Mathematica Aeterna, Vol. 3, 2013, no. 7, 597 - 600

More on Reverses of Minkowski's Integral Inequality

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Abstract

In 2012, Sulaiman proved an integral inequality concerning some reverse of Minkowski's inequality. In this paper, we present a generalization of the inequality.

Mathematics Subject Classification: 26D15

Keywords: Minkowski's inequality, integral inequality

1 Introduction

The Minkowski's inequality states that, for $p \geq 1$, if

$$0 < \int_a^b f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_a^b g^p(x)dx < \infty$$

then

$$\left(\int_a^b (f(x)+g(x))^p \, dx\right)^{1/p} \leq \left(\int_a^b f^p(x)dx\right)^{1/p} + \left(\int_a^b g^p(x)dx\right)^{1/p}.$$

In 2006, Bougoffa [1] presented an integral inequality concerning some reverse of Minkowski's inequality as follows.

For any $f, g > 0$, if $p \geq 1$ and

$$0 < m \leq \frac{f(x)}{g(x)} \leq M$$

for all $x \in [a, b]$, then

$$\left(\int_a^b f^p(x)dx\right)^{1/p} + \left(\int_a^b g^p(x)dx\right)^{1/p} \leq \frac{M(m+1)+M+1}{(m+1)(M+1)} \left(\int_a^b (f(x)+g(x))^p \, dx\right)^{1/p}.$$

In 2012, Sulaiman [2] presented an integral inequality similar to above inequality as follows.

For any $f, g > 0$, if $p \geq 1$ and

$$1 < m \leq \frac{f(x)}{g(x)} \leq M$$

for all $x \in [a, b]$, then

$$\frac{M+1}{M-1}\left(\int_a^b (f(x)-g(x))^p \, dx\right)^{1/p} \leq \left(\int_a^b f^p(x)dx\right)^{1/p} + \left(\int_a^b g^p(x)dx\right)^{1/p} \leq \frac{m+1}{m-1}\left(\int_a^b (f(x)-g(x))^p \, dx\right)^{1/p}. \quad (1)$$

In this paper, we present a generalization of Sulaiman's inequality .

2 Main Results

Proposition 2.1. *Assume that* $0 < c < m \leq M$. *Then*

$$\frac{M+1}{M-c} \leq \frac{m+1}{m-c}.$$

Proof. By the assumption, we have

$$(c+1)m \leq (c+1)M.$$

Then

$$m - cM \leq M - cm.$$

Then

$$(M+1)(m-c) \leq (m+1)(M-c).$$

Thus,

$$\frac{M+1}{M-c} \leq \frac{m+1}{m-c}.$$

This proof is completed. □

Theorem 2.2. *Assume that* $f, g > 0$, $p \geq 1$, *and*

$$0 < c < m \leq \frac{f(x)}{g(x)} \leq M$$

for all $x \in [a, b]$. *Then*

$$\begin{aligned}\frac{M+1}{M-c}\left(\int_a^b (f(x)-cg(x))^p\,dx\right)^{1/p} &\leq \left(\int_a^b f^p(x)dx\right)^{1/p} + \left(\int_a^b g^p(x)dx\right)^{1/p} \\ &\leq \frac{m+1}{m-c}\left(\int_a^b (f(x)-cg(x))^p\,dx\right)^{1/p}.\end{aligned} \tag{2}$$

Proof. By the assumption, we have

$$m - c \leq \frac{f(x)}{g(x)} - c \leq M - c,$$

and then

$$m - c \leq \frac{f(x) - cg(x)}{g(x)} \leq M - c.$$

This implies that

$$\frac{f(x) - cg(x)}{M - c} \leq g(x) \leq \frac{f(x) - cg(x)}{m - c}.$$

Hence,

$$\begin{aligned}&\frac{1}{M-c}\left(\int_a^b (f(x)-cg(x))^p\,dx\right)^{1/p} \\ &\quad \leq \left(\int_a^b g^p(x)dx\right)^{1/p} \leq \frac{1}{m-c}\left(\int_a^b (f(x)-cg(x))^p\,dx\right)^{1/p}.\end{aligned} \tag{3}$$

By the assumption, we have

$$-\frac{1}{m} \leq -\frac{g(x)}{f(x)} \leq -\frac{1}{M}.$$

Then

$$\frac{1}{c} - \frac{1}{m} \leq \frac{1}{c} - \frac{g(x)}{f(x)} \leq \frac{1}{c} - \frac{1}{M}.$$

Thus,

$$\frac{m-c}{cm} \le \frac{f(x)-cg(x)}{cf(x)} \le \frac{M-c}{cM}.$$

This implies that

$$\frac{M}{M-c}\left(f(x)-cg(x)\right) \le f(x) \le \frac{m}{m-c}\left(f(x)-cg(x)\right).$$

Hence,

$$\begin{aligned}\frac{M}{M-c}&\left(\int_a^b \left(f(x)-cg(x)\right)^p dx\right)^{1/p} \\ &\le \left(\int_a^b f^p(x)dx\right)^{1/p} \le \frac{m}{m-c}\left(\int_a^b \left(f(x)-cg(x)\right)^p dx\right)^{1/p}. \qquad (4)\end{aligned}$$

By the inequalities (3) and (4), we obtain the inequality (2). □

By Proposition 2.1, we can guarantee that the inequality (2) is possible. Finally, we will obtain the inequality (1) if we replace c by 1 in the inequality (2).

References

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Received: September, 2013