangle}(T) = 0,\ u^{\triangle\nabla}(0) = 0$, where $\phi_p(u) = |u|^{p-2}u$ with $p > 1$. By using a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson, we prove the m-point boundary value problem has at least triple or arbitrary positive solutions. Our results are new for the special cases of difference equations and differential equations as well as in the general time scale setting. An example illustrates the application of the results obtained.

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1 Introduction

Recently, some authors have obtained many results on the existence of positive solutions to boundary value problems on time scales, for details, see [4, 5, 6, 10, 11, 12, 13, 15] and the references therein. However, there is very little reported work considered the existence of positive solutions to boundary value problems with nonlinear terms involving with the derivative explicitly, see [9, 14].

In [9], Wei Han studied the following m-point p-Laplacian eigenvalue problems

$$\begin{cases} \left(\phi_p\big(u^{\triangle\nabla}(t)\big)\right)^{\nabla} + \lambda f(t, u(t), u^{\triangle}(t)) = 0,\ t \in (0,T)_{\mathbb{T}},\ \lambda > 0 \\ \alpha u(0) - \beta u^{\triangle}(0) = 0,\ u(T) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),\ u^{\triangle\nabla}(0) = 0 \end{cases},$$

the author showed the existence and uniqueness of a nontrivial solution by way of the Leray-Schauder nonlinear alternative.

In [14], You-Hui Su concerned the following p- Laplacian dynamic equation

$$\begin{cases} \left(\phi_p\big(u^{\triangle}(t)\big)\right)^{\nabla} + h(t) f(t, u(t), u^{\triangle}(t)) = 0,\ t \in [0,T]_{\mathbb{T}} \\ u(0) - B_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)) = 0,\ u^{\triangle}(T) = 0 \end{cases}.$$

The author obtained that the boundary value problem has at least triple or arbitrary positive solutions by using a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson.

Motivated by the above mentioned works, in this paper, we consider the boundary value problem

$$\begin{cases} \left(\phi_p\big(u^{\triangle\nabla}(t)\big)\right)^{\nabla} + h(t) f(t, u(t), u^{\triangle}(t)) = 0,\ t \in [0,T]_{\mathbb{T}} \\ u(0) - B_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)) = 0 \\ u^{\triangle}(T) = 0 \\ u^{\triangle\nabla}(0) = 0 \end{cases}, \tag{1.1}$$

where $0, T$ are points in $\mathbb{T}$. By an interval $(0,T)_{\mathbb{T}}$, we always mean $(0,T)\bigcap\mathbb{T}$. Other type of interval are defined similarly. $\xi_i \in [0,T]_{\mathbb{T}}$ such that $0 \le \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(T)$, $\alpha_i \in [0,+\infty)(i = 1,2,\ldots,m-2)$, $\sum_{i=1}^{m-2} \alpha_i \ne 1$, and B_0 satisfies

$$Bx \le B_0(x) \le Ax,\ x \in \mathbb{R}^+, \tag{1.2}$$

where A and B are positive real numbers. We denote the p-Laplacian operator by $\phi_p(u)$, i.e., $\phi_p(u) = |u|^{p-2}u,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1$. By using a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson, we prove that the boundary value problem (1.1) has at least triple or arbitrary positive solutions.

We note that by a solution u of problem (1.1) we mean that $u : \mathbb{T} \to \mathbb{R}$, which is a delta differential, $u^{\triangle}$ and $(\phi_p(u^{\triangle\nabla}))^{\nabla}$ are both continuous on $\mathbb{T}^k \bigcap \mathbb{T}_k$, and u satisfies problems (1.1). The interrelated definitions on time scales can be found in [3].

Throughout this paper, it is assumed that

(H_1) $\eta \in [0, \frac{T}{2}]_{\mathbb{T}}$ and $T \ge 1$;

$(H_2) f : [0,T]_{\mathbb{T}} \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous, and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $\mathbb{R}^+$ denotes the nonnegative real numbers;

$(H_3) h : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous, and does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$ holds.

2 Preliminary Notes

Definition 2.1. Let $\mathbb{E}$ be a real Banach space. A nonempty, closed, convex set $P \subset \mathbb{E}$ is said to be a cone provided that the following conditions are satisfied:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset \mathbb{E}$ induces an ordering in $\mathbb{E}$ given by $x \leq y$ if and only if $y - x \in P$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then, for positive real numbers a, b, c and d, we define the following sets:

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}, P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x), \gamma(x) \leq d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.$$

Now, we give a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson .

Lemma 2.1([2]). Let P be a cone in a real Banach space $\mathbb{E}$ and $\gamma, \theta, \psi, \alpha$ be defined as above. Moreover ψ satisfies $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$ such that, for some positive numbers h and d, $\alpha(x) \leq \psi(x)$ and $\|x\| \leq h\gamma(x)$ for all $x \in \overline{P(\gamma, d)}$. Suppose that $Q : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and that there exist positive real numbers a, b, c with $a < b$ such that:

(i) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Q(x)) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(ii) $\alpha(Qx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Qx) > c$;

(iii) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Qx) < a$ for all $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then Q has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$\gamma(x_i) \leq d$ for $i = 1, 2, 3, b < \alpha(x_1), a < \psi(x_2)$ and $\alpha(x_2) < b$ with $\psi(x_3) < a$.

3 Main Results

In this section, by using a generalization of Leggett-Williams fixed-point theorem due to Avery and Peterson [2], we will discuss the existence of positive

solutions to problem (1.1) under some conditions.
Lemma 3.1. The problem (1.1) have the unique solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)\phi_q\big(I(s)\big)\nabla s + t\int_0^T \phi_q\big(I(s)\big)\nabla s \\ &\quad + B_0\Big(\sum_{i=1}^{m-2}\alpha_i\big(\int_0^T \phi_q\big(I(s)\big)\nabla s - \int_0^{\xi_i}\phi_q\big(I(s)\big)\nabla s\big)\Big), \end{aligned}$$

where

$$I(s) = \int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau.$$

Proof. From (1.1) we know the form of the solution is

$$u(t) = -\int_0^t (t-s)\phi_q\big(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau - E\big)\nabla s + Ft + G. \quad (3.1)$$

Since $u^{\triangle\nabla}(0) = 0$, one can get $E = 0$.

Now, we solve for F, G. By (3.1) we have

$$u^{\triangle}(t) = -\int_0^t \phi_q(I(s))\nabla s + F. \quad (3.2)$$

By $u^{\triangle}(T) = 0$, we can get

$$F = \int_0^T \phi_q(I(s))\nabla s,$$

and $u(0) = G = B_0(\sum_{i=1}^{m-2}\alpha_i u^{\triangle}(\xi_i))$ then

$$G = B_0\Big(\sum_{i=1}^{m-2}\alpha_i\big(\int_0^T \phi_q(I(s))\nabla s - \int_0^{\xi_i}\phi_q(I(s))\nabla s\big)\Big).$$

Substituting F, G in (3.1), one has

$$\begin{aligned} u(t) &= -\int_0^t (t-s)\phi_q\big(I(s)\big)\nabla s + t\int_0^T \phi_q\big(I(s)\big)\nabla s \\ &\quad + B_0\Big(\sum_{i=1}^{m-2}\alpha_i\big(\int_0^T \phi_q\big(I(s)\big)\nabla s - \int_0^{\xi_i}\phi_q\big(I(s)\big)\nabla s\big)\Big). \end{aligned}$$

Now suppose that $u(t)$ is the solution of (1.1). We will show that

$$u(t) \geq 0, \quad u^{\triangle}(t) \geq 0, \quad u^{\triangle\nabla}(t) \leq 0. \quad (3.3)$$

From (3.2), we can get

$$u^{\triangle\nabla}(t) = -\phi_q(I(s)) \le 0,\ t \in [0,T]_{\mathbb{T}}.$$

So we obtain that $u^{\triangle}(t) \ge u^{\triangle}(T) = 0$ for $t \in [0,\mathbb{T}]_{\mathbb{T}}$. Note that

$$u(t) \ge u(0) = B_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)) \ge B(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)) \ge 0,\ t \in [0,\sigma(T)]_{\mathbb{T}}.$$

Then (3.3) holds.

Let the Banach space $\mathbb{E}$ be $C^1_{ld}\big([0,\sigma(T)]_{\mathbb{T}} \to \mathbb{R}\big)$ with the norm

$$\|u\| = max\{sup_{t\in[0,\sigma(T)]_{\mathbb{T}}}|u(t)|, sup_{t\in[0,T]_{\mathbb{T}}}|u^{\triangle}(t)|\}$$

and define the cone $P \subset \mathbb{E}$ by

$P = \{u \in \mathbb{E} : u(t) \ge 0 for\ t \in [0,\sigma(T)]_{\mathbb{T}}\ and\ u^{\triangle}(t) \ge 0, u^{\triangle\nabla}(t) \le 0\ for\ t \in [0,T]_{\mathbb{T}}, u^{\triangle}(T) = 0, u^{\triangle\nabla}(0) = 0\}$.

To obtain our main results, we make use of the following lemmas.

Lemma 3.2 [14]. If $u \in P$, then

(i) $u(t) \ge \frac{t}{\sigma(T)} u(\sigma(T))$ for $t \in [0,\sigma(T)]_{\mathbb{T}}$;

(ii) $su(t) \le tu(s)$ for $s,t \in [0,\sigma(T)]_{\mathbb{T}}$ and $s \le t$.

Lemma 3.3 [14]. For any $u \in P$, there exists a real number $M > 0$ such that $sup_{t\in[0,\sigma(T)]_{\mathbb{T}}}u(t) \le M sup_{t\in[0,T]_{\mathbb{T}}}u^{\triangle}(t)$, where $M = max\{1, \frac{\sigma(T)}{T}(B\sum_{i=1}^{m-2}\alpha_i + T)\}$.

Now, we define the operator $Q : P \to \mathbb{E}$ by

$$\begin{aligned}(Qu)(t) \quad = \quad & -\int_0^t (t-s)\phi_q\big(I(s)\big)\nabla s + t\int_0^T \phi_q\big(I(s)\big)\nabla s \\ & + B_0\Big(\sum_{i=1}^{m-2} \alpha_i\big(\int_0^T \phi_q\big(I(s)\big)\nabla s - \int_0^{\xi_i} \phi_q\big(I(s)\big)\nabla s\big)\Big).\end{aligned}$$

Lemma 3.4. $Q:\ P \to P$ is completely continuous.

Proof. First, it is obvious that $Q:\ P \to P$.

Second, we show that Q maps a bounded set into a bounded set. Assume that $c > 0$ is a constant and $u \in \overline{P}_c = \{u \in P : \|u\| = max\{\sup_{t\in[0,\sigma(t)]_{\mathbb{T}}} |u(t)|, \sup_{t\in[0,T]_{\mathbb{T}}} |u^{\triangle}(t)|\} \le c\}$.

Note that the continuity of f guarantees that there is a constant $D > 0$ such that $f(t,u,u^{\triangle}) \le \phi_p(D)$ for $(t,u,u^{\triangle}) \in [0,T]_{\mathbb{T}} \times [0,c] \times [0,c]$. Hence, for $t \in [0,T]_{\mathbb{T}}$,

$$\left|\int_0^T \phi_q(I(s))\nabla s\right| < +\infty \tag{3.4}$$

and

$$\Big| -\int_0^t (t-s)\phi_q(I(s))\nabla s + t\int_0^T \phi_q(I(s))\nabla s$$

$$+B_0\Big(\sum_{i=1}^{m-2}\alpha_i(\int_0^T\phi_q(I(s))\nabla s-\int_0^{\xi_i}\phi_q(I(s))\nabla s)\Big)\Big|<+\infty. \tag{3.5}$$

Hence, Q maps a bounded set into a bounded set.

Third, for $t_1,t_2\in[0,T]_{\mathbb{T}}$, and we suppose $t_1\le t_2$, we have

$$\begin{aligned}
& \big|(Qu)(t_1)-(Qu)(t_2)\big| \\
= & \ \big|-\int_0^{t_1}(t_1-s)\phi_q(I(s))\nabla s+t_1\int_0^T\phi_q(I(s))\nabla s+\int_0^{t_2}(t_2-s)\phi_q(I(s))\nabla s \\
& -t_2\int_0^T\phi_q(I(s))\nabla s\big| \\
= & \ \big|\int_0^{t_1}(t_2-t_1)\phi_q(I(s))\nabla s+\int_{t_1}^{t_2}(t_2-s)\phi_q(I(s))\nabla s+(t_1-t_2)\int_0^T\phi_q(I(s))\nabla s\big| \\
\le & \ (t_2-t_1)\big|\int_0^T\phi_q(I(s))\nabla s\big|+(t_2-t_1)\big|(t_2-t_1)\phi_q(\int_0^T h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\big| \\
& +(t_2-t_1)\big|\int_0^T\phi_q(I(s))\nabla s\big| \\
= & \ (t_2-t_1)(\big|2\int_0^T\phi_q(I(s))\nabla s\big|+\big|(t_2-t_1)\phi_q(\int_0^T h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau\big|) \\
\to & \ 0\ as\ (t_1\to t_2).
\end{aligned}$$

The Arzela-Ascoli theorem on time scales [7] tells us that $Q\overline{P}_c$ is relatively compact.

We next claim that $Q:\overline{P}_c\to P$ is continuous. Assume that $\{u_n\}_{n=1}^{\infty}\subset\overline{P}_c$ and $\lim_{n\to\infty}\|u_n-u_0\|\to 0$. This means that $\lim_{n\to\infty}|u_n-u_0|\to 0$ and $\lim_{n\to\infty}|u_n^{\triangle}-u_0^{\triangle}|\to 0$. Since $\{(Qu_n)(t)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0,T]_{\mathbb{T}}$, there exists a uniformly convergent subsequence $\{(Qu_n)(t)\}_{n=1}^{\infty}$. Let $\{(Qu_{n(m)})(t)\}_{m=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0,T]_{\mathbb{T}}$. Observe that

$$\begin{aligned}
(Qu_n)(t) &= -\int_0^t(t-s)\phi_q\big(I(s)\big)\nabla s+t\int_0^T\phi_q\big(I(s)\big)\nabla s \\
&\quad +B_0\Big(\sum_{i=1}^{m-2}\alpha_i\big(\int_0^T\phi_q\big(I(s)\big)\nabla s-\int_0^{\xi_i}\phi_q\big(I(s)\big)\nabla s\big)\Big).
\end{aligned}$$

By using (3.4) and (3.5), inserting $u_{n(m)}$ into the above and then letting $m\to\infty$, we obtain

$$\begin{aligned}
v(t) &= -\int_0^t(t-s)\phi_q\big(I(s)\big)\nabla s+t\int_0^T\phi_q\big(I(s)\big)\nabla s \\
&\quad +B_0\Big(\sum_{i=1}^{m-2}\alpha_i\big(\int_0^T\phi_q\big(I(s)\big)\nabla s-\int_0^{\xi_i}\phi_q\big(I(s)\big)\nabla s\big)\Big),
\end{aligned}$$

where we have used Lebesgues dominated convergence theorem on time scales [1]. From the definition of Q, we know that $v(t) = Qu_0(t)$ on $[0,T]_{\mathbb{T}}$. This shows that each subsequence of $\{(Qu_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $\{(Qu_0)(t)\}$, Therefore, the sequence $\{(Qu_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $\{(Qu_0)(t)\}$. This means that Q is continuous at $u_0 \in \overline{P}_c$. So, Q is continuous on $\overline{P}_c$ since u_0 is arbitrary. Thus, Q is completely continuous. The proof is complete.

Now, it is easy to obtain that all the fixed points of the completely continuous operator Q are solutions of the boundary value problem (1.1). Define the nonnegative continuous convex functionals γ, nonnegative continuous concave functional α, θ, and nonnegative continuous functional ψ, respectively, on P by

$$\begin{aligned}
&\gamma(u) = sup_{t\in[0,T]_{\mathbb{T}}} u^{\triangle}(t) = u^{\triangle}(0),\\
&\alpha(u) = inf_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta),\\
&\psi(u) = \theta(u) = sup_{t\in[0,T]_{\mathbb{T}}} u(t) = u(\mathrm{T}).
\end{aligned}$$

By Lemma 3.3, one obtains

$$sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} u(t) \leq M sup_{t\in[0,T]_{\mathbb{T}}} u^{\triangle}(t) = M\gamma(u) \ for\ all\ u \in P.$$

For notational convenience, we denote

$$\begin{aligned}
&L = \int_0^T \phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s, N = \int_0^T s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s,\\
&M^* = \int_0^T s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s + A\sum_{i=1}^{m-2}\alpha_i\Big(\int_0^T \phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s - \int_0^{\xi_i}\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s\Big).
\end{aligned}$$

In the following, we list and prove the results in this subsection.

Theorem 3.5. Suppose that there exist constants a, b, d such that $0 < a < \frac{\eta}{T}b < \frac{\eta}{2T}d$. f satisfies the following conditions:

(i) $f(\tau, u(\tau), u^{\triangle}(\tau)) \leq \phi_p(\frac{d}{L})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [0,T]_{\mathbb{T}} \times [0, Md] \times [0, d]$;

(ii) $f(\tau, u(\tau), u^{\triangle}(\tau)) > \phi_p(\frac{bT}{\eta N})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [\eta, T]_{\mathbb{T}} \times [b, Md] \times [0, d]$;

(iii) $f(\tau, u(\tau), u^{\triangle}(\tau)) < \phi_p(\frac{a}{M^*})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [0,T]_{\mathbb{T}} \times [0, a] \times [0, d]$.

Then problem (1.1) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_i\| \leq d\ for\ i = 1,2,3, b < u_1(\eta), a < u_2(\eta)\ and\ u_2(\eta) < b\ with\ u_3(\eta) < a. \tag{3.6}$$

Proof. By the definition of the completely continuous operator Q and its properties, it suffices to show that all the conditions of Lemma 2.1 hold with respect to Q.

First, we show that $Q : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

For any $u \in \overline{P(\gamma, d)}$, we have $\gamma(u) = sup_{t\in[0,T]_{\mathbb{T}}} u^{\triangle}(t) \leq d$. By Lemma 3.3, one has $sup_{t\in[0,T]_{\mathbb{T}}} u(t) \leq Md$. Assumption (i) implies that $f(\tau, u(\tau), u^{\triangle}(\tau)) \leq \phi_p(\frac{d}{L})$, then

$\gamma(Qu) = sup_{t\in[0,T]_{\mathbb{T}}}(Qu)^{\triangle}(t) = (Qu)^{\triangle}(0) = \int_0^T s\phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s$

$\leq \int_0^T h(\tau)\phi_q(\int_0^s \phi_p(\frac{d}{L})\nabla\tau)\nabla s = \frac{d}{L}\int_0^T \phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s = d.$

Second, we verify that condition (i) of Lemma 2.1 holds. Let $u(t) = -\frac{b}{T^2}(t-T)^2 + \frac{T}{\eta}b$. It's easy to see that

$\alpha(u) = min_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta) = -\frac{b}{T^2}(\eta-T)^2 + \frac{T}{\eta}b = b(-\frac{\eta^2}{T^2} - 1 + \frac{\eta}{T} + \frac{\eta}{T} + \frac{T}{\eta})$

$\geq b(-\frac{\eta^2}{T^2} - 1 + \frac{\eta}{T} + 2) = b[1 + \frac{\eta}{T}(1 - \frac{\eta}{T})] \geq b$, $\theta(u) = \frac{T}{\eta}b$,

and

$\gamma(u) = max_{t\in[\eta,T]_{\mathbb{T}}} u^{\triangle}(t) = u^{\triangle}(0)$, we will prove $\gamma(u) \leq d$ from two cases.

Case (I) Fix $t \in \mathbb{T}^k$. we consider the case where t is right-scattered.

$u^{\triangle}(t) = \frac{u(\sigma(t))-u(t)}{\sigma(t)-t} = \frac{\frac{b}{T^2}(t^2)-\sigma(t)^2-2tT+2\sigma(t)T}{\sigma(t)-t}$, then $u^{\triangle}(0) = \frac{b}{T^2}(2T - \sigma(t)) \leq d$.

Case (II) Let now t be left-dense and right-dense. In this case

$u^{\triangle}(t) = u'(t)) = -\frac{2b}{T^2}(t - T)$, then $u^{\triangle}(0) = u'(0) = \frac{2b}{T} \leq d$.

Thus $\{u \in P(\gamma, \theta, \alpha, b, \frac{T}{\eta}b, d) : \alpha(u) > b\} \neq \varnothing$. For any $\{u \in P(\gamma, \theta, \alpha, b, \frac{T}{\eta}b, d) : \alpha(u) > b\}$, Lemma 3.3 implies that $b \leq u \leq Md$ and $0 \leq u^{\triangle} \leq d$ for all $t \in [\eta, T]_{\mathbb{T}}$.

The properties of u implies $u(t) \geq \frac{t}{T}u(T)$ for $t \in [0, T]_{\mathbb{T}}$ and by assumption (ii) and Lemma 3.2, we have

$$\begin{aligned}
\alpha(Qu) &= inf_{t\in[\eta,T]_{\mathbb{T}}}(Qu)(t) = (Qu)(\eta) \geq \frac{\eta}{T}(Qu)(T) \\
&= \frac{\eta}{T}\Big\{ - \int_0^T (T-s)\phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad +T\int_0^T \phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad +B_0\Big(\sum_{i=1}^{m-2}\alpha_i(\int_0^T \phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad -\int_0^{\xi_i} \phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s)\Big)\Big\} \\
&\geq \frac{\eta}{T}\Big\{\int_0^T s\phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad +B\Big(\sum_{i=1}^{m-2}\alpha_i(\int_0^T \phi_q(\int_0^s h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau)\nabla s
\end{aligned}$$

$$
\begin{aligned}
& -\int_0^{\xi_i}\phi_q(\int_0^s h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\nabla s)\Big)\Big\} \\
\geq\ & \frac{\eta}{T}\Big\{\int_0^T s\phi_q(\int_0^s h(\tau)\phi_p(\frac{bT}{\eta N})\nabla\tau)\nabla s + B\Big(-\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q(\int_0^s h(\tau)\phi_p(\frac{bT}{\eta N})\nabla\tau)\nabla s\Big) \\
& +B\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^T\phi_q(\int_0^s h(\tau)\phi_p(\frac{bT}{\eta N})(\tau))\nabla\tau)\nabla s\Big)\Big\} \\
>\ & \frac{b}{N}\Big\{\int_0^T s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s + B\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^T s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s\Big) \\
& -B\Big(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s)\Big)\Big\} = b.
\end{aligned}
$$

Third, we prove that condition (ii) of Lemma 2.1 holds. For any $u \in P(\gamma,\alpha,b,d)$ with $\theta(Qu) > \frac{T}{\eta}b$, we can get $\alpha(Qu) \geq \frac{\eta}{T}\theta(Qu) > b$.

Finally, we check condition (iii) of Lemma 2.1. Clearly, $\psi(0) = 0 < a$, we have $0 \notin R(\gamma,\psi,a,d)$. If $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = sup_{t\in[0,T]_{\mathbb{T}}}u(t) = u(T) = a$ this yields $0 \leq u \leq a$ for all $t \in [0,T]_{\mathbb{T}}$. In addition, $\gamma(u) = sup_{t\in[0,T]_{\mathbb{T}}}u^{\triangle}(t) \leq d$. Hence, by assumption (iii), we have

$$
\begin{aligned}
\psi(Qu) &= (Qu)(T) = \int_0^T s\phi_q(\int_0^s h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad +B_0\Big(\sum_{i=1}^{m-2}\alpha_i(\int_0^T\phi_q(\int_0^s h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\nabla s \\
&\quad -\int_0^{\xi_i}\phi_q(\int_0^s h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau)\nabla s)\Big) \\
&\leq \int_0^T s\phi_q(\int_0^s h(\tau)\phi_p(\frac{a}{M^*})\nabla\tau)\nabla s - A\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\phi_q(\int_0^s h(\tau)\phi_p(\frac{a}{M^*})\nabla\tau)\nabla s \\
&\quad +A\sum_{i=1}^{m-2}\alpha_i\int_0^T\phi_q(\int_0^s h(\tau)\phi_p(\frac{a}{M^*})\nabla\tau)\nabla s \\
&= \frac{a}{M^*}\{\int_0^T s\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s + A\sum_{i=1}^{m-2}\alpha_i\Big(\int_0^T\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s \\
&\quad -\int_0^{\xi_i}\phi_q(\int_0^s h(\tau)\nabla\tau)\nabla s\Big)\} = a.
\end{aligned}
$$

Consequently, all the conditions of Lemma 2.1 are satisfied. The proof is completed.

Theorem 3.6. Let $i = 1, 2, \cdots, n$ and suppose that there exist constants

a_i, b_i, d_i such that

$$0 < a_1 < \frac{\eta}{T} b_1 < \frac{\eta}{2T} d_1 < a_2 < \frac{\eta}{T} b_2 < \frac{\eta}{2T} d_2 < \cdots < a_n, n \in \mathbb{N}.$$

In addition, f satisfy the following conditions:

(i)$f(\tau, u(\tau), u^{\triangle}(\tau)) \le \phi_p(\frac{d_i}{L})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [0,T]_{\mathbb{T}} \times [0, Md_i] \times [0, d_i]$;

(ii)$f(\tau, u(\tau), u^{\triangle}(\tau)) > \phi_p(\frac{b_i T}{\eta N})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [\eta, T]_{\mathbb{T}} \times [b_i, Md_i] \times [0, d_i]$;

(iii)$f(\tau, u(\tau), u^{\triangle}(\tau)) < \phi_p(\frac{a_i}{M^*})$ for $(\tau, u(\tau), u^{\triangle}(\tau)) \in [0,T]_{\mathbb{T}} \times [0, a_i] \times [0, d_i]$.
Then problem (1.1) has at least $2n+1$ positive solutions.
Proof. When $i = 1$, it is clear that Theorem 3.5 holds. Then we can obtain at least three positive solutions u_1, u_2 and u_3 satisfying (3.6). Hence, we finish the proof by induction.

4 An example

In this section, we present a simple example to explain the main result.
Example 4.1. Let $\mathbb{T} = \{2 - (\frac{1}{3})^{\mathbb{N}_0}\} \bigcup \{0, \frac{1}{4}. \frac{1}{6}, \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\}$.

Consider the following boundary value problem with $p = 6$ and $k \in \mathbb{N}_0$.

$$\begin{cases} \Big(\phi_p(u^{\triangle\nabla}(t))\Big)^{\nabla} + (t+\rho(t)) f(t, u(t), u^{\triangle}(t)) = 0, t \in [0,2]_{\mathbb{T}} \\ u(0) - \frac{1}{1000}(u^{\triangle}(\frac{1}{4}) + u^{\triangle}(\frac{1}{2})) = 0 \\ u^{\triangle}(2) = 0 \\ u^{\triangle\nabla}(0) = 0 \end{cases}, \tag{4.1}$$

where

$$f(t, u(t), u^{\triangle}(t)) = \begin{cases} t + 0.003 + |u^{\triangle}|, \ (t, u, u^{\triangle}) \in [0,2] \times [0,4] \times [0,5]; \\ t + |u^{\triangle}| + p(u), \ (t, u, u^{\triangle}) \in [0,2] \times [4, 4.1] \times [0,5]; \\ t + 2 + |u^{\triangle}|, \ (t, u, u^{\triangle}) \in [0,2] \times [4.1, 20] \times [0,5]. \end{cases}$$

Here $p(u)$ satisfies $p(4) = 0.003$, $p(4.1) = 2$, $p(u) : \mathbb{R} \to \mathbb{R}^+$ is continuous and $p^{\triangle}(u) > 0$.

It is obvious that $A = B = \frac{1}{1000}$ and $a_1 = a_2 = 1$, Choosing $\eta = 1$ and

$r = \frac{3}{2}$, a direct calculation shows that

$$\begin{aligned} L &= \int_0^2 \phi_q(\int_0^s (\tau + \rho(\tau))\nabla\tau)\nabla s \approx 2.039, \\ N &= \int_0^2 s\phi_q(\int_0^s (\tau + \rho(\tau))\nabla\tau)\nabla s \approx 3.772, \\ M^* &= \int_0^2 s\phi_q(\int_0^s (\tau + \rho(\tau))\nabla\tau)\nabla s + A\sum_{i=1}^{m-2} \alpha_i(\int_0^T \phi_q(\int_0^s (\tau + \rho(\tau))\nabla\tau)\nabla s \\ &\quad - \int_0^{\xi_i} \phi_q(\int_0^s (\tau + \rho(\tau))\nabla\tau)\nabla s) \approx 3.776. \end{aligned}$$

If we take $a = 2, b = 4.1, d = 5$, then $0 < a < \frac{\eta}{t}b < \frac{\eta N}{TL}d$. Moreover,

$$f(t, u(t), u^{\triangle}(t)) = t{+}2{+}|u^{\triangle}| > 1.518 \approx \phi_p(\frac{b}{N}),\ (t, u, u^{\triangle}) \in [0, 2]{\times}[4.1, 20]{\times}[0, 5],$$

$$f(t, u(t), u^{\triangle}(t)) = t{+}0.003{+}|u^{\triangle}| < 0.042 \approx \phi_p(\frac{a}{M^*}),\ (t, u, u^{\triangle}) \in [0, 2]{\times}[0, 4]{\times}[0, 5],$$

$$max_{(t,u(t),u^{\triangle}(t))\in[0,2]\times[0,20]\times[0,5]} f(t, u, u^{\triangle}) = t + 2 + |u^{\triangle}| < 88.3 \approx \phi_p(\frac{d}{L}).$$

Therefore, all the conditions of Theorem 3.5 are satisfied. By Theorem 3.5, we see that the boundary value problem (4.1) has at least three positive solutions u_1, u_2, u_3 such that
$\|u_i\| \le 5$ for $i = 1, 2, 3,\ 4.1 < u_1(1), 2 < u_2(1)$ and $u_2(1) < 4.1$ with $u_3(1) < 2$.

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