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Matrix-Geometric Method for Queueing Model with Subject to Breakdown and N-Policy Vacations

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Abstract

In this paper we present matrix-geometric method for analyzing the N-policy multiple vacation queueing model with server breakdown and repair studied in this paper. The service station, the server is subject to breakdown while in operation. Service resumes immediately after a repair process, and a vacation starts. Arrivals follows a Poisson process with rates depending upon the system rate, namely, vacation, service, or breakdown state. The repair times, time to breakdown follows a repair or vacation. Using quasi-birth death process and matrix-geometry model, we may gain the distribution of the steady-state queue system. Furthermore, we derive the formula of expected queue length and expected waiting period. Finally, numerical example are presented.

Key words: Working breakdown, Matrix-Geometry methods, N-Policy Vacation, Queue size, stationary probability.

1 Introduction

Queueing systems with vacation have been investigated for over two decades as a very useful tool for modeling and analysing computer systems, communication networks, manufacturing, Industrial organizations, production system and many other. Server vacation may occur due to empty(idle), server subject to breakdown or another task being assigned to the server. The concept of N-policy multiple vacations was first introduced by Yadin and Naor [14] and later it was invedticated by Lee, Srinivasan [7] and Kella [6] in queueing system. The amount of literature relating to queueing model with vacation is growing rapidly, as can be seen in survey papers by Doshi [1] and Takagi [11]. For general queueing analysis, including the vacation polices, the readers are recommended to Gross and harris [2] and Tian and Zhang [13]. Zeng [15] discussed two-phase service queueing system with server breakdown and vacations. Li,Shi and Chao [8] have considered an $M/G/1$ queueing system with server breakdowns and bernoulli vacations and have discussed the mean number of customers in the system. Numerical investigation and sensitivity analysis of the reliability and availability measures of a repair system were investigated by Ke and Lin [5], in which the servers were imperfect and applied a multiple vacation policy. Tian and Yue [12] discussed the queueing system with variable arrival rate will be studied by using the principle of quasi-birth and death process(QBD) and matrix-geometric method, furthermore, some performance measures are obtained, such as the number of customers in the system in steady-state.

Matrix-geometric methods approach is a useful tool for solving the more complex queueing problems of the rapid growth of the state-space introduced by the need to construct the generator matrix. Matrix-geometric method is applied by many researchers to solve various queueing problems in different frameworks. Neuts [10] explained various matrix geometric solutions of stochastic models. Matrix-geometric approach is utilized to develop the computable explicit formula for the probability distributions of the queue length and other system characteristic. Recently, Jain and Jain [3] analyzed a single server working vacation queueing model with multiple type of server breakdowns. They proposed a matrix-geometric method for computing the stationary queue length distribution.

Recently, Ke [4] suggested the optimal control of an $M/G/1$ queueing system with server vacations, startup and breakdowns. In this paper, we introduced the matrix-geometric method for N-policy multiple vacation analyzing the system breakdown and repair. The server takes a vacation, when the system becomes empty or the server itself breakdown. The instance at which the server comes back from a vacation and the queue length reaches or exceeds N(threshold) customer in the system, it begins service immediately and exhaustively. This type of control policy is also called N-policy queueing system with vacation. In the number of waiting customers in the system at any vacation completion is less than N then the server continuous to be in vacation(multiple vacation).

2 Model Descriptions

In this paper, We consider matrix-geometric method for N-policy multiple vacation queueing model with server breakdown and repair. The basic assumptions are described as follows.

The customers arrive at the system according to the state of servers, namely, vacation state, service state or breakdown state. The customers arrives in a Poisson process with rate λ_v for vacation period and λ for server breakdown. The server goes on vacation at the instant when the system is either empty or the server itself breakdown, we term it as the server vacation. Whenever the system becomes empty, the server is turned off. As soon as the total number of arrivals in the queue reaches or exceeds N(threshold), the server is turned on. After serving the whole customer in the system, the server goes regular to vacation period. This process will be repeated constantly. Vacation times follows an exponential distribution with rate v. The service times follows exponential distribution with rate μ.

The server subject to breakdown during individual service at any time with exponential breakdown rate α. Whenever the server fails, it is immediately sent for repairs during which the server stops providing service and waits for the repair to complete. Repair time is assumed to be exponential distribution with rate β. We assume that the arrival times, the service times, and multiple vacation times are mutually independent. In addition, the service discipline is first come first service(FCFS)

Solution of the models:
(1) $i = 0$, denote the server is on vacation
(2) $i = 1$, denote the server is in working state
(3) $i = 2$, denote the server is breakdown but under repair.
The system states are denoted by

- $P_{i,0,V}$ - Probability that there are i^{th} customer in the system when the server is on vacation.
- $P_{i,1,W}$ - Probability that there are i^{th} customer in the system when the server is on working state.
- $P_{i,2,B}$ - Probability that there are i^{th} customer in the system when the server is breakdown and under repair.

The steady state equation covering the model are constructed as follows:

$$\lambda_v \, P_{0,0,V} = \mu \, P_{1,1,W} \tag{1}$$

$$(\lambda_v + v) \, P_{i,0,V} = \lambda_v \, P_{i-1,0,V}, \quad 1 \le i \le N-1 \tag{2}$$

$$(\lambda + \mu + \alpha) \, P_{1,1,W} = v \, P_{1,0,v} + \mu \, P_{2,1,W} + \beta \, P_{1,2,B} \tag{3}$$

$$(\lambda + \mu + \alpha) \, P_{i,1,W} = \lambda \, P_{i-1,1,W} + v \, P_{i,0,V} + \mu P_{i+1,1,W} + \beta P_{i,2,B} \qquad i \ge 2 \tag{4}$$

$$(\lambda + \mu + \alpha) \, P_{N,1,W} = \lambda \, P_{N-1,1,W} + v \, P_{N,0,V} + \mu P_{N+1,1,W} + \beta P_{N,2,B}, \; i \ge N \tag{5}$$

$$(\lambda + \mu + \alpha) \, P_{i,1,W} = \lambda \, P_{i-1,1,W} + v \, P_{i,0,V} + \mu P_{i+1,1,W} + \beta P_{i,2,B}, \; i \ge N+1 \tag{6}$$

$$(\lambda + \beta) \, P_{1,2,B} = \alpha \, P_{1,1,W} \tag{7}$$

$$(\lambda + \beta) \, P_{i,2,B} = \lambda \, P_{i-1,2,B} + \alpha \, P_{i,1,W}, \qquad i \ge 2 \tag{8}$$

3 Matrix Geometry Processes

This matrix represents Quasi-birth and death process was developed Neuts[10] to solve the stationary state probability for the vector state Markov process with repetitive structure. We develop the steady-state probability. The corresponding transition rate matrix $\mathcal{Q}$ at this Markov chain has the block-tridiogonal form. Consider the generator matrix $\mathcal{Q}$ as shown below

$$\mathcal{Q} = \begin{bmatrix} \mathcal{B}_0 & \mathcal{C}_0 & & & & \\ \mathcal{B}_1 & \mathcal{A}_1 & \mathcal{A}_0 & & & \\ & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_0 & & \\ & & \mathcal{A}_2 & \mathcal{A}_1 & \mathcal{A}_0 & \cdots \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \tag{9}$$

Where

$$\mathcal{B}_0 = [-\lambda_v], \mathcal{C}_0 = [\lambda_v \ \ 0 \ \ 0], \mathcal{B}_1 = \begin{bmatrix} 0 \\ \mu \\ 0 \end{bmatrix}, \mathcal{A}_0 = \begin{bmatrix} \lambda_v & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{A}_1 = \begin{bmatrix} -(\lambda_v + v) & v & 0 \\ 0 & -(\lambda + \mu + \alpha) & \alpha \\ 0 & \beta & -(\lambda + \beta) \end{bmatrix}$$

The matrix $\mathcal{Q}$ has a block-tridiaonal structure with indicates that is a QBD process, see Neuts [10] and Latachi, Ramasami [9]. To analyze this QBD process, it is necessary to solve for the minimal non-neative solution of the matrix quadratic equation as follows,

$$\mathcal{R}^2\mathcal{A}_2 + \mathcal{R}\mathcal{A}_1 + \mathcal{A} = 0 \tag{10}$$

and this solution is called the rate matrix and denoted by $\mathcal{R}$. Obviously, we have the following lemma.

Lemma 3.1. *If $\rho = \frac{\lambda}{\mu} < 1$, the matrix equation* (10) *as the minimal non-negative solution*

$$\mathcal{R} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ 0 & \rho & \mathcal{R}_{23} \\ 0 & 0 & \mathcal{R}_{33} \end{bmatrix}$$

Where

$$
\begin{aligned}
\rho &= \lambda\,\mu^{-1} < 1 \\
\mathcal{R}_{11} &= \left[(\lambda_v + v) - \sqrt{(\lambda_v + v)^2 - 4\lambda_v}/2\lambda_v\right] \\
\mathcal{R}_{12} &= \mathcal{R}_{11} v\beta / \left[\lambda + \mu(1 - \mathcal{R}_{12}(\rho + \mathcal{R}_{11})\right] \\
\mathcal{R}_{13} &= \mathcal{R}_{12}(\mathcal{R}_{23} + \mu) / \left[(\lambda + \beta) - (\mathcal{R}_{11} + \mathcal{R}_{13})\right] \\
\mathcal{R}_{23} &= \mathcal{R}_{22}\beta / \left[(\lambda + \beta) - (\rho + \mathcal{R}_{33})\right] \\
\mathcal{R}_{33} &= \left[(\lambda + \beta) - \sqrt{(\lambda + \beta)^2 - 4\lambda}/2\lambda\right]
\end{aligned}
$$

Proof. We can assume that $\mathcal{R}$ has the structure as,

$$
\mathcal{R} = \begin{bmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ 0 & \mathcal{R}_{22} & \mathcal{R}_{23} \\ 0 & 0 & \mathcal{R}_{33} \end{bmatrix}
$$

and substituting $\mathcal{R}$ into equation (10), then we have

$$\lambda_v + \mathcal{R}_{11}^2 - \mathcal{R}_{11}[\lambda_v + v] = 0 \tag{11}$$

$$\mu\mathcal{R}_{12}[\mathcal{R}_{11} + \mathcal{R}_{22}] + \mathcal{R}_{11} v\beta - \mathcal{R}_{11}[\lambda + \mu + \alpha] = 0 \tag{12}$$

$$[\mathcal{R}_{11}\mathcal{R}_{13} + \mathcal{R}_{12}\mathcal{R}_{23} + \mathcal{R}_{13}\mathcal{R}_{33}] + \mathcal{R}_{12}\mu - \mathcal{R}_{13}[\lambda + \beta] = 0 \tag{13}$$

$$\lambda + \mu\mathcal{R}_{22}^2 - \mathcal{R}_{22}[\lambda + \mu + \alpha] = 0 \tag{14}$$

$$\mathcal{R}_{23}[\mathcal{R}_{22} + \mathcal{R}_{33}] + \mathcal{R}_{22}\beta - \mathcal{R}_{23}[\lambda + \beta] = 0 \tag{15}$$

$$\lambda + \mathcal{R}_{33}^2 - \mathcal{R}_{33}[\lambda + \beta] = 0 \tag{16}$$

□

4 Stationary Distribution

Let $\mathcal{X}$ be the stationary probability vector associated with the transition matrix $\mathcal{Q}$ such that, $\mathcal{X}\mathcal{Q} = 0$, $\mathcal{X}1 = 0$. The matrix-geometric solution method(see Neuts [10]), we partitioned vector $\mathcal{X} = (\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots)$, where each $\mathcal{X}_i = (\mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2}), i \geq 0$. Then we have

$$\mathcal{X}_i = (\mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2}) = (\mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2})\mathcal{R}^{i-1} \tag{17}$$

Using eqations (11) to (16), we can obtain the values of $\mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2}$

$$
\begin{aligned}
\mathcal{X}_{i0} &= \mathcal{X}_{10}\,\mathcal{R}^{i-1}, \quad i \geq 0 \\
\mathcal{X}_{i1} &= \mathcal{X}_{11}\,\mathcal{R}^{i-1}, \quad i \geq 1 \\
\mathcal{X}_{i2} &= \mathcal{X}_{12}\,\mathcal{R}^{i-1}, \quad i \geq 1
\end{aligned}
$$

and $(\mathcal{X}_{00}, \mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2})$ satisfies the set of equations as follows

$$(\mathcal{X}_{00}, \mathcal{X}_{i0}, \mathcal{X}_{i1}, \mathcal{X}_{i2})B[\mathcal{R}] = 0 \tag{18}$$

Where

$$\begin{aligned} B[\mathcal{R}] &= \begin{bmatrix} \mathcal{B}_0 & \mathcal{C}_0 \\ \mathcal{B}_1 & \mathcal{A}_1 + \mathcal{R}\mathcal{A}_2 \end{bmatrix} \\ &= \begin{bmatrix} -\lambda_v & \lambda_v & 0 & 0 \\ 0 & (\mathcal{R}_{11} - (\lambda_v + v)) & \mu\mathcal{R}_{12} + (v+\beta) & 0 \\ \mu & 0 & \mu\mathcal{R}_{33} - (\lambda + \mu + v) & (\alpha + \mathcal{R}_{23}) \\ 0 & 0 & 0 & (\mathcal{R}_{33} - (\lambda + \beta)) \end{bmatrix} \end{aligned} \tag{19}$$

Using equation (19), we obtain the following set of the equation

$$-\lambda_v \mathcal{X}_{00} + \mu \mathcal{X}_{11} = 0 \tag{20}$$

$$\lambda_v \mathcal{X}_{00} + \mathcal{R}_{11} - (\lambda_v + v)\, \mathcal{X}_{10} = 0 \tag{21}$$

$$(\mu\mathcal{R}_{12} + v + \beta)\, \mathcal{X}_{10} + [\mu\mathcal{R}_{33} - (\lambda + \mu + \alpha)]\mathcal{X}_{11} = 0 \tag{22}$$

$$(\alpha + \mathcal{R}_{23})\mathcal{X}_{11} + [\mathcal{R}_{33} - (\lambda + \beta)]\mathcal{X}_{12} = 0 \tag{23}$$

and

$$\mathcal{X}_{i0} = \mathcal{X}_{00} \left(\frac{\lambda_v}{[\lambda_v + v - \mathcal{R}_{11}]} \right) \mathcal{R}^{i-1}, \quad i \geq 0$$

$$\mathcal{X}_{i1} = \mathcal{X}_{00} \left(\frac{\lambda_v}{[\lambda_v + v - \mathcal{R}_{11}]} \right) \left(\frac{v\,\beta + \mu\,\mathcal{R}_{12}}{[\lambda + \mu + \alpha] - \mu\,\mathcal{R}_{22}} \right) \mathcal{R}^{i-1}, \quad i \geq 1$$

$$\mathcal{X}_{i2} = \mathcal{X}_{00} \left[\lambda_v - \left(\frac{\lambda_v}{[\lambda_v + v - \mathcal{R}_{11}]} \right) - \left(\frac{\lambda_v \mu}{[\lambda_v + v - \mathcal{R}_{11}]} \right) \left(\frac{v\,\beta + \mu\mathcal{R}_{12}}{[\lambda + \mu + \alpha] - \mu\mathcal{R}_{22}} \right) \right] \mathcal{R}^{i-1},$$

where

$$\mathcal{X}_{00} = \frac{(1-\mathcal{R}_{11})(1-\mathcal{R}_{12})(1-\mathcal{R}_{13})}{\begin{bmatrix} (1-\mathcal{R}_{11})(1-\mathcal{R}_{22})(1-\mathcal{R}_{33}) + u_0[(1-\mathcal{R}_{22})(1-\mathcal{R}_{33}) + \mathcal{R}_{12} \\ (1-\mathcal{R}_{33}) + \mathcal{R}_{12}\mathcal{R}_{23} + \mathcal{R}_{13}(1-\mathcal{R}_{22})\mathcal{R}_{12}] + u_0 u_1[(1-\mathcal{R}_{11}) \\ (1-\mathcal{R}_{33}) + \mathcal{R}_{23}(1-\mathcal{R}_{11})] + u_2[(1-\mathcal{R}_{11})(1-\mathcal{R}_{33})] \end{bmatrix}}$$

$$u_0 = \left(\frac{\lambda_v}{\lambda_v + v - \mathcal{R}_{11}} \right)$$

$$u_1 = \left(\frac{v\beta + \mu\mathcal{R}_{12}}{(\lambda + \mu + \alpha) - \mu\mathcal{R}_{22}} \right)$$

$$u_2 = [\lambda_v + u_0 - \mu u_0 u_1]$$

Using equation (20) to (23), we can obtain

$$\mathcal{X}_{10} = \mathcal{X}_{00}\, u_0$$
$$\mathcal{X}_{11} = \mathcal{X}_{00}\, u_0\, u_1$$
$$\mathcal{X}_{12} = \mathcal{X}_{00}\, u_2$$

The normalizing condition is

$$\mathcal{X}_{00} + (\mathcal{X}_{10}, \mathcal{X}_{11}, \mathcal{X}_{12})\,(I - \mathcal{R})^{-1} = 0$$

5 Performance Characteristic

The performance characteristic are used to bring out the qualitative behaviour of the queueing model under study. Numerical study has been dealt to find the following measure.
The expression for various performance characteristic of the system are as follows.

- The expected number of customer in the system when the server is on vacation.

$$E(V) = \sum_{i=0}^{\infty} i\mathcal{X}_{i,0}$$

- The expected number of customer in the system when the server is in working state.

$$E(W) = \sum_{i=0}^{\infty} i\mathcal{X}_{i,1}$$

- The expected number of customer in the system when the server is in breakdown state(under repair).

$$E(B) = \sum_{i=0}^{\infty} i\mathcal{X}_{i,2}$$

- The expected number of customer in the system is given by

$$E(N) = E(V) + E(W) + E(B)$$

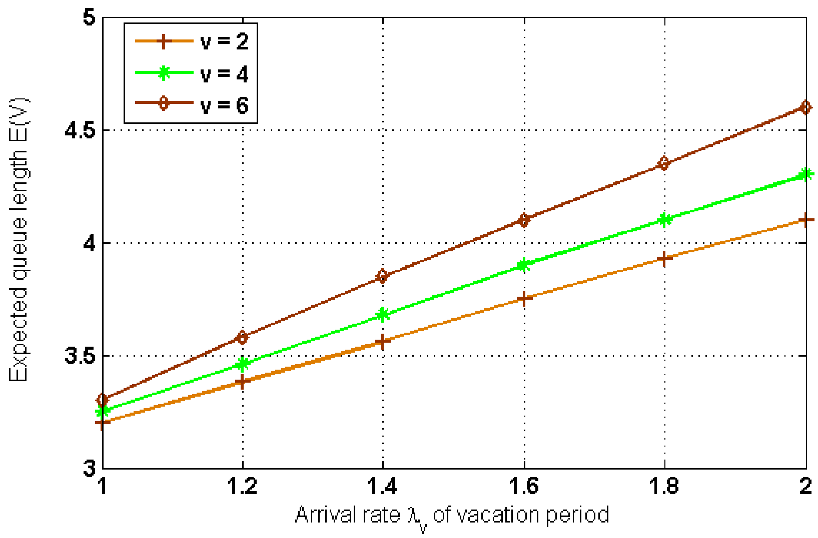

Figure 1: The relation of E(V) with λ_v

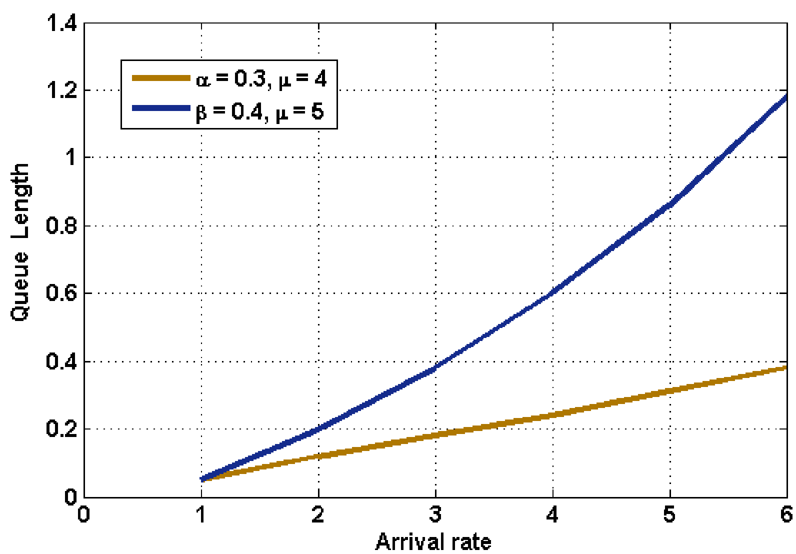

Figure 2: Queue length Vs Arrival rate

6 Numerical Results and Analysis

The value of the mean queue length, proportion time of server when the server is in vacation state, server is in working state have been computed for different values of $\lambda_v, \lambda, \mu, r, \beta$. Some of these are graphed in (20) to (23). In this case

$\lambda = 4, \lambda = 3, \mu = 1, \alpha = 0.5$, some graphs given by using the matlab program.

Figure 1 shows that relative of the expected queue length $E(V)$ and λ_v at the case of $\mu = 4, \beta = 5, \lambda = 3, \mu = 1, \alpha = 1$. We can see that the expected queue length $E(V)$ increases evidently with an increase value of λ_v when v is fixed. This is mainly because that the greater mean vacation rate v is the sorter the vacation time is, which will make less customer arrival of the system. Figure 2 shows that if arrival rate is increase then queue length also increases.

7 Conclusion

In this paper, we have developed a new concept of N-policy multiple vacation with system breakdown and repair. Using QBD process and matrix-geometric solution method to derive the distribution for the stationary queue length and waiting time of a customer of a system. We then developed the matrix-form expression of various system performance measures. Finally, numerical results were provided, and an application example was given to show the potential applicability of this working vacation queue.

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