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Low Dimensional n-Lie Algebras

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Abstract

This paper considers structures of low dimensional n-Lie algebras over a field of characteristic 2. It first refined the classification of $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 2. And then it provided an isomorphic criterion theorem for $(n+2)$-dimensional n-Lie algebras.

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1 Introduction

The notion of n-Lie algebras appeared in three different contexts. The $n = 3$ case first appeared in Nambu's work [1] on simultaneous classical dynamics of three particles. There, Nambu introduced 3-ary multilinear operations and extended the Poisson bracket to the 3-ary bracket. In a mathematical development, Filippov [2] formulated the theory of n-Lie algebras based on the $(2n-1)$-fold Jacobi type identity and classified $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. In 2004, metric n-Lie algebras were obtained in the study of Plücker-type relations for orthogonal planes [3].

In recent years, n-Lie algebras have attracted much attention due to their important applications in various areas such as string and membrane theories. For instance, Bagger and Lambert [4] and Gustavsson [5] proposed a field theory model for multiple M2-branes (BLG model) based on the metric 3-Lie algebras. More applications of n-Lie algebras in string and membrane theories can be found in [6]-[7].

It is known that up to isomorphisms there is a unique simple finite dimensional n-Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero [8], which is $(n+1)$-dimensional. In the paper [9], it was shown that there exist $[\frac{n}{2}] + 1$ classes simple $(n+1)$-dimensional n-Lie algebras over a complete field of characteristic 2, and there are no simple $(n+2)$-dimensional n-Lie algebras. Therefore, A glaring discrepancy has emerged between the structures of n-Lie algebras over the field of characteristic zero and the field of characteristic 2. So it is significant for studying structures of $(n+1)$ and $(n+2)$-dimensional n-Lie algebras. In this paper we first refine the classification of $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 2 given in [9], and then prove the isomorphic criterion theorem for $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 2.

A vector space A over an algebraically field F of characteristic 2 is *an n-Lie algebra* if there is an n-ary multi-linear operation $[\ ,\cdots,\ ]$ satisfying the following identities

$$[x_1,\cdots,x_n]=[x_{\sigma(1)},\cdots,x_{\sigma(n)}],$$

$$[x_1,\cdots,x_i,\cdots,x_j,\cdots,x_n]=0 \text{ if } x_i=x_j \text{ for some } i\neq j, \tag{1.1}$$

$$[[x_1,\ \cdots,\ x_n],\ y_2,\ \cdots,\ y_n]=\sum_{i=1}^{n}[x_1,\ \cdots,\ [x_i,\ y_2,\ \cdots,\ y_n],\ \cdots,\ x_n], \tag{1.2}$$

where σ runs over the symmetric group S_n.

Denote by $[A_1,\ A_2,\ \cdots,\ A_n]$ the subspace of A generated by all vectors $[x_1,\ x_2,\ \cdots,x_n]$, where $x_i \in A_i$, for $i=1,2,\cdots,n$. The subalgebra $A^1=[A,A,\cdots,A]$ is called *the derived algebra* of A. If $A^1=0$, then A is an abelian n-Lie algebra.

In the following, we suppose that F is an algebraically closed field of characteristic 2.

2 Structures of n-Lie Algebras

Theorem 2.1. Let A be an $(n+1)$-dimensional n-Lie algebra over F and $e_1,e_2,\cdots,e_{n+1}$ be a basis of A, then one and only one of the following possibilities holds up to isomorphisms: (a) If $\dim A^1=0$, A is abelian.
(b) If $\dim A^1=1$, $(b_1)[e_2,\cdots,e_{n+1}]=e_1,(b_2)[e_1,\cdots,e_n]=e_1$.
(c) If $\dim A^1=2$, $\beta\in F,\beta\neq 0$,

$$(c_1)\begin{cases}[e_1,e_3,\cdots,e_{n+1}]=e_2,\\ [e_2,\cdots,e_{n+1}]=e_1;\end{cases}\qquad (c_2)\begin{cases}[e_1,e_3,\cdots,e_{n+1}]=e_2,\\ [e_2,\cdots,e_{n+1}]=e_1+\beta e_2;\end{cases}$$

(d) If $\dim A^1 = r \geq 3$,

$$(d_1)\begin{cases} [\hat{e}_1, e_2, \cdots, e_{n+1}] = e_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [e_1, \cdots, \hat{e}_p, \cdots, e_{n+1}] = e_p, \\ [e_1, \cdots, \hat{e}_{p+1}, \cdots, e_{n+1}] = e_r, \\ [e_1, \cdots, \hat{e}_{p+2}, \cdots, e_{n+1}] = e_{r-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [e_1, \cdots, \hat{e}_{p+q}, \cdots, e_{n+1}] = e_{p+1}; \end{cases} \qquad (d_2)\begin{cases} [\hat{e}_1, e_2, \cdots, e_{n+1}] = e_1, \\ [e_1, \hat{e}_2, \cdots, e_{n+1}] = e_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [e_1, \cdots, \hat{e}_{r-1}, \cdots, e_{n+1}] = e_{r-1}, \\ [e_1, \cdots, \hat{e}_r, \cdots, e_{n+1}] = e_r; \end{cases}$$

where q is even, $p+q = r$ and $0 < q \leq r$, the symbol $\hat{e}_i$ means that e_i is omitted in the bracket.

Proof. We only need to prove the case (c) since the other cases are proved in [9]. By Theorem 2.1 in [9], when $\dim A^1 = 2$, the multiplication table of A in the basis $e_1, \cdots, e_{n+1}$ is given by $(c_1)'$ and (c_2), where

$$(c_1)'\begin{cases} [e_1, e_3, \cdots, e_{n+1}] = \alpha e_2, \\ [e_2, \cdots, e_{n+1}] = e_1; \end{cases} \qquad \alpha \in F,\ \alpha \neq 0.$$

Replacing e_2 and e_{n+1} by $\sqrt{\alpha} e_2$ and $\frac{1}{\sqrt{\alpha}} e_{n+1}$ in $(c_1)'$ respectively, we get that $(c_1)'$ is isomorphic to $(c_1)\begin{cases} [e_1, e_3, \cdots, e_{n+1}] = e_2, \\ [e_2, \cdots, e_{n+1}] = e_1. \end{cases}$

We take a linear transformation of the basis $e_1, \cdots, e_{n+1}$ by replacing e_1, e_2 and e_{n+1} by $e_1 + a e_2$, $e_1 + \frac{1}{a} e_2$ and $\frac{1}{a} e_{n+1}$, respectively, then (c_2) is isomorphic to $(c_2)'\begin{cases} [e_1, e_3, \cdots, e_{n+1}] = e_1, \\ [e_2, \cdots, e_{n+1}] = \frac{1}{a^2} e_2; \end{cases}$ where $a \in F$, $a + \frac{1}{a} = \beta$. And the structure of A is completely determined by the action of $\mathrm{ad}(e_3, \cdots, e_{n+1})$ on A^1. From $(c_2)'$, the n-Lie algebras related to the case (c_2) with nonzero coefficients β and β' are isomorphic if and only if there exists a nonzero element $s \in F$ and a nonsingular (2×2) matrix B such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix} = s B^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{a_1^2} \end{pmatrix} B,$$

where $a + \frac{1}{a} = \beta$ and $a_1 + \frac{1}{a_1} = \beta'$. This implies that the n-Lie algebras corresponding to the case (c_2) with nonzero coefficients β and β' are isomorphic if and only if $a = a_1$ and $a = \frac{1}{a_1}$, that is, $\beta = \beta'$ (it is clear that $\beta = \beta'$ if and only if $a = a_1$ and $a = \frac{1}{a_1}$). The result follows.

Suppose $(A, [,\cdots,]_1)$ and $(A, [,\cdots,]_2)$ are n-Lie algebras with two n-ary Lie products $[,\cdots,]_1$ and $[,\cdots,]_2$ on a vector space A and $e_1, e_2, \cdots, e_{n+2}$ be a basis of A. For $1 \leq i < j \leq n+2$, set

$$e_{i,j} = [e_1, \cdots, \hat{e}_i, \cdots, \hat{e}_j, \cdots, e_{n+2}]_1 = \sum_{k=1}^{n+2} b_{i,j}^k e_k,\ b_{i,j}^k \in F, \tag{2.1}$$

$$B=\begin{pmatrix} b_{1,2}^{1} & b_{1,3}^{1} & \cdots & b_{1,n+2}^{1} & b_{2,3}^{1} & \cdots & b_{n+1,n+2}^{1} \\ b_{1,2}^{2} & b_{1,3}^{2} & \cdots & b_{1,n+2}^{2} & b_{2,3}^{2} & \cdots & b_{n+1,n+2}^{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{1,2}^{n+2} & b_{1,3}^{n+2} & \cdots & b_{1,n+2}^{n+2} & b_{2,3}^{n+2} & \cdots & b_{n+1,n+2}^{n+2} \end{pmatrix},$$

then $(e_{1,2},e_{1,3},\cdots,e_{1,n+2},e_{2,3},\cdots,e_{2,n+2},\cdots,e_{n+1,n+2})=(e_1,e_2,\cdots,e_{n+2})B$. The multiplication of $(A,[,\cdots,]_1)$ is determined by $((n+2)\times\frac{(n+1)(n+2)}{2})$ matrix B. And B is called the structure matrix of $(A,[,\cdots,]_1)$ with respect to the basis $e_1,\ e_2,\ \cdots,\ e_{n+2}$.

Similarly, denote $\bar{B}$ is the structure matrix of $(A,[,\cdots,]_2)$ with respect to the basis $e_1,\ e_2,\ \cdots,\ e_{n+2}$, that is,

$$\bar{e}_{ij}=[e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+2}]_2=\sum_{k=1}^{n+2}\bar{b}_{i,j}^{k}e_k,\ \ \bar{b}_{i,j}^{k}\in F, \tag{2.2}$$

$$(\bar{e}_{1,2},\bar{e}_{1,3},\cdots,\bar{e}_{1,n+2},\bar{e}_{2,3},\cdots,\bar{e}_{2,n+2},\cdots,\bar{e}_{n+1,n+2})=(e_1,\cdots,e_{n+2})\bar{B}.$$

By the above notations we have following criterion theorem.

Theorem 2.2. The n-Lie algebras $(A,[,\cdots,]_1)$ and $(A,[,\cdots,]_2)$ with products (2.1) and (2.2) on an $(n+2)$-dimensional linear space A are isomorphic if and only if there exists a nonsingular $((n+2)\times(n+2))$ matrix $T=(t_{i,j})$ such that

$$B=T^{-1}\bar{B}T_*, \tag{2.3}$$

where $T_*=(T_{k,l}^{i,j})$ is an $(\frac{(n+1)(n+2)}{2}\times\frac{(n+1)(n+2)}{2})$ matrix, and $T_{k,l}^{i,j}\in F$ is the determinant defined by (2.5) below for $1\le i,j,k,l\le n+2$.

Proof. If n-Lie algebra $(A,[,\cdots,]_1)$ is isomorphic to $(A,[,\cdots,]_2)$ under an isomorphism σ. Let $e_1,\cdots,e_{n+2}$ be a basis of A, and structure matrices are determined by Eqs (2.1) and (2.2) with respect to the basis $e_1,\cdots,e_{n+2}$ respectively, that is,

$$e_{i,j}=[e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+2}]_1=\sum_{k=1}^{n+2}b_{i,j}^{k}e_k,\ \ B=(b_{i,j}^{k})_{(n+2)\times\frac{(n+2)\times(n+2)}{2}};$$

$$\bar{e}_{i,j}=[e_1,\cdots,\hat{e}_i,\cdots,\hat{e}_j,\cdots,e_{n+2}]_2=\sum_{k=1}^{n+2}\bar{b}_{i,j}^{k}e_k,\ \ \ \bar{B}=(\bar{b}_{i,j}^{k})_{(n+2)\times\frac{(n+2)\times(n+2)}{2}}.$$

Denote $e_i'=\sigma(e_i),\ 1\le i\le n+2$ and the nonsingular $((n+2)\times(n+2))$ matrix $T=(t_{ij})$ is the transition matrix of σ with respect to the basis $e_1,\ e_2,\ \cdots,\ e_{n+2}$, that is,

$$(\sigma(e_1),\cdots,\sigma(e_{n+2}))=(e_1',\cdots,e_{n+2}')=(e_1,e_2,\cdots,e_{n+2})T. \tag{2.4}$$

$$e_{k,l}'=[e_1',\ \cdots,\hat{e'}_k,\ \cdots,\ \hat{e'}_l,\ \cdots,\ e_{n+2}']_2=[\sum_{m=1}^{n+2}t_{m,1}e_m,\sum_{m=1}^{n+2}t_{m,2}e_m,\cdots,$$

$$\sum_{m=1}^{n+2} t_{m,k-1}e_m, \sum_{m=1}^{n+2} t_{m,k+1}e_m, \cdots, \sum_{m=1}^{n+2} t_{m,l-1}e_m, \sum_{m=1}^{n+2} t_{m,l+1}e_m, \ \cdots, \sum_{m=1}^{n+2} t_{m,n+2}e_m]_2$$

$$= T_{k,l}^{1,2}\bar{e}_{1,2} + T_{k,l}^{1,3}\bar{e}_{1,3} + \cdots + T_{k,l}^{1,n+2}\bar{e}_{1,n+2} + T_{k,l}^{2,3}\bar{e}_{2,3} + \cdots + T_{k,l}^{n+1,n+2}\bar{e}_{n+1,n+2},$$

where $T_{k,l}^{i,j}$ is the determinant of the $(n \times n)$-order matrix $W_{k,l}^{i,j}$, and

$$W_{k,l}^{i,j} = \begin{pmatrix} t_{1,1} & \cdot & t_{1,k-1} & t_{1,k+1} & \cdot & t_{1,l-1} & t_{1,l+1} & \cdot & t_{1,n+2} \\ t_{2,1} & \cdot & t_{2,k-1} & t_{2,k+1} & \cdot & t_{2,l-1} & t_{2,l+1} & \cdot & t_{2,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{i-1,1} & \cdot & t_{i-1,k-1} & t_{i-1,k+1} & \cdot & t_{i-1,l-1} & t_{i-1,l+1} & \cdot & t_{i-1,n+2} \\ t_{i+1,1} & \cdot & t_{i+1,k-1} & t_{i+1,k+1} & \cdot & t_{i+1,l-1} & t_{i+1,l+1} & \cdot & t_{i+1,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{j-1,1} & \cdot & t_{j-1,k-1} & t_{j-1,k+1} & \cdot & t_{j-1,l-1} & t_{j-1,l+1} & \cdot & t_{j-1,n+2} \\ t_{j+1,1} & \cdot & t_{j+1,k-1} & t_{j+1,k+1} & \cdot & t_{j+1,l-1} & t_{j+1,l+1} & \cdot & t_{j+1,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n+1,1} & \cdot & t_{n+1,k-1} & t_{n+1,k+1} & \cdot & t_{n+1,l-1} & t_{n+1,l+1} & \cdot & t_{n+1,n+2} \\ t_{n+2,1} & \cdot & t_{n+2,k-1} & t_{n+2,k+1} & \cdot & t_{n+2,l-1} & t_{n+2,l+1} & \cdot & t_{n+2,n+2} \end{pmatrix} \quad (2.5)$$

where $1 \leq i < j \leq n+2, \ 1 \leq k \neq l \leq n+2$. Denote

$$T_* = \begin{pmatrix} T_{1,2}^{1,2} & T_{1,3}^{1,2} & \cdot & T_{1,n+2}^{1,2} & T_{2,3}^{1,2} & \cdot & T_{n+1,n+2}^{1,2} \\ T_{1,2}^{1,3} & T_{1,3}^{1,3} & \cdot & T_{1,n+2}^{1,3} & T_{2,3}^{1,3} & \cdot & T_{n+1,n+2}^{1,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{1,2}^{n,n+2} & T_{1,3}^{n,n+2} & \cdot & T_{1,n+2}^{n,n+2} & T_{2,3}^{n,n+2} & \cdot & T_{n+1,n+2}^{n,n+2} \\ T_{1,2}^{n+1,n+2} & T_{1,3}^{n+1,n+2} & \cdot & T_{1,n+2}^{n+1,n+2} & T_{2,3}^{n+1,n+2} & \cdot & T_{n+1,n+2}^{n+1,n+2} \end{pmatrix}, \quad (2.6)$$

then T_* is a $(\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2})$ matrix, and

$$(e'_{1,2}, e'_{1,3}, \cdots, e'_{1,n+2}, e'_{2,3}, \cdots, e'_{n+1,n+2})$$
$$= (\bar{e}_{1,2}, \bar{e}_{1,3}, \cdots, \bar{e}_{1,n+2}, \bar{e}_{2,3}, \cdots, \bar{e}_{n+1,n+2})T_*.$$

From identities (2.1) and (2.2)

$$(e'_{1,2}, e'_{1,3}, \cdots, e'_{1,n+2}, e'_{2,3}, \cdots, e'_{n+1,n+2}) = (e_1, e_2, \cdots, e_{n+2})\bar{B}T_*. \quad (2.7)$$

$$e'_{k,l} = [e'_1, \ \cdots, \hat{e}'_k, \ \cdots, \ \hat{e}'_l, \ \cdots, \ e'_{n+2}]_2 = [\sigma(e_1), \cdots, \widehat{\sigma(e_k)}, \ \cdots, \widehat{\sigma(e_l)}, \ \cdots,$$

$$\sigma(e_{n+2})]_2 = \sigma([e_1, \ \cdots, \hat{e}_k, \ \cdots, \ \hat{e}_l, \ \cdots, \ e_{n+2}]_1) = \sigma(e_{kl}) = \sum_{s=1}^{n+2}(\sum_{i=1}^{n+2} b_{kl}^i)t_{si}e_s.$$

$$(e'_{1,2}, e'_{1,3}, \cdots, e'_{1,n+2}, e'_{2,3}, \cdots, e'_{n+1,n+2}) = (e_1, e_2, \cdots, e_{n+2})TB. \quad (2.8)$$

It follows (2.7) and (2.8) that

$$TB = \bar{B}T_*, \text{ that is } B = T^{-1}\bar{B}T_*.$$

On the other hand, we take a linear transformation σ of A, such that $\sigma(e_1, \cdots, e_{n+2}) = (\sigma(e_1), \cdots, \sigma(e_{n+2})) = (e_1, \cdots, e_{n+2})T$. By the similar discussion to above, σ is an n-Lie isomorphism from $(A, [, \cdots,]_1)$ to $(A, [, \cdots,]_2)$. The proof is completed.

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