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Minimal Quasi-absorbent in Groupoid-lattice II

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Abstract

Otto. Steinfeld introduced the concept of quasi-ideal in his paper ″On ideal-quotients and prime ideals(1953)″. Much of Steinfeld's contributions to quasi-ideals is contained in his monograph ″Quasi-ideals in rings and semigroups(1978)″. In the paper (with Rédei) ″Einiges über gruppoid-Verbände mit Anwendungen auf Gruppen, Ringe, Halbgruppen (1974)″, the authors generalized concepts from groups, ring, and semigroups to groupoid lattices. In our paper[5], we have introduced the notion of semiprime absorbent in groupoid lattices. Here in this paper we will discuss some properties of minimal quasi-absorbent using semiprime absorbent in groupoid lattices.

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1 Introduction

Definition 1.1 *A partially ordered groupoid is a non-empty set C satisfying the following properties:*
(i) *C is a groupoid w.r.t multiplication* $''\cdot''$,
(ii) *C is a partially ordered set w.r.t a partial ordering* $''\leq''$,
(iii) *If $a \leq b$, then $ca \leq cb$ and $ac \leq bc$ $\forall a, b, c \in C$.*

We know that a lattice "V" is called a complete if every subset of $''V''$ has a least upper bound (join $''\vee''$) and greatest lower bound (meet $''\wedge''$) in $''V''$. The greatest element is denoted by $''e''$ and the least element is denoted by $''0''$.

Definition 1.2 *A groupoid lattice is a partially ordered groupoid $\langle V, \cdot, \leq \rangle$ such that V is a complete lattice with respect to the partial ordering $\leq$ and it has the following properties:*
(iv) $a^2 \leq a \quad \forall a \in V$,
(v) $0 \cdot e = e \cdot 0 = 0$
for the greatest element e and least element 0 of V, where V denotes a groupoid lattice. Condition (iii) and (v) implies,
(vi) $0 \cdot a = a \cdot 0 = 0 \quad \forall a \in V$.

Definition 1.3 *An element b of V is called absorbent of the element a of V if*

(vii) $$b \leq a$$

and

$(viii)$ $$ab \leq b$$

(ix) $$ba \leq b$$

holds, b is a left absorbent of a if (vii) and $(viii)$ holds and right absorbent if (vii) and (ix) holds.

Definition 1.4 *An element k of V is called quasi-absorbent of $a \in V$ if $k \leq a$ and $ka \wedge ak \leq k$.*

Definition 1.5 *By a bi-absorbent of $a \in V$ we mean an element $b \in V$ such that $b \in a$ and $(ba)a \wedge b(ab) \leq b$.*

If $a_\mu (\mu \in \Lambda)$ are elements of the groupoid lattice V, then from (iii), we have

$$(x) \qquad b\Big(\bigwedge_{\mu\in\Lambda} a_\mu\Big) \leq \bigwedge_{\mu\in\Lambda} b a_\mu, \qquad \Big(\bigwedge_{\mu\in\Lambda} a_\mu\Big) b \leq \bigwedge_{\mu\in\Lambda} a_\mu b \qquad \forall b \in V$$

$$(xi) \qquad b\Big(\bigvee_{\mu\in\Lambda} a_\mu\Big) \geq \bigvee_{\mu\in\Lambda} b a_\mu, \qquad \Big(\bigvee_{\mu\in\Lambda} a_\mu\Big) b \geq \bigvee_{\mu\in\Lambda} a_\mu b \qquad \forall b \in V.$$

Definition 1.6 *The quasi-absorbent $k \neq 0$ of an element a of a groupoid lattice V is said to be minimal if a has no non-zero quasi-absorbent k_1 such that $k_1 < k$. A minimal left, right or bi-absorbent of a can be defined analogously.*

Proposition 1.1[1]:If r and l are right- and left-absorbent of $a \in V$, respectively, then $rl \leq r \wedge l$, rl is a bi-absorbent and $r \wedge l$ is a quasi-absorbent of a.

Proposition 1.1[2]: A quasi absorbent k of an element $a \in V$ is minimal if and only if its non-zero elements generate the same left-absorbent and right-absorbent of a.

2 Semiprime absorbent:

Proposition 2.1[5]: Let m be a two-sided absorbent of a groupoid-lattice $a \in V$. The following conditions are equivalent:
(i) If b is a (two-sided) absorbent of a such that $b^2 \leq m$, then $b \leq m$;
(ii) If l is a left-absorbent of a such that $l^2 \leq m$, then $l \leq m$;
(iii) If r is a right-absorbent of a such that $r^2 \leq m$, then $r \leq m$.

Theorem 2.1 [5]*: Every minimal quasi absorbent k of a semiprime groupoid lattice $a \in V$ is the meet of a minimal left absorbent l and minimal right absorbent r of a.*

Proposition 2.2: Every minimal quasi-absorbent k of a semiprime groupoid lattice $a \in V$ has the form $k = ea \wedge af = eaf \quad \forall e^2 = e, f^2 = f$, where ea, af are minimal right and minimal left absorbents of a, respectively.

Proof: By Theorem 2.1, k is the meet of a minimal right absorbent r and minimal left absorbent l of a, that is $k = r \wedge l$. Now we shall prove the existence of non-zero idempotents e, f in a such that $r = ea$ and $l = af$. Let m be two-sided absorbent of a such that $l \leq m$ and x be a non-zero element of l. Then the product mx is a left absorbent of a. Since l is minimal, either $mx = 0$ or $mx = l$. If $mx = 0$, then the set X of all the elements x of l with $mx = 0$ is a non-zero left absorbent of a. By the minimality of l, we have $X = l$, that is $mX = ml = 0$. Since $l \leq m$, we conclude that $l^2 = 0$, which contradicts the condition that a is a semiprime groupoid lattice. So we have $mx = l$ for all non-zero x of l. If m is a two-sided absorbent of a such that $r \leq m$, one can show that $ym = r$ for all non-zero y of r. Now let d be non-zero element of $k = r \wedge l$. From the above we know that

$$(iv) \qquad ada = la = ar \quad \forall d \in k = r \wedge l.$$

Since a is a semiprime groupoid lattice, $r^2 = r$ and $l^2 = l$ must hold. This and (iv) imply

$$l = l^2 \leq la = ada$$

and

$$r = r^2 \leq ar = ada.$$

Proposition 2.3: Let e be an idempotent element of a groupoid lattice $a \in V$ and r, l are left and right absorbent of a, respectively. Then re and el are quasi-absorbents of a such that $re = r \wedge ae$ and $el = ea \wedge l$.

Theorem 2.2 *If a quasi absorbent k of a groupoid lattice a is a division element of a, then k is a minimal quasi absorbent of a.*

Proof: Let $k^{'}$ be a quasi absorbent of a. Such that $0 \neq k^{'} \leq k$. Then $kk^{'} \wedge k^{'}k \leq ak^{'} \wedge k^{'}a \leq k$ implies that $k^{'}$ is a quasi absorbent of a. Since k is a division element and division element has no proper quasi absorbent, we have $k = k^{'}$. Thus k is the minimal quasi absorbent of a.

Proposition 2.4: Let l be a minimal left absorbent of a groupoid lattice $a \in V$. If e is a non-zero idempotent element of l, then el is a division element of a, moreover it is a minimal quasi absorbent of a.

Proof: By proposition 2.3, el is a quasi absorbent of a. Evidently, e is a left identity of el. Let eh be non-zero element of el. Then $l \cdot (eh)$ is a non-zero left absorbent of a such that $l \cdot (eh) \leq l$. By the minimality of l, we have $l \cdot (eh) = l$. Hence $(el) \cdot (eh) = el$. This implies the existence of non-zero element ez of el such that $(ez) \cdot (eh) = e$. Thus $0 \neq el$ is a division element. Hence it is minimal quasi-absorbent of a by Theorem 2.2.

Theorem 2.3 *Let e be a non-zero idempotent of a semiprime groupoid lattice $a \in V$. Then the following conditions are equivalent:*

(i) *ae is a minimal left absorbent of a;*
(ii) *aea is a minimal quasi absorbent of a;*
(iii) *ea is a minimal right absorbent of a.*

Proof: (i) implies (ii): By proposition 2.3, eae is the minimal quasi absorbent of a if ae is a minimal left absorbent of a. (ii) implies (i): Conversely, *let eae be a minimal quasi absorbent of* a*. If* l *is a non-zero left absorbent of* a *such that* $l \leq ea$*, then* $l = le$*. By proposition 2.2, we have* $ele \leq eae$ *is a quasi absorbent of* a*. Since* 0 *is a semiprime absorbent of* a*,* $0 \neq l^2 = (le) \cdot (le)$*, whence* $le \neq 0$*. This fact and the minimality of the quasi absorbent eae imply that* $ele = eae$*. Hence* $e \in ele \leq le = l$*. Therefore,* $ae \leq al \leq l$*, that is,* $ae = l$*. This means that ae is minimal left absorbent of* a*. The implication* (ii) *implies* (iii) *and* (iii) *implies* (ii) *can be prove dually.*

Theorem 2.4 *Let* a *be a semiprime groupoid lattice of* V*. Then the product* k_1k_2 *of any two minimal quasi absorbents* k_1 *and* k_2 *of* a *is either* 0 *or a minimal quasi absorbent of* a*.*

Proof: By Theorem 2.1, there exist minimal right absorbents r_1, r_2 *and minimal left absorbents* l_1, l_2 *of* a *such that* $k_1 = r_1 \wedge l_1$ *and* $k_2 = r_2 \wedge l_2$*. Assume* $k_1k_2 \neq 0$*. Then* $0 \neq k_1k_2 = (r_1 \wedge l_1)(r_2 \wedge l_2) \leq r_1l_2$ *, where* $r_1l_2 \neq 0$*. So by Theorem 2.3,* r_1l_2 *is a minimal quasi absorbent of* a*. Since* $0 \neq k_1k_2 \leq r_1l_2$ *, we need only to show that* k_1k_2 *is a quasi absorbent of* a*. By proposition 2.2, there exist non-zero idempotent element* e_1, e_2 *and* f_1, f_2 *in* a *such that* $k_1 = e_1a \wedge af_1 = e_1af_1$ *and* $k_2 = e_2a \wedge af_2 = e_2af_2$*. Hence* $k_1k_2 = (e_1af_1)(e_2af_2)$*. Therefore by proposition 2.3,* k_1k_2 *is a quasi absorbent of* a*.*

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