**Mathematica Aeterna, Vol. 7, 2017, no. 4, 417 - 438**

# Lévy processes, martingales, reversed martingales and orthogonal polynomials

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**Abstract**

We study class of Lévy processes having distributions being indentifiable by moments. We define system of polynomial martingales $\{M_n(X_t,t),\mathcal{F}_{\leq t}\}_{n\geq 1}$, where $\mathcal{F}_{\leq t}$ is a suitable filtration defined below. We present several properties of these martingales. Among others we show that $M_1(X_t,t)/t$ is a reversed martingale as well as a harness. Main results of the paper concern the question if martingale say $M_i$ multiplied by suitable determinstic function $\mu_i(t)$ is a reversed martingale. We show that for $n\geq 3$ $M_n(X_t,t)$ is a reversed martingale (or orthogonal polynomial) only when the Lévy process in question is Gaussian (i.e. is a Wiener process). We study also a more general question if there are chances for a linear combination (with coefficients depending on $t$) of martingales $M_i$, $i=1,\ldots,n$ to be reversed martingales. We analyze case $n=2$ in detail listing all possible cases.

**Mathematics Subject Classification:** Primary 60G51 60G44; Secondary11B68

**Keywords:** Lévy processes, polynomial martingales, orthogonal polynomials, reversed martingales, harnesses, tangent numbers.

# 1 Introduction

Let us recall that Lévy processes $\{X_t\}_{t\geq 0}$ are such stochastic processes that start from zero i.e. $X_0=0$ a.s. and have stationary and independent increments which means that distribution of $X_t-X_s$ is the same as that of $X_{t-s}$ for all $0\leq s\leq t$ and $X_t-X_s$ is independent on $X_u-X_v$ whenever $0\leq s<t\leq v<u$.

This paper deals with those Lévy processes that posses all moments, more precisely we assume that the distributions of $X_t$, $t\geq 0$ are identifiable by their moments. Among other advantages this assumption allows to define a family of polynomial functions constructed of observations of the process. We examine such properties of these polynomials as being a martingale, a reversed martingale or a harness. The martingale theory is a very developed method of analysis of stochastic processes hence indicating martingales that can be constructed from the Lévy process we broaden the spectrum of tools that are at hand in analysis of a given Lévy process.

One can define many families of polynomials for Lévy processes with existing all moments. The most popular ones are the Kaillath–Segall polynomials (see [9], [7], [10], [20]) connected with a path' structure of the process and the properties of the multiple integrals of the process. There are also so called Teugels polynomials (see [13], [14]) associated with the properties of the Lévy measure of the process.

As stated above we are seeking such polynomial functions $M_n(X_t, t)$ of the process's observations $X_t$ at $t$ that are martingales. We indicate conditions under which these polynomials multiplied by some deterministic functions of the time parameter or their linear combinations with depending on $t$ coefficients are the reversed martingales or constitute a family of orthogonal, polynomial martingales. We give some properties of the so called 'connection coefficients' between polynomial martingales and orthogonal polynomials of the marginal distribution.

We also analyze the structure of the so called 'angular brackets' of the martingales $M_n$ i.e. functions $p_n(t) = EM_n(X_t, t)^2$.

Of course there exist relations of our martingales with Kaillath–Segall polynomials (see [9]) or Yablonski's polynomials (see [20]). In 2011 during a seminar presentations in Innsbruck J.L. Solé constructed polynomial martingales using Bell's (or Yablonski's) polynomials. This was based on two papers [11] and [12]. We present many more properties of these martingales than it was mentioned in Solé's and Utzet papers and presentation. They include expansion of some products of these martingales in linear combinations of them. Those useful technical results are presented in Lemma 4. We study also relation of polynomial martingales $M_n$ to the system of orthogonal polynomials of the marginal distributions. Some results in this topic are presented in Proposition 6. Of course on the way we point out relationship with Yablonski's polynomials.

The paper is organized as follows. The next Section 2 contains our main results. It is divided into two subsections. Subsection 2.1 contains properties of the family of polynomial martingales $\{M_n\}$ while Subsection 2.2 our main results answering questions if polynomial martingales $\{M_n\}$ are harnesses or have reversed martingale property (respectively Theorems 5 and 9). We consider also question when linear combinations of martingales $\{M_n\}$ are reversed martingales (Theorem 7) as well as we study the relationship between polynomials orthogonal with respect to the marginal distributions and polynomial martingales $\{M_n\}$ . Section 3 contains some open problems that can be solved using technic presented in the paper. Finally Section 4 contains some technical, auxiliary results as well as longer, tedious proofs.

At last let us mention the fact that while analyzing consequences of the assumption that $\mu(t)M_2(X_t,t)$ is the reversed martingale we had to prove, believed to be new, interesting property of the so called tangent numbers (see (28)), numbers closely related to Bernoulli numbers.

# 2 Polynomial martingales

Let us formulate assumptions that will be in force throughout the paper.

On the probability space $(\Omega,\mathcal{F},P)$ there is defined a Lévy stochastic process $\mathbf{X}=(X_t)_{t\geq 0,}$ i.e. time homogeneous process with independent increments, continuous with probability.

We define filtrations $\mathcal{F}_{\leq s}=\sigma(X_u:u\leq s)$ for $s>0$, $\mathcal{F}_{\geq s}=\sigma(X_u:u\geq s)$ and $\mathcal{F}_{s,u}=\sigma(X_t:t\notin(s,u))$.

We want to stress that all equalities between random variables are understood to be with probability 1. Hence we drop abbreviation a.s. usually following equality between random variables for the clarity of exposition.

We will be interested only in those Lévy processes which posses all moments. Such processes constitute a subclass of the class of all Lévy processes and the main tool of analyzing them are the moment functions. Hence we will not refer to the Lévy measure which is traditionally used in the analysis of Lévy processes. Instead we will use Kolmogorov's characterization of the infinitely divisible distributions as presented e.g. in [5] to study our class of Lévy processes. Of course the two approaches are closely related since one can get all moment functions of the process knowing its characteristic function. We will use moment functions since they constitute a very natural tool of examining the analyzed class of processes, for the sake of completeness of the paper and also in order to illustrate the usage of the recently obtained results of the paper [15].

Let us denote by $m_n(t)$ the $n-$th moment of the process i.e. $m_n(t)=EX_t^n$. We will assume that for all $n\geq 0$ functions $m_n(t)$ exist and are well defined.

Let us recall that a sequence $\{\alpha_n\}_{n\geq 0}$ of real numbers is called *a moment sequence* iff every $(n+1)\times(n+1)-$ matrix defined by $[\alpha_{i+j}]_{0\leq i,j\leq n}$ is positive definite. It is known that, when the above holds, there exists a positive measure $d\beta$ such that $\alpha_n=\int x^n d\beta(x)$. Let us also recall that not every moment sequence defines uniquely the measure whose moments the elements of this sequence constitute. In order that this measure be uniquely defined certain restrictions on he moment sequence have to be imposed. The most popular one is the Carleman's condition stating that if

$$\sum_{n\geq 0}\frac{1}{\alpha_{2n}^{1/2n}}=\infty, \tag{1}$$

then the moment sequence $\{\alpha_n\}_{n\geq 0}$ defines its measure uniquely. Another criterion is that

$$\int \exp\left(y\left|x\right|\right)d\beta\left(x\right)<\infty,$$

for some $y>0$.

In the sequel we will assume that $\forall t\geq 0$ sequence $\{EX_t^n\}_{n\geq 0}$ defines marginal measure uniquely. For the compact introduction see e.g. first sections of [17]. The discussion of how assumptions we are making in order to assure the existence of characteristic function of moments and the above mentioned assumptions assuring identifiability of distribution by its moments is done in Remark 7.

## 2.1 General properties

We have the following set of easy observations some of which are known. We present them here for the completeness of the paper.

**Proposition 1** *i)*

$$m_n(s+t)=\sum_{j=0}^{n}\binom{n}{j}m_j(s)m_{n-j}(t), \tag{2}$$

*for all $n\geq 0$ and $s,t\geq 0$.*

*ii) Let $Q(t;x)=\sum_{j\geq 0}m_j(t)x^j/j!$ be the characteristic function of the moment functions, then*

$$Q(t;x)=\exp(tf(x)),$$

*with $f(x)=\sum_{k\geq 1}c_kx^k/k!$. Coefficients $c_i$, $i=1,\ldots$ are such that for every $t\geq 0$ the sequence $\{m_n(t)\}_{n\geq 0}$ is the moment sequence.*

*iii) $c_1t=EX_t$, $\operatorname{var}(X_t)=c_2t$. Let $\hat{m}_n(t)=E(X_t-c_1t)^n$ be the central moment sequence. Then $\sum_{j\geq 0}\hat{m}_n(t)x^j/j!=\exp(t(f(x)-c_1x))$.*

*iv) Moment functions $m_n(t)$ satisfy the following set of differential equations: $\forall n>0$, $t>0$*

$$m_n^{'}(t)=\sum_{j=1}^{n}\binom{n}{j}c_jm_{n-j}(t). \tag{3}$$

*v) $\sum_{j=0}^{n}\binom{n}{j}m_{n-j}(-s)m_{j+i}(s)=\frac{\partial^n}{\partial u^n}(\exp(-sf(u))\frac{\partial^i}{\partial u^i}\exp(sf(u)))\Big|_{u=0}$*

**Proof.** Is shifted to Section 4. ■

**Remark 1** *Looking at assertion ii) and confronting it with the well known definition of so called cumulants i.e. coefficients of the power series expansion of the function* $\log\int\exp(xy)d\beta(y)$ *we see that coefficients* $c_i$ *are cumulants of the distribution of* $X_1$.

Let us remark that (2) is well known. We have recalled it here for the sake of completeness.

**Remark 2** *Since* $c_0 = 0$ *from the expansion* $\exp(tf(x)) = \sum_{n\geq 0}(tx)^n(f(x)/x)^n/n!$ *we deduce that coefficient by* $x^n$ *is a polynomial in* $t$ *of degree at most* $n$. *Thus we can define moment functions for non-positive* $t$. *Consequently (2) is true for all* $t, s \in \mathbb{R}$.

Hence as a corollary we have the following observation.

**Proposition 2** *i) Let us define for all* $n$ *and* $t > 0$ :

$$M_n(x,t) = \sum_{j=0}^{n}\binom{n}{j}m_{n-j}(-t)x^j. \tag{4}$$

*Then for all* $n$ *and* $t > s > 0$ :

$$E(M_n(X_t,t)|\mathcal{F}_{\leq s}) = M_n(X_s,s). \tag{5}$$

*ii) Characteristic function of polynomials* $\{M_n(x,t)\}$ *is the following:*

$$\sum_{n\geq 0}\frac{r^n}{n!}M_n(x,t) = \exp(rx - tf(r)) \stackrel{df}{=} \mathcal{N}_t(x,r). \tag{6}$$

*iii) We have for* $t > s > 0$ :

$$E(\mathcal{N}_t(X_t,r)|\mathcal{F}_{\leq s}) = \mathcal{N}_s(X_s,r),$$

*hence* $(\mathcal{N}_t(X_t,r),\mathcal{F}_{\leq t})$ *is a martingale known as 'exponential martingale'. Besides* $E\mathcal{N}_t(X_t,r) = 1$.

**Proof.** We have $E(M_n(X_t,t)|\mathcal{F}_{\leq s}) = \sum_{j=0}^{n}\binom{n}{j}m_{n-j}(-t)E(X_t - X_s + X_s)^j = \sum_{j=0}^{n}\binom{n}{j}m_{n-j}(-t)\sum_{k=0}^{j}\binom{j}{k}X_s^k m_{k-j}(t-s) = \sum_{k=0}^{n}\binom{n}{k}X_s^k\sum_{j=k}^{n}\binom{n-k}{j-k}m_{k-j}(t-s)m_{n-j}(-t) = \sum_{k=0}^{n}\binom{n}{k}X_s^k m_{n-k}(-s)$ by (2).

ii) It easily follows from Proposition 1, ii) and (4). iii) Follows directly from (5). ∎

**Remark 3** *Notice that polynomial martingales $\{M_n\}_{n\geq 0}$ are not the only polynomial martingales of the given Lévy process. In fact family of polynomials defined by*

$$\tilde{M}_n(X_t, t) = \sum_{j=1}^{n} b_{n,j} M_j\left(X_t, t\right),$$

*where coefficients $\{b_{n,j}\}$ do not depend on $t$, constitute another family of polynomial martingales.*

**Remark 4** *Assertion ii) of the above mentioned Proposition appeared earlier in [11].*

**Remark 5** *Notice that from (6) one can easily deduce that*

$$\frac{d}{dx} M_n(x, t) = n M_{n-1}(x, t).$$

*This means that polynomial martingales are the so called Appell polynomials. In particular they are not in general orthogonal. The only case when they are the case when $M_n(x, t)$ is the Hermite polynomial. This fact has some probabilistic consequences that will be exposed below.*

**Remark 6** *Coefficients $c_i$ can be identified with the moments of Kolmogorov's measure $dK$ of the analyzed Lévy process. Recall that since we deal with the process that has finite variance we can use the Lévy canonical form of the infinitely divisible distribution in the equivalent (Kolmogorov's) form (see e.g. [5], p.93, (10)). Applying appropriate formula for $t = -ix$ we get*

$$E\exp(xX_t) = \exp(tf(x)) = \exp(tc_1 x + t\int_{-\infty}^{\infty} \frac{(\exp(xy) - 1 - xy)}{y^2} dK(y), \quad (7)$$

*where $K(y)$ is a non-decreasing function with bounded variation such that $K(-\infty) = 0$ and $K(\infty) = \int_{\mathbb{R}} dK(y) = \operatorname{var}(X_1) = c_2$.*

Following this remark we have the following Proposition exposing relationship between coefficients $\{c_j\}_{j\geq 1}$ and moments of the measure $dK$.

**Proposition 3** *i)*

$$c_i = \int_{\mathbb{R}} y^{i-2} dK(y),$$

*consequently $c_i/c_2$ is the $i-2$ moment of the probability measure $\frac{1}{c_2}dK(y)$, for $i \geq 3$.*

*ii)* $(c_4/c_2)^{1/2} \leq (c_6/c_2)^{1/4} \leq \ldots \leq (c_{2k+2}/c_2)^{1/2k} \leq \ldots$

*iii)* $c_4 - c_3^2/c_2 \geq 0$ *since* $c_4/c_2 - (c_3/c_2)^2$ *is the variance of the measure* $\frac{1}{c_2}dK$.

*iv) If $c_{2k} = 0$ for some $k \geq 2$ then $dK$ must be degenerated and concentrated at $0$ (consequently $c_i = 0$ for $i \geq 3$) so we deal with the Gaussian case since*

$$\exp(c_1 xt + tc_2 x^2/2) = \int \exp(xy) \frac{1}{\sqrt{2\pi c_2 t}} \exp(-\frac{(y - xc_1 t)^2}{2c_2 t}) dy.$$

*v) If $c_4/c_2 - (c_3/c_2)^2 = 0$ then $dK$ is degenerated and concentrated at $\frac{c_3}{c_2}$, consequently $c_i = c_3^{i-2}/c_2^{i-3}$ for $i \geq 3$. We deal in this case with a mixture of the modified Poisson (i.e. concentrated at points $nc_3/c_2$, $n \geq 3$) and Gaussian distributions. The mixture depends on the relationship between $c_2$ and $c_3$.*

**Proof.** i) We confront (7) with the definition of the coefficients $c_i$. ii) We use Jensen's inequality. iii), iv), Are trivial. v) We confront assertion iii) with (7). ■

**Remark 7** *Let us recall result from [8] stating that measure $K$ defines one dimensional marginal measures uniquely. Hence if Kolmogorov measure $K$ is unidentifiable by moments then the same must be true with marginal measures and conversely. Notice also that if measure $K$ is identifiable by moments then expression $\int_{-\infty}^{\infty} \frac{(\exp(xy)-1-xy)}{y^2} dK(y)$ is finite, consequently $\log(E\exp(xX_t))$ is finite in some neighborhood of zero and we deal with the so called 'small exponential moments' case, the situation often considered by researchers working on Lévy processes.*

Following the above mentioned Remarks and interpretation of the coefficients $c_i$ we will assume from now on that these coefficients are such that the Kolmogorov's measure $dK$ is determined by them completely.

Since coefficients $c_i$, $i \geq 1$ determine Lévy process with finite all moments completely we will use notation $\mathbf{X}(\{c_i\})$, $\mathbf{X}(\mathbf{c})$, or finally $\mathbf{X}(\{c_1, c_2, \ldots\})$ to denote Lévy process with parameters $\{c_1, c_2, \ldots\}$.

**Remark 8** *Taking into account interpretation and properties of the coefficients $c_i$ given above we can refer to the martingale characterizations given by Wesołowski in [19]. One of them is obviously wrong. Namely the characterization of the Poisson process by the form of first three polynomial martingales is not true. This is so since from the martingale conditions considered by Wesołowski in Theorem 1. of [19] it follows that $c_1 = c_2 = c_3 = 1$. As the above Remark shows it is not enough to impose that all $c_i = 1$ for $i \geq 4$ which would lead to the Poisson process with parameter $1$ as indicated in Proposition 3, iv).*

*On the other hand the second martingale characterization of the Wiener process (within the class of Lévy processes) by the first four polynomial martingales given by Theorem 3. of [19] is true since the form of these martingales impose that $c_3 = c_4 = 0$. As it can be seen from Proposition 3, iii) it is enough to deduce that then all $c_i = 0$ for $i \geq 4$.*

**Remark 9** *In [20](2.1) Yablonski defined family of polynomials $P_n(x_1,\ldots,x_n)$ of the increasing numbers of variables by the expansion*

$$\exp(\sum_{k\geq 1}\frac{(-1)^{k-1}x_k}{k}z^k)=\sum_{n\geq 0}z^nP_n(x_1,\ldots,x_n). \tag{8}$$

*He proved validity of the above expansion for $|z|<1/\limsup_{k\longrightarrow\infty}|x_k|^{1/k}$ and also gave some properties of these polynomials. Comparing (8) with Proposition 1,ii) we see that*

$$\begin{aligned} x_k &= (-1)^k c_k/(k-1)!, &(9)\\ m_n(t) &= n!P_n(c_1t,-c_2t,c_3t/2,\ldots,(-1)^{n-1}tc_n/(n-1)!), &(10)\end{aligned}$$

*where $P_n$ is the mentioned above Yablonski's polynomial. In view of (9) we see that the condition $\limsup_{k\longrightarrow\infty}|x_k|^{1/k}<\infty$ is equivalent to the following one: $\limsup_{k\longrightarrow\infty}|c_k|^{1/k}/k<\infty$. However as Proposition 1.5 of [17] shows it can happen that in the case of deterministic moment problem (i.e. when coefficients $c_k$ fully determine distribution $dK$) $\limsup_{k\longrightarrow\infty}|c_k|^{1/k}/k$ can be finite or infinite. Hence existence of expansion (8) has nothing to do with determinacy of the Lévy process by its moments.*

*Following formulae ([20],(2.2)–(2.4)) and using our notation given by (10) we have the following properties of moments $m_n(t)$ which we quote here for completeness of the paper:*

$$\begin{aligned} m_{n+1}(t) &= t\sum_{j=0}^{n}\binom{n}{j}c_{j+1}m_{n-j}(t), &(11)\\ \frac{\partial m_n(t)}{\partial c_l} &= \begin{cases} 0 & if\quad l>n\\ ntm_{n-l}(t) & if\quad l\leq n\end{cases}, &(12)\\ m_n(t;\mathbf{c}+\mathbf{d}) &= \sum_{k=0}^{n}\binom{n}{k}m_k(t;\mathbf{c})m_{n-k}(t;\mathbf{d}), &(13)\\ m_n(t;(c_1\alpha,c_2\alpha^2,\ldots)) &= \alpha^n m_n(t;(c_1,c_2,\ldots)). &(14)\end{aligned}$$

*where we denoted $m_n(t;\mathbf{c})$ $n-th$ moment of the Lévy process with parameters $\mathbf{c}=(c_1,c_2,\ldots)$.*

*Finally let us remark that as shown in [10] Yablonski's polynomials $P_n$ are closely related to the Kailath–Segall polynomials (see [9]) that are used to study the path properties of Lévy processes. Hence our results give new interpretation of these polynomials.*

Using this formula and (11) we have the following set of useful relationships:

**Lemma 4** *i)*

$$M_1(x,t)M_n(x,t) = M_{n+1}(x,t) + t\sum_{k=1}^{n}\binom{n}{k}c_{k+1}M_{n-k}(x,t).$$

*Thus in particular* $EM_1(X_t,t)M_n(X_t,t) = tc_{n+1}$.

*ii)*

$$M_2(x,t)M_n(x,t) = M_{n+2}(x,t) + 2nc_2tM_n(x,t) + t\sum_{k=2}^{n+1}(\binom{n}{k-1}$$
$$+2\binom{n}{k})c_{k+1}M_{n-k+1}(x,t) + t^2\sum_{l=2}^{n}\binom{n}{l}M_{n-l}(x,t)\sum_{k=1}^{l-1}\binom{l}{k}c_{k+1}c_{l-k+1}.$$

*In particular* $EM_2(X_t,t)M_n(X_t,t) = tc_{n+2} + t^2\sum_{k=1}^{n-1}\binom{n}{k}c_{k+1}c_{n+1-k}$.

*iii)* $\forall n,k \geq 0, t \geq 0:$

$$E(M_k(X_t,t)M_n(X_t,t)) = \frac{\partial^n\partial^k}{\partial u^n\partial v^k}\exp(t(f(u+v)-f(u)-f(v))\bigg|_{u=v=0},$$

*consequently*

$$EM_n(X_t,t)M_k(X_t,t) = \sum_{j=1}^{\min(k,n)} d_j^{(k,n)}t^j,$$

*with*

$$d_j^{(k,n)} = \frac{d^{n+k-j}}{dx^{n+k-j}}\left(h(x)\right)^j\bigg|_{x=0}, \tag{15}$$

*where we denoted* $h(x) = \sum_{k\geq 2}c_kx^{k-1}/(k-1)! = f'(x) - c_1$. *In particular coefficient by* $t$ *is equal to* $c_{n+k}$, *by* $t^2$ $\frac{d^{n+k-2}}{dx^{n+k-2}}\left(h(x)\right)^2\Big|_{x=0}$ *and by* $t^{\min(n,k)}$ *is equal to* $\frac{d^{\max(n,k)}}{dx^{\max(n,k)}}\left(h(x)\right)^{\min(n,k)}\Big|_{x=0}$. *If* $n = k$ *coefficient by* $t^k$ *is equal to* $k!c_2^k > 0$, .

**Proof.** Rather tedious proof is shifted to Section 4. ■

## 2.2 Harnesses, reversed martingales and orthogonal polynomials

As a immediate corollary we get the following nice property of the Lévy processes which was observed by Jacod and Protter in [6]. We cite this result for completeness of the paper.

**Theorem 5** *Let $\mathbf{X}(\{c_1, c_2, \ldots\})$ be some Lévy process defined on $(0, \infty)$ and let $M_1(X_t, t)$ be the first of the polynomial martingales defined by Proposition (2).*

*Then $(M_1(X_t, t)/t, \mathcal{F}_{\leq t})$ is the reversed martingale and $M_1(X_t, t)$ has the harness property that is:.*

$$\begin{aligned}\frac{1}{s}E(M_1(X_s, s)|\mathcal{F}_{\geq t}) &= \frac{1}{t}M_1(X_t, t),\\ E(M_1(X_t, t)|\mathcal{F}_{s,u}) &= \frac{u-t}{u-s}M_1(X_s, s) + \frac{t-s}{u-s}M_1(X_u, u),\end{aligned}$$

*where $s < t < u$, and $\mathcal{F}_{s,u} = \sigma(X_v; v \in (0, s] \cup [u, \infty))$.*

**Proof.** The proof is in [6], however basing on Lemma 4,i) one can give an alternative one. ■

Let us denote by $\{Q_j(x, t)\}_{j\geq 0}$ system of monic polynomials orthogonal with respect to marginal measure of $X_t$. By assumption they are linearly independent and we have the following two expansions:

$$\begin{aligned}M_n(x, t) &= \sum_{j=0}^{n} \hat{b}_{n,j}(t)Q_j(x, t),\\ Q_n(x, t) &= \sum_{j=0}^{n} b_{n,j}(t)M_j(x, t).\end{aligned}$$

We have the following simple observation:

**Proposition 6** *i) $\forall n \geq 1 : b_{n,n}(t) = \hat{b}_{n,n}(t) = 1$, $b_{n,0}(t) = \hat{b}_{n,0}(t) = 0$, hence in particular: $Q_1(x, t) = M_1(x, t)$, $Q_2(x, t) = M_2(x, t) - c_3M_1(x, t)/c_2$,*

*ii) $\forall n \geq 2 : \hat{b}_{n,1}(t) = c_{n+1}/c_2$, hence in particular $M_2(x, t) = Q_2(x, t) + c_3Q_1(x, t)/c_2$,*

*iii) $\forall n \geq 2 : \hat{b}_{n,2}(t) = (tc_2 \sum_{k=1}^{n-1} \binom{n}{k} c_{k+1}c_{n+1-k} + c_{n+2}c_2 - c_3c_{n+1})/(2tc_2^3 + c_2c_4 - c_3^2)$.*

*iv) The only Lévy process with all moments existing for whom polynomial martingales $\{M_n(X_t, t)\}_{n\geq 0}$ are orthogonal is the Wiener process with the variance equal to $c_2$.*

**Proof.** i) Since we have both $EM_n(X_t, t) = EQ_n(X_t, t) = 0$ for all $n \geq 1$ we deduce that both $b_{n,0}(t) = \hat{b}_{n,0}(t) = 0$. Also since both systems of polynomials $\{Q_j\}$ and $\{M_j\}$ are monic then $\hat{b}_{n,n}(t) = b_{n,n}(t) = 1$. Hence in particular $Q_1(x, t) = M_1(x, t)$.

ii) On one hand by assertion i) of Lemma 4 we have $EQ_1(X_t, t)M_n(X_t, t) = tc_{n+1}$ while by assumption concerning polynomials $Q_n$ we get: $\hat{b}_{n,1}(t)EQ_1^2(X_t, t) = \hat{b}_{n,1}(t)tc_2$. Hence $\hat{b}_{n,1}(t) = c_{n+1}/c_2$.

iii) We have $Q_2(x,t) = M_2(x,t) - c_3 M_1(x,t)/c_2$ and consequently: $E(Q_2^2(X_t,t)) = tc_4 + 2t^2c_2^2 - 2tc_3^2/c_2 + tc_3^2c_2/c_2^2 = t(2tc_2^3 + c_2c_4 - c_3^2)/c_2$ and $EM_n(X_t,t)Q_2(X_t,t) = tc_{n+2} + t^2\sum_{k=1}^{n-1}\binom{n}{k}c_{k+1}c_{n+1-k} - c_3tc_{n+1}/c_2 = t(tc_2\sum_{k=1}^{n-1}\binom{n}{k}c_{k+1}c_{n+1-k} + c_{n+2}c_2 - c_3c_{n+1})/c_2$.

iv) One can see that condition $Q_n(x,t) = M_n(x,t)$ for all $n \geq 1$ is satisfied by assertion ii) of Proposition 6 only if $c_i = 0$ for all $i \geq 3$. On the other hand for the Wiener process Hermite polynomials that generate martingales by the formula $(c_2t)^{n/2} H_n(x/\sqrt{c_2t}) = M_n(x,t)$ constitute also family of orthogonal polynomials of the marginal distribution which is of course $N(0, c_2t)$. ■

Our main concern in this paper is to select those Lévy processes with all moments existing that have also polynomial reversed martingales and orthogonal martingales (that necessarily are also reversed martingales as remarked in [15], Corollary 5).

The problems that we will approach now are the following:

Fix $n$. Can we find such rational (in $t$) function $\mu_n(t)$, such that $\mu_n(t)M_n(t)$ is a reversed martingale.

The next problem is a generalization of the above mentioned problem.

Fix $n$. Can we find such rational (in $t$) functions $\mu_k(t)$, $k = 1, 3, \ldots, n$ that

$$R_n(X_t,t) = \sum_{k=1}^{n} \mu_k(t) M_k(X_t,t), \tag{16}$$

is a reversed martingale.

**Remark 10** *As it can be easily noticed technically the reversed martingale property is equivalent to the following condition: for all* $0 < s < t$, $l \geq 1$ *:*

$$\mu_n(s) EM_n(X_s,s) M_l(X_s,s) = \mu_n(t) EM_n(X_t,t) M_l(X_t,t),$$

*in case of Problem 2.2 and for all* $0 < s < t$, $l \geq 1$ *:*

$$\sum_{k=1}^{n} \mu_k(s) EM_k(X_s,s) M_l(X_s,s) = \sum_{k=1}^{n} \mu_k(t) EM_k(X_t,t) M_l(X_t,t). \tag{17}$$

*in the case of Problem 2.2.*

**Proof.** In case of Problem 2.2 we have $E(\mu_n(s)M_n(X_s,s)|\mathcal{F}_{\geq t}) = \mu_n(t)M_n(X_t,t)$. Multiplying both sides by $M_l(X_t,t)$ and taking expectation we get right hand side while for the l-st we have $E(\mu_n(s)M_n(X_s,s)M_l(X_t,t) = E(\mu_n(s)M_n(X_s,s)M_l(X_s,s)$ since $M_l(X_s,s)$ is the martingale. The second case is treated similarly. ■

We will solve the Problem 2.2 completely (Thm. 9) while Problem 2.2 only partially. Namely for $n = 2$. It is too complex to be solved in full generality in a short paper.

We will also consider the following simplified version of the above mentioned general reversed martingale problem.

Namely we select those polynomial martingales $M_n(x,t)$ that multiplied by some deterministic function $\mu_n(t)$ constitute a reversed martingale.

One of our main result states that for $n \geq 3$ within the class of Lévy processes with all moments only the ones with all parameters $c_i$ equal to zero for $i \geq 3$ have this property.

First let us solve Problem 2.2 for $n = 2$.

We have the following result:

**Theorem 7** *Suppose that $\mathbf{X}(\{c_1, c_2, \ldots\})$ be some Lévy process defined on $(0,\infty)$. Let $\{M_i(X_t,t)\}_{i\geq 1}$ be its polynomial martingales defined by (4), then $\sum_{k=1}^{2}\mu_k(t)M_k(X_t,t)$ is a reversed martingale for some functions $\mu_k(t)$, $k = 1, 2$ iff functions $\mu_1(t)$ and $\mu_2(t)$ are the following:*

$$\mu_2(t) = \frac{c_2 - \beta c_3}{t(2c_2^3 t + c_2 c_4 - c_3^2)}, \tag{18}$$

$$\mu_1(t) = \frac{\beta(2c_2^2 t + c_4) - c_3}{t(2c_2^3 t + c_2 c_4 - c_3^2)}, \tag{19}$$

*where $\beta$ is a constant and either of the following following cases happen:*

*1) $c_3 = 0$, then*

$$\exp(tf(x)) = e^{c_1 tx}(\cos(x\sqrt{\frac{c_4}{2c_2}}))^{-2tc_2^2/c_4}, \tag{20}$$

*for $|x| < \frac{\pi}{2}\sqrt{\frac{2c_2}{c_4}}$. In particular assuming for simplicity that $c_1 = 0$ the distribution of $X_t$ for $t = \frac{c_4}{2c_2^2}$ has density $h(y)$ equal to*

$$h(y) = \frac{\sqrt{c_4}}{\sqrt{8c_2}\cosh(\frac{\pi y\sqrt{2c_2}}{2\sqrt{c_4}})};\ y \in \mathbb{R}. \tag{21}$$

*and is identifiable by moments.*

*2) $c_4/c_2 = c_3^2/c_2^2$ then Lévy measure of such a process is degenerated, concentrated at $c_3/c_2$ and consequently $\mathbf{X}(\{c_1, c_2, \ldots\})$ is in this case the mixture of Poisson and Gaussian processes depending if $c_3 = c_2$ (pure Poisson case) or $c_3 = c_4 = 0$ pure Gaussian case or $\frac{c_3}{c_2} \neq 0$ or $1$ the nontrivial mixture.*

*3) $2c_4/c_2 = c_3^2/c_2^2$ then*

$$\exp(tf(x)) = e^{(c_1 - 2c_3/c_2)tx}\left(\frac{1}{1 - c_3 x/(2c_2)}\right)^{4tc_3^2/c_2^2}$$

*that is one dimensional distributions are of shifted gamma type.*

*4)* $2c_4/c_2 > c_3^2/c_2^2$, *then*

$$\exp(tf(x)) = \exp(xt(c_1 - \frac{c_3c_2}{c_4c_2 - c_3^2})) \tag{22}$$

$$\times \left(\frac{1+\frac{\chi_3}{2\alpha}\tan(x\alpha)}{1-\frac{\chi_3}{2\alpha}\tan(x\alpha)}\frac{1}{2\alpha^2-\chi_3^2/2+(2\alpha^2+\chi_3^2/2)\cos 2x\alpha}\right)^{2t/(4\alpha^2-\chi_3^2)}, \tag{23}$$

*where we denoted* $\alpha = \frac{1}{2}\sqrt{2\frac{c_4}{c_2} - 3\chi_3^2}$ *and* $\chi_3 = c_3/c_2$.

*5)* $2c_4/c_2 < c_3^2/c_2^2$, *then*

$$\exp(tf(x)) = \exp(xt(c_1 - \frac{c_3c_2}{c_4c_2 - c_3^2})) \tag{24}$$

$$\times \left(\frac{1+\frac{\chi_3}{2\alpha}\tanh(x\alpha)}{1-\frac{\chi_3}{2\alpha}\tanh(x\alpha)}\frac{1}{2\alpha^2-\chi_3^2/2+(2\alpha^2+\chi_3^2/2)\cosh 2x\alpha}\right)^{2t/(4\alpha^2-\chi_3^2)}. \tag{25}$$

**Proof.** is shifted to Section 4. ■

Notice that even if both $\sum_{k=1}^{2}\mu_k(t)M_k(X_t,t)$ and $M_1(X_t,t)$ are the reversed martingales it does not mean that for some function $\tilde{\mu}(t)$ $\tilde{\mu}(t)M_2(X_t,t)$ is a reversed martingale. As it will follow from the observations below the property that $\tilde{\mu}_l(t)M_l(X_t,t)$ is a reversed martingale for some function $\tilde{\mu}_l(t)$ is somewhat independent from the property that linear combination of martingales $M_i$, $i = 1,\ldots,l$ (such as (16)) is a reversed martingale.

It is so since we have the following observations.

**Lemma 8** *Let* $\mathbf{X}(\{c_1,c_2,\ldots\})$ *be Lévy process defined on* $(0,\infty)$ *and let* $\{M_n(X_t,t)\}_{n\geq 1}$ *be polynomial martingales defined by (4). Suppose for* $k \geq 2 : \mu(t)M_k(X_t,t)$ *is the reversed martingale, then*
*i) for all* $l = 1,2,\ldots$

$$\mu(s)EM_l(X_s,s)M_k(X_s,s) = \mu(t)EM_l(X_t,t)M_k(X_t,t), \tag{26}$$

*where* $\mu(t) = 1/EM_k(X_t,t)M_k(X_t,t)$, *(*$EM_l(X_t,t)M_k(X_t,t)$*, are given by Lemma 4, iii)),*

*ii)* $c_j = 0$, $j = \max(3,k-1),\ldots,2k-1$.

**Proof.** Is shifted to Section 4. ■

**Remark 11** *Let us notice that polynomials* $p_k(t) = EM_k(X_t,t)M_k(X_t,t)$ *are in fact the so called 'angular brackets' of the polynomial martingales* $M_k(X_t,t)$*. We know that they are non-decreasing functions of* $t$ *and Lemma 4, iii) gives its precise form.*

As an immediate corollary of Lemma 8,ii) and Remark 6,iii) we have the following result.

**Theorem 9** *For $k \geq 3$ there does not exist function $\mu(t)$ such that $\mu(t)M_k(t)$ is a reversed martingale unless $c_i = 0$ for $i \geq 3$.*

**Proof.** By Lemma 8 we know that parameters $c_3, c_4, \ldots, c_{2k-1}$ are equal to zero. In particular we have $c_4 = 0$ which leads by Remark 6,iii) to the conclusion that $c_i = 0$ for $i \geq 3$. ■

**Remark 12** *Notice that to have orthogonal polynomial martingales we have to have $EM_l(X_s, s)M_k(X_s, s) = 0$ for $k \neq l$. The presented above consideration show that it is possible only iff $c_i = 0$ for $i \geq 3$. This corresponds with the assertion iv) of the Proposition 6.*

Thus it remains to consider the case $k = 2$.

**Remark 13** *The fact that $(\mu(t)M_2(X_t, t), \mathcal{F}_{\leq t})$ is a reversed martingale implies by Lemma 8,ii) that $c_3 = 0$. Further from the proof of Theorem 7 it follows that if $c_3 = 0$ then $\mu_1(t) = \beta/t$ and $\mu_2(t) = 1/(2c_2^2t^2 + tc_4)$. Hence if $c_3 = 0$ and $\sum_{i=1}^{2} \mu_i(t)M_i(X_t, t)$ is a reverse martingale then $\mu_2(t)M_2(X_t, t)$ must also be a reversed martingale since $M_1(X_tt)/t$ is.*

**Remark 14** *Just for curiosity notice that it follows from Theorem 7,1) that if $c_1 = 0$ the moment generating function of the process in this case is symmetric consequently that coefficients $c_j$ with odd numbers are equal to zero and moreover numbers $\chi_n = c_n/c_2$ satisfy the following recursion:*

$$\chi_{2(k+1)} = \frac{\chi_4}{2} \sum_{j=0}^{k-1} \binom{2k}{2j+1} \chi_{2(j+1)} \chi_{2(k-j)}, \tag{27}$$

*which after denoting $T_j = \chi_{2j}(\frac{2}{\chi_4})^{j-1}$ can be reduced to the following one:*

$$T_{k+1} = \sum_{j=1}^{k} \binom{2k}{2k-1} T_j T_{k-j+1}. \tag{28}$$

*Little reflections shows that numbers $T_k$ are the so called tangent numbers*[1] *which surprisingly come to the Lévy processes scene.*

**Remark 15** *As a corollary we can now refer to the third martingale characterization of the Wiener process done by Wesołowski in [18]. It states that if a square integrable process $\mathbf{X} = (X_t)_{t\geq 0}$ has the property that $(X_t, \mathcal{F}_{\leq t})$ and $(X_t^2 - t, \mathcal{F}_{\leq t})$ are martingales and $(X_t/t, \mathcal{F}_{\geq t})$ and $((X_t^2 - t)/t^2, \mathcal{F}_{\geq t})$ are reversed martingales then the process is a Wiener process. It was shown in [16]*

[1] seq A000182 on http://oeis.org

*that this is not true characterization. Namely a counterexample with dependent increments was shown.*

*If we however we confine ourselves to the class of Lévy processes having all moments then this characterization is true. Since as shown above for our class of Lévy processes with* $c_1 = 0$, $c_2 = 1$, $(X_t, \mathcal{F}_{\leq t})$ *and* $(X_t^2 - t, \mathcal{F}_{\leq t})$ *are martingales and* $(X_t/t, \mathcal{F}_{\geq t})$ *is the reversed martingale only condition that* $((X_t^2 - t)/t^2, \mathcal{F}_{\geq t})$ *is a reversed martingale matters. Comparing this requirement with Theorem 13 we see that we must have* $c_4 = 0$ *to fulfill the requirement. But* $c_4 = 0$ *leads to* $c_i = 0$, *for all* $i \geq 3$ *by Proposition 3,iii).*

# 3 Open problems

First of all let us ask the following general question. Theorem 5 was proved under assumption that we deal with the Lévy process with all moments existing. The proof of this result was simple because it strongly depended on this assumption.

Can we weaken this assumption? That is can we prove assertions of Theorem 5 assuming that say the Lévy process has only first $m$ ($m$ some fixed integer) moments? Can we prove harness property of $M_1$ assuming only existence of the first $m$ moments and say knowing that $E(X_s|\mathcal{F}_{\geq t})$ for $s < t$ is a linear function of $X_t$?

Let us return the general 'reversed martingale' Problem 2.2.

The case $n = 2$ was examined in Theorem 7.

What about $n > 2$ can we find such functions $\mu_k(t)$, $k = 1, \ldots, n$ that $R_n$ (defined by 16) is the reversed martingale?

Similarly one can pose the following problem concerning the so called quadratic harnesses among Lévy process the problem inclusively studied recently by Bryc, Wesołowski and Matysiak (see [1],[2],[3]).

Find all Lévy process (i.e. coefficients $c_i$, $i \geq 3$) such that $M_2(X_t, t)$ is a quadratic harness i.e.

$$\begin{aligned} E(M_2(X_t,t)|\mathcal{F}_{s,u}) &= AM_2(X_s,s) + BM_1(X_s,s)M_1(X_u,u) \\ &+ CM_2(X_u,u) + DM_1(X_s,s) + EM_1(X_u,u) - c_2 s^2, \end{aligned}$$

where $0 < s < t < u$, $A$, $B$, $C$, $D$, $E$ are some functions of $s, t, u$ only. Note that $A$, $B$, $C$, $D$, $E$ can be relatively easily found by solving system of 5 linear equations obtained by multiplying the above equality by $M_2(X_s, s)$, $M_2(X_u, u)$, $M_1(X_s, s)M_1(X_u, u)$, $M_1(X_s, s)$ and $M_1(X_u, u)$ and calculating expectation of both sides and utilizing the fact that $M_i(X_t, t)$, $i = 1, 2$ are martingales (as done in [15]). Having $A$, $B$, $C$, $D$, $E$ we multiply both sides of this equality by $M_l(X_s, s)M_k(X_u, u)$ and calculate their expectations. On the way we use

Lemma 4i)-ii) and Lemma 8,i). In this way we get system of recursions to be satisfied by coefficients $c_i$, $i \geq 4$.

What about extension of these results to processes with nonhomogeneous, independent increments. A stem in this direction is done in [12].

# 4 Proofs

**Proof.** [Proof of Proposition 1] i) We have $m_n(s+t) = EX_{t+s}^n = E(X_{t+s} - X_s + X_s)^n = \sum_{j=0}^{n} \binom{n}{j} m_j(t) m_{n-j}(s)$ since $E(X_{t+s} - X_s)^n = m_n(t)$ for the Lévy processes.

ii) Let us define $Q(t;x) = \sum_{j\geq 0} m_n(t) x^j/j!$ . Following i) we get

$$Q(t+s;x) = Q(t;x)Q(s;x).$$

Since for fixed $x$ function the $Q$ is continuous in the first argument by assumption we are dealing with multiplicative Cauchy equation. Hence $Q(t;x)$ $= \exp(tf(x))$ for some constant $f(x)$ depending on $x$. Since $Q(t;x)$ is analytic with respect to $x$ and also since $Q(t;0) = 1$ we can expand function $f$ in a power series of the form $f(x) = \sum_{k\geq 1} c_k x^k/k!$. Following definition of the function $Q$ we get further statements of ii).

iii) We have by direct calculation: $m_1(t) = EX_t = \frac{\partial}{\partial x} \exp(tf(x))\big|_{x=0} = c_1 t$ and $m_2(t) = EX_t^2 = \frac{\partial^2}{\partial x^2} \exp(tf(x))\Big|_{x=0} = c_1^2 t + c_2 t$. Now let us consider sequence $\hat{m}_n(t)$. We have $\hat{m}_n(t) = \sum_{i=0}^{n} \binom{n}{i} m_{n-i}(t)(-1)^i (c_1 t)^i$ and also $\sum_{i\geq 0} (-1)^i (c_1 t)^i \frac{x^i}{i!} = \exp(-c_1 tx)$. Hence $\sum_{j\geq 0} \hat{m}_n(t) \frac{x^n}{n!} = \exp(tf(x) - c_1 tx))$.

iv) First of all let us notice that following definition of the function $Q$ we have $m_n'(t) = \frac{\partial^n \partial Q(t;x)}{\partial x^n \partial t}\Big|_{x=0} = \frac{\partial^n}{\partial x^n}(f(x)\exp(tf(x))\big|_{x=0}$ . Now we apply Leibnitz formula for $n-$th derivative of the product of two differentiable functions. On the way we have to remember that $\frac{d^n}{dx_n} f(x)\Big|_{x=0} = c_n$.

v) We have:

$$\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{j=0}^{n} \binom{n}{j} m_{n-j}(-s) m_{j+i}(s) =$$
$$\exp(-sf(x)) \sum_{j=0}^{\infty} \frac{u^j}{j!} m_{j+i}(s) = \exp(-sf(u)) \frac{\partial^i}{\partial u^i} \exp(sf(u)).$$

■

**Proof.** [Proof of Lemma 4] i) First observe that $M_1(x,t)\mathcal{N}_t(x,r) = (x - c_1 t)\mathcal{N}_t(x,r) = \frac{\partial}{\partial r}\mathcal{N}_t(x,r) + t(f'(r) - c_1)\mathcal{N}_t(x,r)$, where $\mathcal{N}_t(x,r)$ is the defined in Proposition 2, ii) characteristic functions of polynomials $M_n$. Hence using

Leibnitz's rule we get:

$$M_1(x,t)M_n(x,t) = \frac{\partial^n}{\partial r^n} M_1(x,t)\mathcal{N}_t(x,r)\bigg|_{r=0}$$

$$= \frac{\partial^{n+1}}{\partial r^{n+1}}\mathcal{N}_t(x,r)\bigg|_{r=0} + t\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^j}{\partial r^j}(f'(r)-c_1)\frac{\partial^{n-j}}{\partial r^{n-j}}\mathcal{N}_t(x,r)\bigg|_{r=0}$$

$$= M_{n+1}(x,t) + t\sum_{j=0}^{n}\binom{n}{j}c_{j+1}M_{n-j}(x,t),$$

since obviously $\frac{\partial^k}{\partial r^k}\mathcal{N}_t(x,r)\Big|_{r=0} = M_k(x,t)$.

ii) Recall that $M_2(x,t) = M_1(x,t)^2 - c_2 t$, hence using i) we get

$$M_1(x,t)^2 M_n(x,t) = M_1(x,t)M_{n+1}(x,t) + t\sum_{k=1}^{n}\binom{n}{k}c_{k+1}M_{n-k}(x,t)M_1(x,t)$$

$$= M_{n+2}(x,t) + t\sum_{k=1}^{n+1}\binom{n+1}{k}c_{k+1}M_{n+1-k}(x,t) + t\sum_{k=1}^{n}\binom{n}{k}c_{k+1}M_{n-k+1}(x,t) +$$

$$t^2\sum_{k=1}^{n-1}\binom{n}{k}c_{k+1}\sum_{j=1}^{n-k}\binom{n-k}{j}c_{j+1}M_{n-k-j}(x,t)$$

$$= M_{n+2}(x,t) + t\sum_{k=1}^{n+1}(\binom{n}{k-1} + 2\binom{n}{k})c_{k+1}M_{n-k+1}(x,t) +$$

$$t^2\sum_{l=2}^{n}\binom{n}{l}M_{n-l}(x,t)\sum_{k=1}^{l-1}\binom{l}{k}c_{k+1}c_{l-k+1}$$

Since $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

iii)Notice that on one hand $EM_n(X_t,t)M_k(X_t,t)$ that is equal to

$$E\ \frac{\partial^n\partial^k}{\partial u^n\partial v^k}\mathcal{N}_t(X_t,u)\mathcal{N}_t(X_t,v)\bigg|_{u=v=0} = \frac{\partial^n\partial^k}{\partial u^n\partial v^k}E\mathcal{N}_t(X_t,u)\mathcal{N}_t(X_t,v)\bigg|_{u=v=0}.$$

Now notice that $E\mathcal{N}_t(X_t,u)\mathcal{N}_t(X_t,v) = E\exp((u+v)X_t - t(f(u)+f(v)) = \exp(t(f(u+v)-f(u)-f(v))$ by Proposition 2, ii). Notice that

$$\frac{\partial^k}{\partial v^k}\exp(t(f(u+v)-f(u)-f(v))\bigg|_{u=0} = 0$$

for $k \geq 1$. Further notice that $\frac{\partial^k}{\partial v^k}\exp(t(f(u+v)-f(u)-f(v))$ is a product of two expressions : first being a polynomial in $t$ of order $k$ with coefficients being some differential expressions of $f(u+v)-f(v)$ and the second

$\exp(t(f(u+v)-f(u)-f(v))$. Consequently upon applying Leibnitz rule to this product and setting $u=v=0$ we see that only the first expression matters. The assertion follows the fact that $\frac{\partial^n}{\partial u^n}(f^{(j)}(u+v)-f^{(j)}(v))\big|_{u=v=0}$ $=\frac{\partial^n}{\partial u^n}(f^{(j)}(u+v)-f^{(j)}(v)\big|_{v=0})\big|_{u=0}$ for $j=1,\ldots,k$. Firstly we observe that $n-$th derivative of $\exp(tf(x))$ with respect to $x$ is of the form $(tf^{(n)}(x)+\ldots+t^n(f'(x))^n)\exp(tf(x))$. The independence of $c_1$ follows the fact that $\exp(t(f(u+v)-f(u)-f(v))=\exp(t(f(u+v)-c_1(u+v)-(f(u)-c_1u)-(f(v)-c_1v)))$ hence does not depend on $c_1$. Thus visibly $EM_n(X_t,t)M_k(M_t,t)$ is a polynomial in $t$ of order $\min(n,k)$ with coefficient by $t^j$ equal to $\frac{d^{n+k-j}}{dx^{n+k-j}}\left(f'(x)\right)^j\Big|_{x=0}$ for $j=0,\ldots,\min(n,k)$. ■

**Proof.** [Proof of Lemma 8] First of all notice that if $\mu(t)M_k(t)$ is a reversed martingale then $E(\mu(s)M_k(X_s,s)|\mathcal{F}_{\geq t})=\mu(t)M_k(X_t,t)$ a.s., hence multiplying both sides by $M_l(X_t,t)$ and taking expectation of both sides we get $\mu(s)EM_k(X_s,s)M_l(X_t,t)=\mu(t)EM_k(X_t,t)M_l(X_t,t)$. Finally we use the fact that $M_l$ is a martingale. Thus we get (26). By Lemma 4,iii) we know that $EM_k(X_t,t)M_l(X_t,t)$ is a polynomial of order $\min(k,l)$ in $t$. Moreover if $l=k$ coefficient by $t^k$ is equal to $k!c_2^k>0$. Secondly notice that quantity $\mu(t)EM_k(X_t,t)M_l(X_t,t)$ has to be independent on $t$, thus since for $l=k$ $EM_k(X_t,t)M_l(X_t,t)$ is a polynomial in $t$ of exactly $k-$th order we deduce that $\mu(t)$ must be proportional to the inverse of $EM_k(X_t,t)M_k(X_t,t)$.

i) By Lemma 4,iii) we know that for $l<k$ $EM_k(X_t,t)M_l(X_t,t)$ is a polynomial in $t$ of order $l$, so if $\mu(t)EM_k(X_t,t)M_l(X_t,t)$ is to be independent of $t$ $EM_k(X_t,t)M_l(X_t,t)$ must be zero polynomial.

ii) The fact that $c_{k+l}=0$, $l=1,\ldots,k-1$ follows formula $d_1^{(k,l)}=c_{k+l}$ and the fact that $EM_k(X_t,t)M_l(X_t,t)$ for $l<k$ must be zero polynomial in particular its coefficients by $t$ (which are equal to $c_{k+l}$) must be equal to zero. In this way we get the case $k=2$. Let us now consider coefficient in $EM_k(X_t,t)M_l(X_t,t)$ by $t^2$. It is equal to $\sum_{j=1}^{l+k-3}\binom{l+k-2}{j}c_{j+1}c_{l+k-j-1}$ as indicated by Lemma 4,ii). Let us now take into account the fact that $c_{k+1},\ldots,c_{2k-1}$ are equal to zero. It means that in fact we have to have: $\sum_{j=l-1}^{k-1}\binom{l+k-2}{j}c_{j+1}c_{l+k-j-1}=0$. Now we change index of summation to $s=j-l+1$ and get that for all $l=2,\ldots,k-1$ we have to have $\sum_{s=0}^{k-l}\binom{k+l-2}{s+l-1}c_{s+l}c_{k-s}$ $=0$. Let us consider $l=k-1$ and $k-2$. From the first equality we deduce that $c_kc_{k-1}=0$ and from the second that $(\binom{2k-4}{k-3}+\binom{2k-4}{k-1})c_{k-2}c_k+\binom{2k-4}{k-2}c_{k-1}^2=0$. Now if $k=3$ and $c_2>0$ we deduce that $c_3=0$ when $k=3$. Thus let us take $k\geq 4$. By multiplying both sides of the last equality by $c_{k-1}$ we deduce that since $c_{k-1}c_k=0$ that $c_{k-1}=0$, or equivalently that $c_{k-2}c_k=0$. Let us consider now $l=k-3$. We get $((\binom{2k-5}{k-4}+\binom{2k-5}{k-1})c_{k-3}c_k+(\binom{2k-5}{k-3}+\binom{2k-5}{k-2})c_{k-2}c_{k-1}=$ $0$. Hence $c_kc_{k-3}=0$ and so on. But after $k-2$ such steps we will get $c_2c_k=$ $0$. But $c_2>0$. So we deduce that $c_k=0$. ■

**Proof.** [Proof of Theorem 7] First of all let us notice that condition (17) for

$l = 1, 2$ leads to the following two linear equations:

$$\begin{array}{rcl} \mu_1 EM_1^2(X_t.t) + \mu_2 EM_1(X_t,t)M_2(X_t,t) & = & \beta, \\ \mu_1 EM_1(X_t.t)M_2(X_t,t) + \mu_2 EM_2^2(X_t,t) & = & 1 \end{array}$$

for the functions $\mu_1$ and $\mu_2$. Hence indeed they are given by (18) and (19). Now notice that in order that say right hand side of (17) be independent on $t$ for $l \geq 3$ we have to have:

$$\begin{array}{rcl} -2\beta c_2^2 c_{l+1} + (\beta c_3 - c_2) \sum_{k=1}^{l-1} \binom{l}{k} c_{k+1} c_{n+1-k})) & = & 2A_l c_2^3, \\ (c_3 - \beta c_4) c_{l+1} + (-c_2 + \beta c_3) c_{l+2} & = & A_l (c_2 c_4 - c_3^2), \end{array}$$

for some constant $A_l$. Let us denoting $\chi_l = c_l / c_2$ eliminate $A_l$ from the above equations. We will get then:

$$(\chi_4 - \chi_3^2)(-\beta\chi_{l+1} + \frac{(\beta\chi_3 - 1)}{2} \sum_{k=1}^{l-1} \binom{l}{k} \chi_{k+1}\chi_{n+1-k}) = \chi_{l+2}(\beta\chi_3 - 1) + \chi_{l+1}(\chi_3 - \beta\chi_4).$$

This equation is equivalent to the following relationship:

$$(1 - \beta\chi_3)(\chi_{l+2} - \chi_{l+1}\chi_3 - \frac{(\chi_4 - \chi_3^2)}{2} \sum_{k=1}^{l-1} \binom{l}{k} \chi_{k+1}\chi_{l+1-k}) = 0$$

$(1 - \beta\chi_3) = 0$ leads to $\mu_2 = 0$ so let us assume that

$$\chi_{l+2} = \chi_3\chi_{l+1} + \frac{(\chi_4 - \chi_3^2)}{2} \sum_{k=1}^{l-1} \binom{l}{k} \chi_{k+1}\chi_{l+1-k}.$$

Let us denote $\varphi(r) = \sum_{n=2}^{\infty} \frac{r^{n-2}}{(n-2)!}\chi_n$. Comparing this definition with (7) we see that $\varphi(r) = f''(r)/c_2$. Notice that $\varphi(0) = 1$. Multiplying both sides by $\frac{r^{l-1}}{(l-1)!}$ and summing by $l$ from 1 to $\infty$.

$$\begin{array}{rcl} \varphi'(r) & = & \chi_3\varphi(r) + \frac{(\chi_4 - \chi_3^2)}{2} \sum_{l=2}^{\infty} \frac{r^{l-1}}{(l-1)!} \sum_{k=1}^{l-1} \binom{l}{k} \chi_{k+1}\chi_{l+1-k} \\ & = & \chi_3\varphi(r) + \frac{(\chi_4 - \chi_3^2)}{2} \sum_{k=1}^{\infty} \frac{r^{k-1}}{k!}\chi_{k+1} \sum_{l=k+1}^{\infty} \frac{l r^{l-k}}{(l-k)!}\chi_{l+1-k} \\ & = & \chi_3\varphi(r) + \frac{(\chi_4 - \chi_3^2)}{2} \sum_{k=1}^{\infty} \frac{r^{k-1}}{k!}\chi_{k+1} \sum_{m=1}^{\infty} \frac{(k+m) r^m}{m!}\chi_{m+1} \\ & = & \chi_3\varphi(r) + \frac{(\chi_4 - \chi_3^2)}{2} \sum_{k=1}^{\infty} \frac{r^{k-1}}{k!}\chi_{k+1}(k \sum_{m=1}^{\infty} \frac{r^m}{m!}\chi_{m+1} + \sum_{m=1}^{\infty} \frac{r^m}{(m-1)!}\chi_{m+1}) \\ & = & \chi_3\varphi(r) + \frac{(\chi_4 - \chi_3^2)}{2} 2\varphi(r) \int_0^r \varphi(x)dx. \end{array}$$

So we have end up with the following differential equation:

$$\psi''(r) - \chi_3\psi'(r) - v\psi'(r)\psi(r) = 0 \tag{29}$$

where we denoted $\psi(r) = \int_0^r \varphi(x)dx = f'(r)/c_2$, $v = (\chi_4 - \chi_3^2)$ with initial conditions $\psi(0) = 0$, $\psi'(0) = 1$. Before solving this equation in full generality let us consider particular cases.

1. Let us assume that $v > 0$ and $\chi_3 = 0$. Our equation now becomes

   $$\psi''(r) - v\psi'(r)\psi(r) = 0,$$

   which leads to $\psi(r) = \sqrt{\frac{2C_1}{v}}\tan(\sqrt{\frac{C_1}{2v}}(r + C_2))$. Taking into account initial conditions we get $\psi(r) = \sqrt{\frac{2}{\chi_4}}\tan(\frac{r}{\sqrt{2\chi_4}})$ and consequently recalling that $\int\tan(ax)dx = -\log\cos(ax)/a$ we get (20).

2. $v = 0$ or $c_4/c_2 = c_3^2/c_2^2$ which means that (recalling interpretation of coefficients $c_n$ presented in Remark 6) variance of the Lévy measure of our process is equal to zero consequently that Lévy is degenerated and concentrated at the point $c_3/c_2$. In this case equation (29) is reduced to the following: $\psi''(r) - \chi_3\psi'(r) = 0$ which gives (after taking into account initial conditions) $\psi(r) = \exp(r\chi_3)/\chi_3$. Hence we get the assertion.

3. If $v = \chi_3^2/2$ then one can easily check that the following function:

   $$\psi(x) = \frac{x}{1 - \chi_3 x/2}$$

   satisfies conditions $\psi(0) = 0$ and $\psi'(0) = 1$ and moreover satisfies differential equation (29) with $v = \chi_3^2/2$. Hence $f(x) = c_1 x + \frac{-2xc_2}{c_3} - 4c_2^2\ln(1 - xc_3/(2c_2))/c_3^2$

4. If $2v > \chi_3^2$, then by solving (29) and then imposing initial conditions we get

   $$\psi(x) = \frac{2\sin r\alpha}{2\alpha\cos r\alpha - \chi_3\sin r\alpha},$$

   where we denoted $\alpha = \frac{1}{2}\sqrt{2v - \chi_3^2} = \frac{1}{2}\sqrt{2\chi_4 - 3\chi_3^2}$. Thus $f(x) = x(c_1 - \frac{\chi_3}{2\alpha^2+\chi_3^2/2}) + \frac{1}{v}(2\,\mathrm{arctanh}(\frac{\chi_3}{2\alpha}\tan(x\alpha)) - \ln(2\alpha^2 - \chi_3^2/2 + (2\alpha^2 + \chi_3^2/2)\cos 2x\alpha)$. Now recall that $\mathrm{arctanh}\, x = \frac{1}{2}\ln\frac{1+x}{1-x}$ and we get (22) and (23)

5. If $2v < \chi_3^2$, then we argue in the same way as in the previous case but in this case parameter $\alpha$ is imaginary and we get hyperbolic functions.

■

# References


[1] Bryc, Włodzimierz, Wesołowski Jacek, Stitching pairs of Lévy processes into harnesses, *Stochastic Processes and their Applications*, **122**(2012), str. 2854 - 2869

[2] Bożejko, M. Bryc, W. (2006), On a Class of Free Levy Laws Related to a Regression Problem, *Journal of Functional Analysis* **236** , 59-77.

[3] Wlodzimierz Bryc, Wojciech Matysiak, Jacek Wesolowski. Quadratic Harnesses, $q-$commutations, and orthogonal martingale polynomials. *Trans. Amer. Math. Soc.* **359** (2007), no. 11, 5449–5483

[4] Chaumont, L.; Yor, M. Exercises in probability. A guided tour from measure theory to random processes, via conditioning. Cambridge Series in Statistical and Probabilistic Mathematics, 13. *Cambridge University Press*, Cambridge, 2003. xvi+236 pp. ISBN: 0-521-82585-7 MR2016344 (2004m:60001)

[5] Gniedenko, B. W. i Kolmogorov, A. N. Rozkłady graniczne sum zmiennych losowych niezależnych, *PWN*, Warszawa, 1957 (in Polish)

[6] Jacod, Jean; Protter, Philip. Time reversal on Lévy processes. *Ann. Probab.* 16 (1988), no. 2, 620–641. MR0929066 (89j:60102)

[7] Lin, T. F. Multiple integrals of a homogeneous process with independent increments. *Ann. Probab.* **9** (1981), no. 3, 529–532. MR0614639 (82h:60141)

[8] Sato, Ken-iti. Lévy processes and infinitely divisible distributions. Translated from the 1990 Japanese original. Revised by the author. *Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press,* Cambridge, 1999. xii+486 pp. ISBN: 0-521-55302-4 MR1739520 (2003b:60064)

[9] Segall, Adrian; Kailath, Thomas. Orthogonal functionals of independent-increment processes. *IEEE Trans. Information Theory* IT-**22** (1976), no. 3, 287–298. MR0413257 (54 #1374)

[10] Solé, Josep Lluís; Utzet, Frederic. On the orthogonal polynomials associated with a Lévy process. *Ann. Probab.* **36** (2008), no. 2, 765–795. MR2393997 (2009d:60145)

[11] Solé, Josep Lluís; Utzet, Frederic. Time-space harmonic polynomials relative to a Lévy process. *Bernoulli* **14** (2008), no. 1, 1–13. MR2401651 (2009b:60147)

[12] Solé, Josep Lluís; Utzet, Frederic. A family of martingales generated by a process with independent increments. *Theory Stoch. Process.* **14** (2008), no. 2, 139–144. MR2479740 (2010b:60142)

[13] Schoutens, Wim; Teugels, Jozef L. Lévy processes, polynomials and martingales. Special issue in honor of Marcel F. Neuts. *Comm. Statist. Stochastic Models* **14** (1998), no. 1-2, 335–349. MR1617536 (99k:60118)

[14] Schoutens, Wim. Stochastic processes and orthogonal polynomials. *Lecture Notes in Statistics*, **146**. Springer-Verlag, New York, 2000. xiv+163 pp. ISBN: 0-387-95015-X MR1761401 (2001f:60095)

[15] Paweł J. Szabłowski, On Markov processes with polynomials conditional moments, Trans of AMS in print, http://arxiv.org/abs/1210.6055

[16] Szabłowski, Paweł J. $q-$Wiener and $(\alpha,q)-$ Ornstein–Uhlenbeck processes. A generalization of known processes, *Theory of Probability and Its Applications,* **56** (4), 2011, 742–772, http://arxiv.org/abs/math/0507303

[17] Simon, Barry. The classical moment problem as a self-adjoint finite difference operator. *Adv. Math.* **137** (1998), no. 1, 82–203. MR1627806 (2001e:47020)

[18] Wesołowski, Jacek. A martingale characterization of the Wiener process. *Statist. Probab. Lett.* **10** (1990), no. 3, 213–215. MR1072511 (91i:60211)

[19] Wesołowski, Jacek. A martingale characterization of the Poisson process. *Bull. Polish Acad. Sci. Math.* **38** (1990), no. 1-12, 49–53. MR1194245 (93g:60094)

[20] Yablonski, Aleh. The calculus of variations for processes with independent increments. *Rocky Mountain J. Math.* **38** (2008), no. 2, 669–701. MR2401574 (2009c:60143)

[21] Mansuy, Roger; Yor, Marc. Harnesses, Lévy bridges and Monsieur Jourdain. *Stochastic Process. Appl.* 115 (2005), no. 2, 329–338. MR2111197 (2005m:60104)

**Received: December 12, 2017**