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Multi-dimensional Fuzzy Euler Approximation

Yangyang Hao
College of Mathematics and Information Science
Hebei University, Baoding 071002, China
hdhyywa@163.com
Cuilian You
College of Mathematics and Information Science
Hebei University, Baoding 071002, China
yycclian@163.com

Abstract

Multi-dimensional Fuzzy differential equations driven by multi-dimensional Liu process, have been intensively applied in many fields. However, we can not obtain the analytic solution of every multi-dimensional fuzzy differential equation. Then, it is necessary for us to discuss the numerical results in most situations. This paper focuses on the numerical method of multi-dimensional fuzzy differential equations. The multi-dimensional fuzzy Taylor expansion is given, based on this expansion, a numerical method which is designed for giving the solution of multi-dimensional fuzzy differential equation via multi-dimensional Euler method will be presented, and its local convergence also will be discussed.

Mathematics Subject Classification: 34A07

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1 Introduction

As a self-dual measure, credibility measure was presented by Liu and Liu [6] in 2002. Later, credibility theory which is an efficient way for us to study the behavior of fuzzy phenomena was founded by Liu [7] in 2004 and was refined by Liu [8] in 2007. Based on credibility theory, some definitions and theorems have been provided, such as fuzzy variable was proposed by Liu [11], it is a function from credibility space to the set of real numbers. The concept of fuzzy process was provided by Liu [11] to describe dynamic fuzzy phenomena. As a special fuzzy process with stationary and independent normal fuzzy process, Liu process was designed by Liu [10]. Just like the assumption that stock price follows geometric Brownian motion, an alternative assumption that stock price follows geometric Liu process was introduced by Liu [10], and the European options pricing formula for Liu's stock model was given by Qin and Li [13]. Gao [4] initiated a new stock model incorporating the mean reversion. In order learn more about Liu process, Dai [2] proved that all sample paths of Liu process are Lipschitz continuous. A reflection principle related to Liu process was given by Dai [3]. Qin and Wen [12] extended some properties of Liu process to the case of complex Liu process. You, Huo and Wang [14] considered multi-dimensional Liu process.

After the concept of Liu process was put forward, many mathematical tools base on Liu process have been developed. As fuzzy counterparts of Ito integral and Ito formula, Liu integral and Liu formula were introduced by Liu [10]. Multi-dimensional Liu integral and multi-dimensional Liu formal which are the tool to deal with the problems with several dynamic factors were studied by You, Wang and Huo [14]. Later, You and Wang [16] considered the properties of Liu integral, and the properties of complex Liu differential was discussed by You and Wang [17]. The definition of generalized Liu integral was presented by You, Ma and Huo [18]. In You, Huo and Wang [14], Liu integral and Liu formula were extended to the case of multi-dimensional Liu integral and multi-dimensional Liu formula.

Fuzzy differential equations with fuzzy parameters or fuzzy initial condition have already got a better development in the past years. For example, the Cauchy problem for fuzzy differential equations was showed by Kaleva [5]. In 2008, Liu [10] introduced the concept of fuzzy differential equation driven by Liu process. Different from fuzzy differential equations with fuzzy parameters or fuzzy initial condition, the fuzziness of fuzzy differential equation driven by Liu process is not only in parameters, in fuzzy initial condition, but also in the driven process, thus fuzzy differential equation driven by Liu process is a general case of fuzzy differential equations. Over the past years, many scholars have conducted a lot of researches on fuzzy differential equation driven by Liu process. A existence and uniqueness theorem for homogeneous fuzzy differential equation was studied in You, Wang and Huo [15]. Chen and Qin [1] considered a new existence and uniqueness theorem that can deal with more general cases of fuzzy differential equation. Zhu [21] discussed the stability for fuzzy differential equations driven by Liu process. Stability in distance for fuzzy differential equations was proposed by You and Hao [19].

You, Wang and Huo [16] figured out the analytic solution of linear fuzzy differential equation. However, we can not obtain the analytic solution of every fuzzy differential equation. Then, it is necessary for us to discuss the numerical results in most situations. A numerical method that was designed for giving the solution of fuzzy differential equation via Euler method was introduced by You and Hao [20].

In this paper, we will extend the numerical method to the case of multi-dimensional fuzzy differential equation. The rest of the paper is organized as follows. In Section 2, some basic knowledge on credibility theory will be introduced. In Section 3, we will give the multi-dimensional fuzzy Taylor expansion, this expansion is the key to the multi-dimensional fuzzy numerical analysis. The multi-dimensional fuzzy Euler method will appear in Section 4, and its local convergence will be also proved in this section.

2 Preliminary Notes

In this section, we will first review some basic knowledge on credibility theory for further understanding.

Let T be an index set and $(\Theta, \mathcal{P}, \mathrm{Cr})$ be a credibility space, where Θ is a nonempty set, $\mathcal{P}$ is the power set of Θ, and Cr is a credibility measure. A fuzzy process $X(t,\theta)$ is a function of two variables from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers. This fuzzy process $X(t^*,\theta)$ is a fuzzy variable for each fixed t^*. For each fixed θ^*, the function $X(t,\theta^*)$ is called a sample path of the fuzzy process. Instead of longer notation $X(t,\theta)$, we will use the symbol X_t in the following sections.

Definition 2.1 *(Liu [9]) Suppose that $\xi, \xi_1, \xi_2, \cdots$ are fuzzy variables defined on credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. The sequence ξ_i is said to be convergent a.s. to ξ if and only if there exists an event A with $\mathrm{Cr}\{A\} = 1$ such that*

$$\lim_{i\to\infty} |\xi_i(\theta) - \xi(\theta)| = 0.$$

Definition 2.2 *(Liu [10]) A fuzzy process C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$.

Note that Liu process is said to be a standard Liu process if $e = 0$ and $\sigma = 1$.

Definition 2.3 *(You, Wang and Huo [14]) If $C_{it}, i = 1, 2, \cdots, m$ are Liu processes, then $\boldsymbol{C_t} = (C_{1t}, C_{2t}, , C_{mt})^T$ is called an m-dimensional Liu process.*

Theorem 2.1 *(Dai [2]) Let C_t be a Liu process. For any given $\theta \in \Theta$ with $\mathrm{Cr}\{\theta\} > 0$, the path $C_t(\theta)$ is Lipschitz continuous, i.e. there exists a Lipschitz constant $K(\theta)$ satisfying*

$$|C_t - C_s| \leq K(\theta)|t - s|.$$

Theorem 2.2 *(Liu Formula, Liu [10]) Let C_t be a standard Liu process, and let $h(t,c)$ be a continuously differentiable function. Define $X_t = h(t, C_t)$. Then we have the following chain rule*

$$\mathrm{d}X_t = \frac{\partial\, h}{\partial\, t}(t, C_t)\mathrm{d}t + \frac{\partial\, h}{\partial\, c}(t, C_t)\mathrm{d}C_t.$$

Theorem 2.3 *(Multi-dimensional Liu Formula, You, Wang and Huo [14]) Let $\boldsymbol{C_t} = (C_{1t}, C_{2t}, , C_{mt})^T$ be an m-dimensional standard Liu process, and let*

$$\boldsymbol{h}(t, x_1, \cdots, x_n) = (h_1(t, x_1, \cdots, x_n), h_2(t, x_1, \cdots, x_n), \cdots h_p(t, x_1, \cdots, x_n))^T,$$

where $h_i(t, x_1, \cdots, x_n), i = 1, 2 \cdots, p$ are multivariate continuously differentiable functions. If fuzzy process $(X_{1t}, X_{2t}, \cdots, X_{nt})^T$ is given by

$$\begin{cases} \mathrm{d}X_{1t} = u_{1t}\mathrm{d}t + v_{11t}\mathrm{d}C_{1t} + \cdots + v_{1mt}\mathrm{d}C_{mt} \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\ \mathrm{d}X_{nt} = u_{nt}\mathrm{d}t + v_{n1t}\mathrm{d}C_{1t} + \cdots + v_{nmt}\mathrm{d}C_{mt} \end{cases},$$

where u_{it}, v_{ijt} are absolutely integrable fuzzy processes and Liu integrable fuzzy process, respectively. Define $\boldsymbol{Y_t} = \boldsymbol{h}(t, X_{1t}, X_{2t}, \cdots, X_{nt})$. Then

$$\mathrm{d}\boldsymbol{Y_t} = \begin{pmatrix} \mathrm{d}Y_{1t} \\ \mathrm{d}Y_{2t} \\ \vdots \\ \mathrm{d}Y_{pt} \end{pmatrix} = \begin{pmatrix} \frac{\partial\, h_1}{\partial\, t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^n \frac{\partial\, h_1}{\partial\, x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \\ \frac{\partial\, h_2}{\partial\, t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^n \frac{\partial\, h_2}{\partial\, x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \\ \vdots \\ \frac{\partial\, h_p}{\partial\, t}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}t + \sum_{i=1}^n \frac{\partial\, h_p}{\partial\, x_i}(t, X_{1t}, X_{2t}, \cdots, X_{nt})\mathrm{d}X_{it} \end{pmatrix}.$$

Definition 2.4 *(Liu Integral, Liu [10]) Let C_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval$[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} \mid t_{i+1} - t_i \mid .$$

Then the fuzzy integral of fuzzy process X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is a fuzzy variable.

Definition 2.5 *(Multi-dimensional Liu Integral, You, Wang and Huo [14]) Let $\boldsymbol{C_t} = (C_{1t}, C_{2t}, \cdots, C_{mt})^T$ be an m-dimensional standard Liu process, and* *let $\nu^{n\times m}$ denote the set of $n \times m$ matrices $v_t = [v_{ijt}]$, where each entry v_{ijt} is a Liu integrable fuzzy process. Suppose $a < b$. If $v_t \in \nu^{n\times m}$, the m-dimensional Liu integral is defined, using matrix notation*

$$\int_a^b v_t d\boldsymbol{C_t} = \int_a^b \begin{pmatrix} v_{11t} & v_{12t} & \cdots & v_{1mt} \\ v_{21t} & v_{22t} & \cdots & v_{2mt} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1t} & v_{n2t} & \cdots & v_{nmt} \end{pmatrix} \begin{pmatrix} \mathrm{d}C_{1t} \\ \mathrm{d}C_{2t} \\ \vdots \\ \mathrm{d}C_{mt} \end{pmatrix}$$

as the $n \times 1$ matrix whose ith component is the following sum of Liu integrals:

$$\sum_{i=1}^{m} \int_a^b v_{ijt} dC_{jt}.$$

Definition 2.6 *(Fuzzy Differential Equation, Liu [10]) Suppose C_t is a standard Liu fuzzy process, and f and g are some given functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called a fuzzy differential equation driven by Liu process. A solution is a fuzzy process X_t that satisfies above equality identically in t.

Definition 2.7 *(Multi-dimensional Fuzzy Differential Equation, Zhu [21]) Let $\boldsymbol{C_t}$ be a standard n-dimensional Liu process. Then*

$$\mathrm{d}\boldsymbol{X_t} = \boldsymbol{f}(t, \boldsymbol{X_t})\mathrm{d}t + \boldsymbol{g}(t, \boldsymbol{X_t})\mathrm{d}\boldsymbol{C_t}, X_{t0} = x_0 \tag{2}$$

is called an n-dimensional fuzzy differential equation driven by a n-dimensional standard Liu process, where $\boldsymbol{X_t}$ is the n-dimensional state vector, x_0 is the crisp n-dimensional initial state vector, $\boldsymbol{f}(t, \boldsymbol{X_t})$ is some given vector function of time t and state X_t, and $\boldsymbol{g}(t, \boldsymbol{X_t})$ is some given matrix-valued function of time t and state X_t in $C^{n\times n}$.

3 Multi-dimensional fuzzy Taylor expansion

Fuzzy Tayor expansion is the necessary tool to construct numerical method of fuzzy differential equations. In order to obtain Euler method of multi-dimensional fuzzy differential equations, we will give the multi-dimensional fuzzy Tayor expansion in this section.

Consider the following multi-dimensional fuzzy differential equation

$$\mathrm{d}\boldsymbol{X_t} = \boldsymbol{f}(\boldsymbol{X_t})\mathrm{d}t + \boldsymbol{g}(\boldsymbol{X_t})\mathrm{d}\boldsymbol{C_t}, \tag{1}$$

where

$$\boldsymbol{f} = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_n(\boldsymbol{x}))^T$$

and

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{pmatrix} g_{11}(\boldsymbol{x}) & g_{12}(\boldsymbol{x}) & \cdots & g_{1n}(\boldsymbol{x}) \\ g_{21}(\boldsymbol{x}) & g_{22}(\boldsymbol{x}) & \cdots & g_{2n}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(\boldsymbol{x}) & g_{n2}(\boldsymbol{x}) & \cdots & g_{nn}(\boldsymbol{x}) \end{pmatrix}$$

are some given continuously differentiable functions, $\boldsymbol{X_t} = (X_{1t}, X_{2t}, \cdots, X_{nt})$ is a multi-dimensional fuzzy process, $\boldsymbol{C_t} = (C_1, C_2, \cdots, C_n)^T$ is a multi-dimensional standard Liu process. Then the solution $\boldsymbol{X_t}$ is to be interpreted as

$$\boldsymbol{X_t} = \begin{pmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{pmatrix} = \begin{pmatrix} X_{1t_0} + \int_{t_0}^t f_1(\boldsymbol{X_s})\mathrm{d}s + \sum_{j=1}^n \int_{t_0}^t g_{1j}(\boldsymbol{X_s})\mathrm{d}C_{js} \\ \vdots \qquad\qquad \vdots \\ X_{nt_0} + \int_{t_0}^t f_n(\boldsymbol{X_s})\mathrm{d}s + \sum_{j=1}^n \int_{t_0}^t g_{nj}(\boldsymbol{X_s})\mathrm{d}C_{js} \end{pmatrix}. \tag{2}$$

According to multi-dimensional Liu formula, the ith component X_{it} of solution $\boldsymbol{X_t}$ of the fuzzy differential equation can be rewritten as the following form

$$a(\boldsymbol{X_t}) = a(\boldsymbol{X_{t_0}}) + \int_{t_0}^t L^0 a(\boldsymbol{X_s})\mathrm{d}s + \sum_{j=1}^n \int_{t_0}^t L^j a(\boldsymbol{X_s})\mathrm{d}C_{js}, \tag{3}$$

where $L^0 a(\boldsymbol{x}) = \frac{\partial\ a}{\partial\ x_i} f_i$, $L^j a(\boldsymbol{x}) = \frac{\partial\ a}{\partial\ x_j} g_{ij}$, $a(\boldsymbol{x})$ is a function with n variables.

(i) Take $a = f_i$, $a = g_{ij}, j = 1, 2, \cdots, n$, respectively, we have

$$f_i(\boldsymbol{X_t}) = f_i(\boldsymbol{X_{t_0}}) + \int_{t_0}^t L^0 f_i(\boldsymbol{X_s})\mathrm{d}s + \sum_{j=1}^n \int_{t_0}^t L^j f_i(\boldsymbol{X_s})\mathrm{d}C_{js}, \tag{4}$$

$$g_{ij}(\boldsymbol{X_t}) = g_{ij}(\boldsymbol{X_{t_0}}) + \int_{t_0}^t L^0 g_{ij}(\boldsymbol{X_s})\mathrm{d}s + \sum_{l=1}^n \int_{t_0}^t L^l g_{ij}(\boldsymbol{X_s})\mathrm{d}C_{ls}. \tag{5}$$

Substituting (4) and (5) into the ith component of (2), the component of n-dimensional fuzzy Taylor expansion as follows

$$\begin{aligned} X_{it} = X_{it_0} &+ \int_{t_0}^t \{f_i(\boldsymbol{X_{t_0}}) + \int_{t_0}^s L^0 f_i(\boldsymbol{X_{s_1}})\mathrm{d}s_1 + \sum_{j=1}^n \int_{t_0}^s L^j f_i(\boldsymbol{X_{s_1}})\mathrm{d}C_{js_1}\}\mathrm{d}s \\ &+ \sum_{j=1}^n \int_{t_0}^t \{g_{ij}(\boldsymbol{X_{t_0}}) + \int_{t_0}^s L^0 g_{ij}(\boldsymbol{X_{s_1}})\mathrm{d}s_1 + \sum_{l=1}^n \int_{t_0}^s L^l g_{ij}(\boldsymbol{X_{s_1}})\mathrm{d}C_{ls_1}\}\mathrm{d}C_{js} \\ &= X_{it_0} + f_i(\boldsymbol{X_{t_0}}) \int_{t_0}^t \mathrm{d}s + \sum_{j=1}^n g_{ij}(\boldsymbol{X_{t_0}}) \int_{t_0}^t \mathrm{d}C_{js} + \tilde{R}_1, \end{aligned} \tag{6}$$

where

$$\tilde{R}_1 = \int_{t_0}^{t}\int_{t_0}^{s} L^0 f_i(\boldsymbol{X_{s_1}})\mathrm{d}s_1\mathrm{d}s + \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s} L^j f_i(\boldsymbol{X_{s_1}})\mathrm{d}C_{js_1}\mathrm{d}s$$
$$+ \sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s} L^0 g_{ij}(\boldsymbol{X_{s_1}})\mathrm{d}s_1 dC_{js} + \sum_{j=1}^{n}\sum_{l=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s} L^l g_{ij}(\boldsymbol{X_{s_1}})\mathrm{d}C_{ls_1}\mathrm{d}C_{js}.$$

This is the simplest multi-dimensional fuzzy Taylor expansion.

(ii) Let $a = f'_{i x_l} f_l$, $a = g'_{ij x_l} f_l$, $a = f'_{i x_l} g_{il}$, $a = g'_{ij x_p} g_{ip}$, respectively, then we have

$$\begin{aligned} f'_{i x_l}(\boldsymbol{X_t})f_l(\boldsymbol{X_t}) =& f'_{i x_l}(\boldsymbol{X_{t_0}})f_l(\boldsymbol{X_{t_0}}) + \int_{t_0}^{t} L^0 f'_{i x_l}(\boldsymbol{X_s})f_l(\boldsymbol{X_s})\mathrm{d}s \\ &+ \sum_{k=1}^{n}\int_{t_0}^{t} L^k f'_{i x_l}(\boldsymbol{X_s})f_l(\boldsymbol{X_s})\mathrm{d}C_{ks}, \end{aligned} \tag{7}$$

$$\begin{aligned} g'_{ij x_l}(\boldsymbol{X_t})f_l(\boldsymbol{X_t}) =& g'_{ij x_l}(\boldsymbol{X_{t_0}})f_l(\boldsymbol{X_{t_0}}) + \int_{t_0}^{t} L^0 g'_{ij x_l}(\boldsymbol{X_s})f_l(\boldsymbol{X_s})\mathrm{d}s \\ &+ \sum_{k=1}^{n}\int_{t_0}^{t} L^k g'_{ij x_l}(\boldsymbol{X_s})f_l(\boldsymbol{X_s})\mathrm{d}C_{ks}, \end{aligned} \tag{8}$$

$$\begin{aligned} f'_{i x_l}(\boldsymbol{X_t})g_{il}(\boldsymbol{X_t}) =& f'_{i x_l}(\boldsymbol{X_{t_0}})g_{il}(\boldsymbol{X_{t_0}}) + \int_{t_0}^{t} L^0 f'_{i x_l}(\boldsymbol{X_s})g_{il}(\boldsymbol{X_s})\mathrm{d}s \\ &+ \sum_{k=1}^{n}\int_{t_0}^{t} L^k f'_{i x_l}(\boldsymbol{X_s})g_{il}(\boldsymbol{X_s})\mathrm{d}C_{ks}, \end{aligned} \tag{9}$$

$$\begin{aligned} g'_{ij x_p}(\boldsymbol{X_t})g_{ip}(\boldsymbol{X_t}) =& g'_{ij x_p}(\boldsymbol{X_{t_0}})g_{ip}(\boldsymbol{X_{t_0}}) + \int_{t_0}^{t} L^0 g'_{ij x_p}(\boldsymbol{X_s})g_{ip}(\boldsymbol{X_s})\mathrm{d}s \\ &+ \sum_{m=1}^{n}\int_{t_0}^{t} L^m g'_{ij x_p}(\boldsymbol{X_s})g_{ip}(\boldsymbol{X_s})\mathrm{d}C_{ms}. \end{aligned} \tag{10}$$

Substituting (7), (8), (9) and (10) into (6), we have

$$\begin{aligned} X_{it} =& X_{it_0} + f_i(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\mathrm{d}s + \sum_{j=1}^{n} g_{ij}(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\mathrm{d}C_{js} + f'_{i x_l}(\boldsymbol{X_{t_0}})f_l(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\int_{t_0}^{s}\mathrm{d}s_1\mathrm{d}s \\ &+ \sum_{j=1}^{n} g'_{ij x_l}(\boldsymbol{X_{t_0}})f_l(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\int_{t_0}^{s}\mathrm{d}C_{js_1}\mathrm{d}s + \sum_{j=1}^{n} f'_{i x_l}(\boldsymbol{X_{t_0}})g_{il}(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\int_{t_0}^{s}\mathrm{d}s_1\mathrm{d}C_{js} \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} g'_{ij x_p}(\boldsymbol{X_{t_0}})g_{ip}(\boldsymbol{X_{t_0}})\int_{t_0}^{t}\int_{t_0}^{s}\mathrm{d}C_{ls_1}\mathrm{d}C_{js} + \tilde{R}_2, \end{aligned}$$

where

$$\begin{aligned}\tilde{R}_2 = &\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^0 f'_{ix_l}(\boldsymbol{X_{s_2}})f_l(\boldsymbol{X_{s_2}})\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s + \sum_{k=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^k f'_{ix_l}(\boldsymbol{X_{s_2}})f_l(\boldsymbol{X_{s_2}})\mathrm{d}C_{ks_2}\mathrm{d}s_1\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^0 g'_{ijx_l}(\boldsymbol{X_{s_2}})f_l(\boldsymbol{X_{s_2}})\mathrm{d}s_2\mathrm{d}C_{js_1}\mathrm{d}s\\
&+\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^k g'_{ijx_l}(\boldsymbol{X_{s_2}})f_l(\boldsymbol{X_{s_2}})\mathrm{d}C_{ks_2}\mathrm{d}C_{js_1}\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^0 f'_{ix_l}(\boldsymbol{X_{s_2}})g_{il}(\boldsymbol{X_{s_2}})\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}C_{js}\\
&+\sum_{j=1}^{n}\sum_{k=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^k f'_{ix_l}(\boldsymbol{X_{s_2}})g_{il}(\boldsymbol{X_{s_2}})\mathrm{d}C_{ks_2}\mathrm{d}s_1\mathrm{d}C_{js}\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^0 g'_{ijx_p}(\boldsymbol{X_{s_2}})g_{ip}(\boldsymbol{X_{s_2}})\mathrm{d}s_2\mathrm{d}C_{ls_1}\mathrm{d}C_{js}\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{s_1} L^m g'_{ijx_p}(\boldsymbol{X_{s_2}})g_{ip}(\boldsymbol{X_{s_2}})\mathrm{d}C_{ms_2}\mathrm{d}C_{ls_1}\mathrm{d}C_{js}.\end{aligned}$$

This is the second-order multi-dimensional fuzzy Taylor expansion.

Continuing this process, thus the high order multi-dimensional fuzzy Taylor expansion can be obtained.

4 Multi-dimensional fuzzy Euler method

As a numerical method for estimating the solution of differential function, Euler method plays an important role in differential function. When the fuzzy Taylor expansion of multi-dimensional fuzzy differential function is given, we can obtain the Euler method by truncating multiple integral as the following form

$$\boldsymbol{X_{t_{m+1}}} = \begin{pmatrix} X_{1t_{m+1}} \\ \vdots \\ X_{nt_{m+1}} \end{pmatrix} = \begin{pmatrix} X_{1t_m} + f_1(\boldsymbol{X_{t_m}})h + \sum_{j=1}^{n} g_{1j}(\boldsymbol{X_{t_m}})\Delta C_{jt_m} \\ \vdots \qquad\qquad\qquad \vdots \\ X_{nt_m} + f_n(\boldsymbol{X_{t_m}})h + \sum_{j=1}^{n} g_{nj}(\boldsymbol{X_{t_m}})\Delta C_{jt_m} \end{pmatrix},$$

where $h = t_{m+1} - t_m,\ \triangle C_{jt_m} = C_{jt_{m+1}} - C_{jt_m}$.

Example 4.1 In order to illustrate multi-dimensional fuzzy differential equation, let us consider the 2-dimensional fuzzy differential equation

$$\mathrm{d}\boldsymbol{X_t} = \begin{pmatrix} a_1\boldsymbol{X_t} \\ a_2\boldsymbol{X_t} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} b_1\boldsymbol{X_t} & 0 \\ 0 & b_2\boldsymbol{X_t} \end{pmatrix}\mathrm{d}\boldsymbol{C_t}.$$

where $\boldsymbol{X_t} = (X_{1t}, X_{2t})^T$, $\boldsymbol{C_t} = (C_{1t}, C_{2t})^T$.

When the interval $[t_0, t]$ was divided into n parts, we can obtain the numerical solution of the 2-dimensional fuzzy equation easily via multi-dimensional fuzzy Euler method

$$\boldsymbol{X_t} = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} X_{1t_{n-1}} + a_1 \boldsymbol{X_{1t_{n-1}}} h + b_1 \boldsymbol{X_{t_{n-1}}} \Delta C_{1t} \\ X_{2t_{n-1}} + a_2 \boldsymbol{X_{1t_{n-1}}} h + b_2 \boldsymbol{X_{t_{n-1}}} \Delta C_{2t} \end{pmatrix},$$

where $h = t_n - t_{n-1}, \Delta C_{1t} = C_{1t_n} - C_{1t_{n-1}}, \Delta C_{2t} = C_{2t_n} - C_{2t_{n-1}}$, $X_{it_{n-1}}$ is ith component of analytic solution of the 2-dimensional fuzzy differential equation when $t = t_{j-1}$, $\boldsymbol{X_{t_{n-1}}}$ is the analytic solution of the 2-dimensional fuzzy differential equation when $t = t_{j-1}$.

Theorem 4.1 *Considering mutil-dimensional fuzzy differential equation (2). Let f_i and g_{ij} be given functions that satisfy the Linear growth condition*

$$|f_i(\boldsymbol{x})|^2 \vee |g_{ij}(\boldsymbol{x})|^2 \leq L_1(1 + \|\boldsymbol{x}\|^2),$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$. Then there exists constant $N_1(\theta)$ such that $X_{it}^2 \leq 3(n+2)N_1(\theta)$ for the ith component X_{it} of solution of (2), where $N_1(\theta)$ just relies on integrating range and initial value only.

Proof: We shall rewrite the component X_{it} of solution of (2) as the integral form

$$X_{it} = X_{it_0} + \int_{t_0}^{t} f_i(\boldsymbol{X_s})\mathrm{d}s + \int_{t_0}^{t} g_{i1}(\boldsymbol{X_s})\mathrm{d}C_{1s} + \cdots + \int_{t_0}^{t} g_{in}(\boldsymbol{X_s})\mathrm{d}C_{ns}. \quad (11)$$

Taking square of both sides and applying the inequality $(a_1 + a_2 + \cdots + a_n)^2 \leq na_1^2 + na_2^2 + \cdots + na_n^2$, we have

$$X_{it}^2 \leq (n+2)[X_{it_0}^2 + (\int_{t_0}^{t} f_i(\boldsymbol{X_s})\mathrm{d}s)^2 + \sum_{j=1}^{n}(\int_{t_0}^{t} g_{ij}(\boldsymbol{X_s})\mathrm{d}C_{js})^2]. \quad (12)$$

.

Then it follows from Lipschitz continuity of Liu process that

$$\begin{aligned} |\int_{t_0}^{t} g_{ij}(\boldsymbol{X_s})\mathrm{d}C_{js}| &= |\lim_{\Delta \to 0} \sum_{i=0}^{n-1} g_{ij}(\boldsymbol{X_{t_i}})(C_{jt_{i+1}} - C_{jt_i})| \\ &\leq |K_j(\theta)|| \lim_{\Delta \to 0} \sum_{k=0}^{n-1} g_{ij}(\boldsymbol{X_{t_k}})(t_{k+1} - t_k)| \\ &\leq |K_j(\theta)|| \int_{t_0}^{t} g_{ij}(\boldsymbol{X_s})\mathrm{d}s|. \end{aligned}$$

It is easy to rewrite (12) as the following inequality

$$X_{it}^2 \leq (n+2)[X_{it_0}^2 + (\int_{t_0}^t f_i(\boldsymbol{X_s})\mathrm{d}s)^2 + \sum_{j=1}^n (K_j(\theta))^2 (\int_{t_0}^t g_{ij}(\boldsymbol{X_s})\mathrm{d}s)^2].$$

By Hölder inequality and Linear growth condition, we have

$$\begin{aligned}(\int_{t_0}^t f_i(\boldsymbol{X_s})\mathrm{d}s)^2 &\leq (t-t_0)\int_{t_0}^t f_i^2(\boldsymbol{X_s})\mathrm{d}s \\ &\leq (t-t_0)\int_{t_0}^t L_1(1+\|\boldsymbol{X_s}\|^2)\mathrm{d}s \\ &= (t-t_0)^2 L_1 + (t-t_0)L_1 \int_{t_0}^t \|\boldsymbol{X_s}\|^2 \mathrm{d}s.\end{aligned}$$

In a similar proof, we have

$$K_j^2(\theta)(\int_{t_0}^t g_{ij}(\boldsymbol{X_s})\mathrm{d}s)^2 \leq K_j^2(\theta)(t-t_0)^2 L_1 + K_j^2(\theta)(t-t_0)L_1 \int_{t_0}^t \|\boldsymbol{X_s}\|^2 \mathrm{d}s.$$

Thus

$$\begin{aligned}X_{it}^2 \leq &(n+2)[X_{it_0}^2 + (t-t_0)^2 L_1 + L_1(t-t_0)\int_{t_0}^t \|\boldsymbol{X_s}\|^2 \mathrm{d}s \\ &+ \sum_{j=1}^n [K_j^2(\theta)(t-t_0)^2 L_1 + (t-t_0)L_1 K_j^2(\theta)\int_{t_0}^t \|\boldsymbol{X_s}\|^2 \mathrm{d}s].\end{aligned}$$

According to the Grownwall inequality, we obtain

$$X_{it}^2 \leq (n+2)[X_{it_0}^2 + M_0 \exp[M_0(t-t_0)] + \sum_{j=1}^n M_j(\theta)\exp[M_j(\theta)(t-t_0)].$$

Let $N_1(\theta) = max\{X_{it_0}^2, M_0\exp[M_0(t-t_0)], \sum_{j=1}^n M_j(\theta)[M_j(\theta)(t-t_0)]\}$. The inequality is obtained.

Theorem 4.2 *Considering mutil-dimensional fuzzy differential equation (2). Let f_i and g_{ij} be given functions that satisfy the Linear growth condition*

$$|f_i(\boldsymbol{x})|^2 \vee |g_{ij}(\boldsymbol{x})|^2 \leq L_1(1+\|\boldsymbol{x}\|^2),$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$. Then for $\forall\ 0 \leq s \leq t \leq T$, there exists a constant $N_2(\theta)$ such that $(X_{it} - X_{is})^2 \leq N_2(\theta)(t-s)^2$, where X_{it}, X_{is} are the ith component of solution of (2) and $N_2(\theta)$ just relies on s, t and initial value only.

Proof: For any $0 \le s \le t \le T$,

$$X_{it} - X_{is} = \int_s^t f_i(\boldsymbol{X_\tau})\mathrm{d}\tau + \sum_{j=1}^n \int_s^t g_{ij}(\boldsymbol{X_\tau})\mathrm{d}C_{j\tau}.$$

Taking square of both sides and applying the inequality $(a_1 + a_2 + \cdots + a_n)^2 \le na_1^2 + na_2^2 + \cdots + na_n^2$, we have

$$(X_{it} - X_{is})^2 \le (n+1)[\int_s^t f_i(\boldsymbol{X_\tau})\mathrm{d}\tau)^2 + \sum_{j=1}^n (\int_s^t g_{ij}(\boldsymbol{X_\tau})\mathrm{d}C_{j\tau})^2].$$

Then it follows from the Hölder inequality and Linear growth condition that

$$\begin{aligned}
(\int_s^t f_i(\boldsymbol{X_\tau})\mathrm{d}\tau)^2 &\le (t-s)\int_s^t f_i^2(\boldsymbol{X_\tau})\mathrm{d}\tau \\
&\le (t-s)\int_s^t L_1(1 + \|\boldsymbol{X_\tau}\|^2)\mathrm{d}\tau \\
&= L_1(t-s)^2 + L_1(t-s)\int_s^t \|\boldsymbol{X_\tau}\|^2\mathrm{d}\tau.
\end{aligned}$$

In a similar proof, we have

$$K_j^2(\theta)(\int_s^t g_{ij}(\boldsymbol{X_\tau})\mathrm{d}C_{j\tau})^2 \le L_1(t-s)^2 K_j^2(\theta) + L_1 K_j^2(\theta)(t-s)\int_s^t \|\boldsymbol{X_\tau}\|^2\mathrm{d}\tau.$$

According to Theorem 4.1, we have

$$(X_{it} - X_{js})^2 \le L_1(n+1)(t-s)^2\{1 + 3(n+2)N_1(\theta) + \sum_{j=1}^n [K_j^2(\theta) + 3K_j^2(\theta)(n+2)N_1(\theta)]\}.$$

Let

$$N_2(\theta) = L_1(n+1)\{1 + 3(n+2)N_1(\theta) + \sum_{j=1}^n [K_j^2(\theta) + 3K_j^2(\theta)(n+2)N_1(\theta)]\}.$$

Then we have $(X_{it} - X_{is})^2 \le N_2(\theta)(t-s)^2$.
The theorem is proved.

Theorem 4.3 *Considering mutil-dimensional fuzzy differential equation (2). Let f_i and g_{ij} be given functions and satisfy the Linear growth condition*

$$|f_i(\boldsymbol{x})|^2 \vee |g_{ij}(\boldsymbol{x})|^2 \le L_1(1 + \|\boldsymbol{x}\|^2),$$

and Global Lipschitz condition

$$|f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| \vee |g_{ij}(\boldsymbol{x}) - g_{ij}(\boldsymbol{y})| \le L_2\|\boldsymbol{x} - \boldsymbol{y}\|,$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$. Then the mutil-dimensional Euler approximation converges almost surely locally.

Proof: According to the Euler approximation, the local error of the solution X_t can be written as

$$\begin{aligned}\|\boldsymbol{\delta_{n+1}}\| &= \|\boldsymbol{X_{t_{n+1}}} - \tilde{\boldsymbol{X}}_{\boldsymbol{t_{n+1}}}\| \\ &= \sqrt{\sum_{i=1}^{n}(X_{it_{n+1}} - \widetilde{X}_{it_{n+1}})^2}\end{aligned}$$

where $X_{it_{n+1}}$ is analytic solution of (11) when $t = t_{n+1}$, $\tilde{X}_{it_{n+1}}$ is numerical solution that just used Euler approximation once, i.e.

$$\tilde{X}_{it_{n+1}} = X_{it_n} + \int_{t_n}^{t_{n+1}} f_i(\boldsymbol{X_{s_n}})\mathrm{d}s + \sum_{j=1}^{n}\int_{t_n}^{t_{n+1}} g_{ij}(\boldsymbol{X_{s_n}})\mathrm{d}C_{js_n},$$

where X_{it_n} is the analytic solution of (11) when $t = t_n$.
Applying Global Lipschitz condition, Theorem 4.1 and Theorem 4.2, we have

$$\begin{aligned}\lim_{h\to 0}\|\boldsymbol{\delta_{n+1}}\| &= \lim_{h\to 0}\sqrt{\sum_{i=1}^{n}(X_{it_{n+1}} - \widetilde{X}_{it_{n+1}})^2} \\ &= \lim_{h\to 0}\sqrt{\sum_{i=1}^{n}\{\int_{t_n}^{t_{n+1}}[f_i(\boldsymbol{X_s}) - f_i(\boldsymbol{X_{t_n}})]\mathrm{d}s + \sum_{j=1}^{n}\int_{t_n}^{t_{n+1}}[g_{ij}(\boldsymbol{X_s}) - g_{ij}(\boldsymbol{X_{t_n}})]\mathrm{d}C_{js}\}^2} \\ &\le \lim_{h\to 0}\sqrt{\sum_{i=1}^{n}[L_2\sqrt{nN_2(\theta)}h^2 + L_2\sum_{j=1}^{n}K_j(\theta)\sqrt{nN_2(\theta)}h^2]^2} \\ &\le \lim_{h\to 0}h^2\sqrt{\sum_{i=1}^{n}[L_2\sqrt{nN_2(\theta)} + L_2\sum_{j=1}^{n}K_j(\theta)\sqrt{nN_2(\theta)}]^2} \\ &= 0.\end{aligned}$$

Then the theorem is proved.

The effectiveness of Theorem 4.3 can be illustrated by the following example.

Example 4.2 Consider Example 4.1, since $f_i(\boldsymbol{x}) = a_i\boldsymbol{x}$ and $g_i(\boldsymbol{x}) = b_i(\boldsymbol{x})$ satisfy the Linear growth condition

$$|f_i(\boldsymbol{x})|^2 \vee |g_i(\boldsymbol{x})|^2 \le max\{a_i^2, b_i^2\}(1 + \|\boldsymbol{x}\|^2),$$

and Global Lipschitz condition

$$|f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})| \vee |g_i(\boldsymbol{x}) - g_i(\boldsymbol{y})| \le max\{a_i, b_i\}\|\boldsymbol{x} - \boldsymbol{y}\|,$$

for $i = 1, 2$. Then the numerical solution converges almost surely locally according to Theorem 4.3.

5 Conclusions

Multi-dimensional fuzzy differential equation driven by multi-dimensional Liu process, is an important tool to deal with multi-dimensional dynamic system in fuzzy environment. In many cases, unfortunately, it is difficult to find analytic solution of multi-dimensional fuzzy differential equation. This paper gave a multi-dimensional Euler method which was obtained by truncating multiple integral of multi-dimensional Taylor expansion for solving multi-dimensional fuzzy differential equation. Furthermore, the local convergence of multi-dimensional Euler method was proved.

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