Lie symmetries of the canonical geodesic equations for four-dimensional Lie groups

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Abstract

For each of the four-dimensional indecomposable Lie algebras the geodesic equations of the associated canonical Lie group connection are given. In each case a basis for the associated Lie algebra of symmetries is constructed and the corresponding Lie brackets are written down.

1 Introduction

In this paper we continue with our investigation of the Lie symmetries of the geodesic system of the natural linear torsion-free connection associated to any Lie group. See [2] [3] [4] [5] and [7] for further details about this connection. In [4] and [7] the main geometrical properties of this connection have been listed and proofs supplied. Indeed [4] was concerned with geodesic systems in dimensions two and three. The present article extends the investigation to indecomposable groups and Lie algebras in dimensions four. Matrix representations for all such groups and algebras may be found in, for example,

in [7]. Here we shall be content simply to supply the brackets of each four-dimensional Lie algebra.

The notation for the Lie groups in dimension four and their associated Lie algebras is taken from [6]. However, in the interests of efficiency, we prefer to consolidate cases $4.8, 4.9$ and $4.10, 4.11$ into single cases and hence we discern ten classes of Lie algebra in dimension four. An important point to note is that several classes of Lie algebra in dimension four depend on one or two parameters. The symmetry may be broken in the sense that for exceptional values of the parameters, the symmetry algebra may have a higher dimension.

We use $\mathbb{R}^m \rtimes \mathbb{R}^n$ to denote a semi-direct product of abelian Lie algebras in which $\mathbb{R}^m$ is a subalgebra and $\mathbb{R}^n$ an ideal. We will also use H to stand for the three-dimensional Heisenberg Lie algebra.

2   Four-dimensional canonical geodesic systems and their symmetry algebras

$A4.1 : [e_2, e_4] = e_1, [e_3, e_4] = e_2$

Geodesics:

$$\ddot{y} = \dot{z}\dot{w}$$
$$\ddot{z} = 0 \tag{1}$$
$$\ddot{w} = 0$$

Symmetries and Lie algebra:

$e_1 = D_t, \ e_2 = tD_x, \ e_3 = D_x, \ e_4 = D_y, e_5 = D_z, \ e_6 = D_w, \ e_7 = zD_x,$
$e_8 = wD_x, \ e_9 = wD_x, \ e_{10} = wD_t, \ e_{11} = zD_t, \ e_{12} = yD_x + zD_y,$
$e_{13} = \frac{w^2}{2}D_x + wD_y, \ e_{14} = wzD_x + 2zD_y,$
$e_{15} = \frac{w^3}{6}D_x + \frac{w^2}{2}D_y + wD_z,$
$e_{16} = \frac{t}{2}D_t + (\frac{w^2z}{8} - \frac{yw}{4} + x)D_x + (\frac{wz}{4} + \frac{y}{2})D_y + zD_z,$
$e_{17} = \frac{t}{2}D_t + (-\frac{w^2z}{4} + \frac{yw}{2} + 2x)D_x + (\frac{wz}{4} + \frac{y}{2})D_y + wD_w,$
$e_{18} = tD_t + (\frac{yw}{2} - \frac{w^2z}{4})D_x + (y - \frac{wz}{2})D_y,$
$e_{19} = (\frac{wz}{2} - y)D_t \ e_{19} = \frac{tw}{2}D_x + tD_y,$
$e_{20} = tD_t + (\frac{w^2z}{4} - \frac{yz}{2})D_x + (\frac{wz}{2} - y)D_y$

$[e_1, e_4] = e_5, [e_1, e_{15}] = \frac{1}{2}e_1, [e_1, e_{16}] = \frac{1}{2}e_1, [e_1, e_{17}] = e_1, [e_1, e_{19}] = e_6 + \frac{1}{2}e_8,$
$[e_1, e_{20}] = e_1, [e_2, e_7] = e_5, [e_2, e_{10}] = e_1, [e_2, e_{11}] = e_6, [e_2, e_{13}] = e_8 + 2e_6, [e_2, e_{15}]$
$= e_2 + \frac{1}{4}e_{12}, [e_2, e_{16}] = \frac{1}{4}e_{12}, [e_2, e_{17}] = -\frac{1}{2}e_{12}, [e_2, e_{18}] = \frac{1}{2}e_9, [e_2, e_{20}] = \frac{1}{2}e_{12},$
$[e_3, e_8] = e_5, [e_3, e_9] = e_1, [e_3, e_{12}] = e_6 + e_8, [e_3, e_{13}] = e_7, [e_3, e_{14}] = e_2 + e_{12}, [e_3, e_{15}] = \frac{1}{4}e_{13} - \frac{1}{4}e_{11}, [e_3, e_{16}] = \frac{1}{4}e_{13} + e_3 - \frac{1}{4}e_{11}, [e_3, e_{17}] = -\frac{1}{2}e_{13} + \frac{1}{2}e_{11}, [e_3, e_{18}] = \frac{1}{2}e_{10}, [e_3, e_{19}] = \frac{1}{2}e_4, [e_3, e_{20}] = \frac{1}{2}e_{13} - \frac{1}{2}e_{11}, [e_4, e_9] = -e_8, [e_4, e_{10}] =$

$-e_7, [e_4, e_{15}] = \frac{1}{2}e_4, [e_4, e_{16}] = \frac{3}{2}e_4, [e_4, e_{17}] = -e_4, [e_4, e_{18}] = e_{11} - \frac{1}{2}e_{13}, [e_4, e_{20}] =$
$-e_4, [e_5, e_{15}] = e_5, [e_5, e_{16}] = 2e_5, [e_6, e_{11}] = e_5, [e_6, e_{15}] = -\frac{1}{4}e_8 + \frac{1}{2}e_6, [e_6, e_{16}] =$
$-\frac{1}{4}e_8 + \frac{1}{2}e_6, [e_6, e_{17}] = e_6 + \frac{1}{2}e_8, [e_6, e_{18}] = -e_1, [e_6, e_{20}] = -\frac{1}{2}e_8 - e_6, [e_7, e_{14}] =$
$-e_8, [e_7, e_{16}] = 2e_7, [e_8, e_{15}] = e_8, [e_8, e_{16}] = e_8, [e_9, e_{15}] = \frac{1}{2}e_9, [e_9, e_{16}] = -\frac{1}{2}e_9,$
$[e_9, e_{17}] = e_9, [e_9, e_{19}] = e_{12}, [e_9, e_{20}] = e_9, [e_{10}, e_{14}] = -e_9, [e_{10}, e_{15}] = -\frac{1}{2}e_{10},$
$[e_{10}, e_{16}] = \frac{1}{2}e_{10}, [e_{10}, e_{17}] = e_{10}, [e_{10}, e_{19}] = \frac{1}{2}e_{13}, [e_{10}, e_{20}] = e_{10}, [e_{11}, e_{12}] =$
$-e_8, [e_{11}, e_{13}] = -2e_7, [e_{11}, e_{14}] = -e_{12}, [e_{11}, e_{15}] = -\frac{1}{2}e_{13} + \frac{1}{2}e_{11}, [e_{11}, e_{16}] =$
$-\frac{1}{2}e_{13} + \frac{3}{2}e_{11}, [e_{11}, e_{17}] = e_{13} - e_{11}, [e_{11}, e_{18}] = -e_{10}, [e_{11}, e_{19}] = -e_4, [e_{11}, e_{20}] =$
$e_{11} - e_{13}, [e_{12}, e_{15}] = \frac{1}{2}e_{12}, [e_{12}, e_{16}] = -\frac{1}{2}e_{12}, [e_{12}, e_{17}] = e_{12}, [e_{12}, e_{18}] = -e_9,$
$[e_{12}, e_{20}] = -e_{12}, [e_{13}, e_{14}] = -2e_{12}, [e_{13}, e_{15}] = -\frac{1}{2}e_{13}, [e_{13}, e_{16}] = \frac{1}{2}e_{13}, [e_{13}, e_{17}] =$
$e_{13}, [e_{13}, e_{18}] = -2e_{10}, [e_{13}, e_{20}] = -e_{13}, [e_{14}, e_{15}] = e_{14}, [e_{14}, e_{16}] = -e_{14}, [e_{18}, e_{19}] =$
$e_{20}, [e_{18}, e_{20}] = 2e_{18}, [e_{19}, e_{20}] = -2e_{19}.$

The Levi decomposition $\mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes N_{14})$ is a semi-direct product of $\mathfrak{sl}(2, \mathbb{R})$
spanned by e_{18}, e_{19}, e_{20} and a 17 dimensional solvable algebra, which itself is a
semi direct product of abelian $\mathbb{R}^3$, spanned by e_{15}, e_{16}, e_{17} and a 14 dimensional
nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8,$
$e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$, respectively.

$A4.2a \neq 0 : [e_1, e_4] = ae_1, [e_2, e_4] = e_2[e_3, e_4] = e_2 + e_3$

Geodesics:

$$\ddot{x} = a\dot{x}\dot{w}$$
$$\ddot{y} = (\dot{y} + \dot{z})\dot{w}$$
$$\ddot{z} = \dot{z}\dot{w} \tag{2}$$
$$\ddot{w} = 0$$

Symmetries and Lie algebra $a \neq \pm 1$:

$e_1 = -zD_y, \ e_2 = D_z, \ e_3 = e^w(D_z + wD_y), e_4 = D_y, \ , e_5 = e^w D_y, \ e_6 = wD_t,$
$e_7 = D_t, \ e_8 = D_x, \ e_9 = e^{aw}D_x, \ e_{10} = yD_y + zD_z, \ e_{11} = -D_w, \ e_{12} = tD_t,$
$e_{13} = xD_x.$

$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_{10}] = e_2, [e_3, e_{10}] = e_3, [e_3, e_{11}] = e_3 + e_5, [e_4, e_{10}] =$
$e_4, [e_5, e_{10}] = e_5, [e_5, e_{11}] = e_5, [e_6, e_{11}] = e_7, [e_6, e_{12}] = e_6, [e_7, e_{12}] = e_7, [e_8, e_{13}] =$
$e_8, [e_9, e_{11}] = ae_9, [e_9, e_{13}] = e_9.$

The Lie algebra $\mathbb{R}^4 \ltimes (\mathbb{R}^2 \oplus H_5)$ is 13-dimensional solvable. It has a 9-
dimensional non-abelian nilradical $H_5 \oplus \mathbb{R}^4$. Here H_5 denotes the 5-dimensional
Heisenberg algebra and is spanned by e_1, e_2, e_3, e_4, e_5 and the $\mathbb{R}^4$ summand is
spanned by e_6, e_7, e_8, e_9. The 4-dimensional abelian complement to $H_5 \oplus \mathbb{R}^4$ is
spanned by $e_{10}, e_{11}, e_{12}, e_{13}$.

Symmetries $a = \pm 1$:

$$e_1 = D_x, \quad e_2 = e^{aw}D_x, \quad e_3 = zD_y, \quad e_4 = D_z, e_5 = e^w(D_z + wD_y), \quad e_6 = D_y,$$
$$e_7 = e^wD_y, \quad e_8 = xe^{\frac{(1-a)w}{2}}D_y, \quad e_9 = ze^{\frac{(a-1)w}{2}}D_x, \quad e_{10} = wD_t,$$
$$e_{11} = D_t, \quad e_{12} = yD_y + zD_z, \quad e_{13} = D_w, \quad e_{14} = tD_t, \quad e_{15} = xD_x.$$

Symmetry Lie algebra $a = 1$:

$[e_1, e_8] = e_6, [e_1, e_{15}] = e_1, [e_2, e_8] = e_7, [e_2, e_{13}] = -e_2, [e_2, e_{15}] = e_2, [e_3, e_4] = -e_6, [e_3, e_5] = -e_7, [e_4, e_9] = e_1, [e_4, e_{12}] = e_4, [e_5, e_9] = e_2, [e_5, e_{12}] = e_5, [e_5, e_{13}] = -e_5 - e_7, [e_6, e_{12}] = e_6, [e_7, e_{12}] = e_7, [e_7, e_{13}] = -e_7, [e_8, e_9] = -e_3, [e_8, e_{12}] = e_8, [e_8, e_{15}] = -e_8, [e_9, e_{12}] = -e_9, [e_9, e_{15}] = e_9, [e_{10}, e_{13}] = -e_{11}, [e_{10}, e_{14}] = e_{10}, [e_{11}, e_{14}] = e_{11}.$

Symmetry Lie algebra $a = -1$:

$[e_1, e_8] = e_7, [e_1, e_{15}] = e_1, [e_2, e_8] = e6, [e_2, e_{13}] = e_2, [e_2, e_{15}] = e_2, [e_3, e_4] = -e_6, [e_3, e_5] = -e_7, [e_4, e_9] = e_2, [e_4, e_{12}] = e_4, [e_5, e_9] = e_1, [e_5, e_{12}] = e_5, [e_5, e_{13}] = -e5 - e7, [e_6, e_{12}] = e_6, [e_7, e_{12}] = e_7, [e_7, e_{13}] = -e_7, [e_8, e_9] = -e_3, [e_8, e_{12}] = e_8, [e_8, e_{13}] = -e_8, [e_8, e_{15}] = -e_8, [e_9, e_{12}] = -e_9, [e_9, e_{13}] = e_9, [e_9, e_{15}] = e_9, [e_{10}, e_{13}] = -e_{11}, [e_{10}, e_{14}] = e_{10}, [e_{11}, e_{14}] = e_{11}.$

In each case the Lie algebra $\mathbb{R}^4 \rtimes (N_9 \oplus \mathbb{R}^2)$ is 15-dimensional solvable. It has an 11-dimensional decomposable nilradical, $N_9 \oplus \mathbb{R}^2$, where N_9 is is 9-dimensional indecomposable nilpotent spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and $\mathbb{R}^2$ by e_{10}, e_{11} and a 4-dimensional abelian complement spanned by $e_{12}, e_{13}, e_{14}, e_{15}$. $A4.3 : [e_1, e_4] = e_1, [e_3, e_4] = e_2$

Geodesics:

$$\begin{aligned} \ddot{x} &= \dot{x}\dot{w} \\ \ddot{y} &= \dot{z}\dot{w} \\ \ddot{z} &= 0 \\ \ddot{w} &= 0 \end{aligned} \tag{3}$$

Symmetries and Lie algebra

$$e_1 = wD_t, \quad e_2 = \tfrac{w^2}{2}D_y + wD_z, \quad e_3 = e^wD_x, \quad e_4 = D_y, e_5 = D_x, \quad e_6 = D_z,$$
$$e_7 = D_t, \quad e_8 = wD_y, \quad e_9 = tD_t + yD_y + zD_z, \quad e_{10} = D_w + \tfrac{z}{2}D_y, \quad e_{11} = xD_x,$$
$$e_{12} = (wz - y)D_y + zD_z, \quad e_{13} = \tfrac{1}{3}(2tD_t - yD_y - zD_z), \quad e_{14} = tD_y,$$
$$e_{15} = zD_t, \quad e_{16} = \tfrac{tw}{2}D_y + tD_z, \quad e_{17} = (wz - 2y)D_t,$$
$$e_{18} = zD_y, \quad e_{19} = \left(\tfrac{zw^2}{2} - yw\right)D_y + (wz - 2y)D_z$$

$[e_1, e_9] = e_1, [e_1, e_{10}] = -e_7, [e_1, e_{13}] = \frac{2}{3}e_1, [e_1, e_{14}] = e_8, [e_1, e_{16}] = e_2, [e_2, e_9] = e_2, [e_2, e_{10}] = -\frac{1}{2}e_8 - e_6, [e_2, e_{12}] = e_2, [e_2, e_{13}] = -\frac{1}{3}e_2, [e_2, e_{15}] = e_1, [e_2, e_{18}] = e_8, [e_3, e_{10}] = -e_3, [e_3, e_{11}] = e_3, [e_4, e_9] = e_4, [e_4, e_{12}] = -e_4, [e_4, e_{13}] = -\frac{1}{3}e_4, [e_4, e_{17}] = -2e_7, [e_4, e_{19}] = -e_8 - 2e_6, [e_5, e_{11}] = e_5, [e_6, e_9] = e_6, [e_6, e_{10}] = \frac{1}{2}e_4, [e_6, e_{12}] = e_6 + e_8, [e_6, e_{13}] = -\frac{1}{3}e_6, [e_6, e_{15}] = e_7, [e_6, e_{17}] = e_1, [e_6, e_{18}] = e_4, [e_6, e_{19}] = e_2, [e_7, e_9] = e_7, [e_7, e_{13}] = \frac{2}{3}e_7, [e_7, e_{14}] = e_4, [e_7, e_{16}] = e_6 + \frac{1}{2}e_8, [e_8, e_9] = e_8, [e_8, e_{10}] = -e_4, [e_8, e_{12}] = -e_8, [e_8, e_{13}] = -\frac{1}{3}e_8, [e_8, e_{17}] = -2e_1, [e_8, e_{19}] = -2e_2, [e_{12}, e_{14}] = e_{14}, [e_{12}, e_{15}] = e_{15}, [e_{12}, e_{16}] = -e_{16}, [e_{12}, e_{17}] = -e_{17}, [e_{12}, e_{18}] = 2e_{18}, [e_{12}, e_{19}] = -2e_{19}, [e_{13}, e_{14}] = e_{14}, [e_{13}, e_{15}] = -e_{15}, [e_{13}, e_{16}] = e_{16}, [e_{13}, e_{17}] = -e_{17}, [e_{14}, e_{15}] = -e_{18}, [e_{14}, e_{17}] = -e_{12} - 3e_{13}, [e_{14}, e_{19}] = -2e_{16}, [e_{15}, e_{16}] = -\frac{3}{2}e_{13} + \frac{1}{2}e_{12}, [e_{15}, e_{19}] = -e_{17}, [e_{16}, e_{17}] = -e_{19}, [e_{16}, e_{18}] = e_{14}, [e_{17}, e_{18}] = 2e_{15}, [e_{18}, e_{19}] = -2e_{12}.$

The symmetry algebra is $\mathfrak{sl}(3, \mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes \mathbb{R}^8)$ where $\mathfrak{sl}(3, \mathbb{R})$ is spanned by $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}$, the $\mathbb{R}^3$ factor is spanned by e_9, e_{10}, e_{11} and the nilradical $\mathbb{R}^8$ is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$.

$A4.4 : [e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2 [e_3, e_4] = e_2 + e_3$

Geodesics:

$$\ddot{x} = (\dot{x} + \dot{y})\dot{w}$$
$$\ddot{y} = (\dot{y} + \dot{z})\dot{w}$$
$$\dot{z} = \dot{z}\dot{w}$$
$$\ddot{w} = 0$$

(4)

Symmetries and Lie algebra

$e_1 = D_x, \; e_2 = D_y, \; e_3 = D_z, \; e_4 = zD_x, \; e_5 = yD_x + zD_y, \; e_6 = e^w D_x,$
$e_7 = e^w((w - 1)D_x + D_y), \; e_8 = e^w(\frac{1}{2}(w^2 - 2w + 2)D_x + (w - 1)D_y + D_z),$
$e_9 = D_t, \; e_{10} = wD_t, \; e_{11} = tD_t, \; e_{12} = D_w, \; e_{13} = xD_x + yD_y + zD_z.$

$[e_1, e_{13}] = e_1, [e_2, e_5] = e_1, [e_2, e_{13}] = e_2, [e_3, e_4] = e_1, [e_3, e_5] = e_2, [e_3, e_{13}] = e_3, [e_4, e_8] = -e_6, [e_5, e_7] = -e_6, [e_5, e_8] = -e_7, [e_6, e_{12}] = -e_6, [e_6, e_{13}] = e_6, [e_7, e_{12}] = -e_6 - e_7, [e_7, e_{13}] = e_7, [e_8, e_{12}] = -e_7 - e_8, [e_8, e_{13}] = e_8, [e_9, e_{11}] = e_9, [e_{10}, e_{11}] = e_{10}, [e_{10}, e_{12}] = -e_9.$

The symmetry algebra is a 13-dimensional indecomposable solvable algebra $\mathbb{R}^3 \ltimes (N_8 \oplus \mathbb{R}^2)$. Its nilradical is 10-dimensional decomposable, a direct sum of an 8-dimensional nilpotent N_8 spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and $\mathbb{R}^2$ spanned by e_9, e_{10}. The complement to the nilradical is abelian spanned by e_{11}, e_{12}, e_{13}.

$A4.5ab : [e_1, e_4] = e_1, [e_2, e_4] = ae_2[e_3, e_4] = be_3$

Geodesics:

$$\ddot{x} = \dot{x}\dot{w}$$
$$\ddot{y} = a\dot{y}\dot{w}$$
$$\ddot{z} = b\dot{z}\dot{w} \qquad (5)$$
$$\ddot{w} = 0$$

Generic case: $ab \neq 0, -1 < a < b < 1$

Symmetries and Lie algebra

$e_1 = D_x, \ e_2 = D_y, \ e_3 = D_z, \ e_4 = e^w D_x, e_5 = e^{aw} D_y, e_6 = e^{bw} D_z, \ e_7 = D_t,$

$e_8 = wD_t, \ e_9 = xD_x, \ e_{10} = yD_y e_{11} = zD_z, \ D_{12} = tD_t, \ e_{13} = D_w$

$[e_1, e_9] = e_1, [e_2, e_{10}] = e_2, [e_3, e_{11}] = e_3, [e_4, e_9] = e_4, [e_4, e_{13}] = -e_4, [e_5, e_{10}] = e_5, [e_5, e_{13}] = -ae_5, [e_6, e_{11}] = e_6, [e_6, e_{13}] = -be_6, [e_7, e_{12}] = e_7, [e_8, e_{12}] = e_8, [e_8, e_{13}] = -e_7.$

It is a 13-dimensional indecomposable solvable Lie algebra $\mathbb{R}^8 \rtimes \mathbb{R}^5$. It has an 8-dimensional abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and 5-dimensional abelian complement spanned by $e_9, e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.5ab(a = 1, b = 1)$

Symmetries and Lie algebra

$e_1 = D_x, \ e_2 = D_y, \ e_3 = D_z, \ e_4 = e^w D_x, e_5 = e^w D_y, \ e_6 = e^w D_z, e_7 = D_t,$
$e_8 = wD_t, \ e_9 = xD_x + yD_y + zD_z, \ e_{10} = tD_t, \ e_{11} = D_w, \ e_{12} = yD_x,$
$e_{13} = xD_y, \ e_{14} = zD_y, \ e_{15} = yD_z, \ e_{16} = xD_z, \ e_{17} = zD_x,$
$e_{18} = xD_x - zD_z, \ e_{19} = yD_y - zD_z.$

$[e_1, e_9] = e_1, [e_1, e_{13}] = e_2, [e_1, e_{16}] = e_3, [e_1, e_{18}] = e_1, [e_2, e_9] = e_2, [e_2, e_{12}] = e_1, [e_2, e_{15}] = e_3, [e_2, e_{19}] = e_2, [e_3, e_9] = e_3, [e_3, e_{14}] = e_2, [e_3, e_{17}] = e_1, [e_3, e_{18}] = -e_3, [e_3, e_{19}] = -e_3, [e_4, e_9] = e_4, [e_4, e_{11}] = -e_4, [e_4, e_{13}] = e_5, [e_4, e_{16}] = e_6, [e_4, e_{18}] = e_4, [e_5, e_9] = e_5, [e_5, e_{11}] = -e_5, [e_5, e_{12}] = e_4, [e_5, e_{15}] = e_6, [e_5, e_{19}] = e_5, [e_6, e_9] = e_6, [e_6, e_{11}] = -e_6, [e_6, e_{14}] = e_5, [e_6, e_{17}] = e_4, [e_6, e_{18}] = -e_6, [e_6, e_{19}] = -e_6, [e_7, e_{10}] = e_7, [e_8, e_{10}] = e_8, [e_8, e_{11}] = -e_7, [e_{12}, e_{13}] = e_{19} - e_{18}, [e_{12}, e_{14}] =$

$-e_{17}, [e_{12}, e_{16}] = e_{15}, [e_{12}, e_{18}] = e_{12}, [e_{12}, e_{19}] = -e_{12}, [e_{13}, e_{15}] = e_{16}, [e_{13}, e_{17}] = -e_{14}, [e_{13}, e_{18}] = -e_{13}, [e_{13}, e_{19}] = e_{13}, [e_{14}, e_{15}] = -e_{19}, [e_{14}, e_{16}] = -e_{13}, [e_{14}, e_{18}] = e_{14}, [e_{14}, e_{19}] = 2e_{14}, [e_{15}, e_{17}] = e_{12}, [e_{15}, e_{18}] = -e_{15}, [e_{15}, e_{19}] = -2e_{15}, [e_{16}, e_{17}] = e_{18}, [e_{16}, e_{18}] = -2e_{16}, [e_{16}, e_{19}] = -e_{16}, [e_{17}, e_{18}] = 2e_{17}, [e_{17}, e_{19}] = e_{17}.$

It is a 19-dimensional indecomposable Lie algebra with a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(3, \mathbb{R})$ and is spanned by $e_{12}, e_{13}, e_{14}, e_{15,},$ $e_{16}, e_{17}, e_{18}, e_{19}$. The radical is a semi direct product $\mathbb{R}^8 \rtimes \mathbb{R}^3$ with abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and 3-dimensional abelian complement spanned by e_9, e_{10}, e_{11}.

$A4.5ab(a = 1, b = -1)$

Symmetries and Lie algebra

$e_1 = D_x, \; e_2 = D_y, \; e_3 = D_z, \; e_4 = e^w D_x, e_5 = e^w D_y, \; e_6 = e^{-w} D_z, e_7 = D_t,$
$e_8 = w D_t, \; e_9 = t D_t, \; e_{10} = 3D_w + x D_x + y D_y - 2z D_z, \; e_{11} = x D_x + y D_y + z D_z,$
$e_{12} = z e^w D_x, \; e_{13} = z e^w D_y, \; e_{14} = x e^{-w} D_z, \; e_{15} = y e^{-w} D_y,$
$e_{16} = y D_x, \; e_{17} = x D_y, \; e_{18} = x D_x - z D_z, \; e_{19} = y D_y - z D_z$

$[e_1, e_{10}] = e_1, [e_1, e_{11}] = e_1, [e_1, e_{14}] = e_6, [e_1, e_{17}] = e_2, [e_1, e_{18}] = e_1, [e_2, e_{10}] = e_2, [e_2, e_{11}] = e_2, [e_2, e_{15}] = e_6, [e_2, e_{16}] = e_1, [e_2, e_{19}] = e_2, [e_3, e_{10}] = -2e_3, [e_3, e_{11}] = e_3, [e_3, e_{12}] = e_4, [e_3, e_{13}] = e_5, [e_3, e_{18}] = -e_3, [e_3, e_{19}] = -e_3, [e_4, e_{10}] = -2e_4, [e_4, e_{11}] = e_4, [e_4, e_{14}] = e_3, [e_4, e_{17}] = e_5, [e_4, e_{18}] = e_4, [e_5, e_{10}] = -2e_5, [e_5, e_{11}] = e_5, [e_5, e_{15}] = e_3, [e_5, e_{16}] = e_4, [e_5, e_{19}] = e_5, [e_6, e_{10}] = e_6, [e_6, e_{11}] = e_6, [e_6, e_{12}] = e_1, [e_6, e_{13}] = e_2, [e_6, e_{18}] = -e_6, [e_6, e19] = -e_6, [e_7, e_9] = e_7, [e_8, e_9] = e_8, [e_8, e_{10}] = -3e_7, [e_{12}, e_{14}] = -e_{18}, [e_{12}, e_{15}] = -e_{16}, [e_{12}, e_{17}] = e_{13}, [e_{12}, e_{18}] = 2e_{12}, [e_{12}, e_{19}] = e_{12}, [e_{13}, e_{14}] = -e_{17}, [e_{13}, e_{15}] = -e_{19}, [e_{13}, e_{16}] = e_{12}, [e_{13}, e_{18}] = e_{13}, [e_{13}, e_{19}] = 2e_{13}, [e_{14}, e_{16}] = -e_{15}, [e_{14}, e_{18}] = -2e_{14}, [e_{14}, e_{19}] = -e_{14}, [e_{15}, e_{17}] = -e_{14}, [e_{15}, e_{18}] = -e_{15}, [e_{15}, e_{19}] = -2e_{15}, [e_{16}, e_{17}] = e_{19} - e_{18}, [e_{16}, e_{18}] = e_{16}, [e_{16}, e_{19}] = -e_{16}, [e_{17}, e_{18}] = -e_{17}, [e_{17}, e_{19}] = e_{17}.$

It is a 19-dimensional indecomposable Lie algebra with a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(3, \mathbb{R})$ and is spanned by $e_{12}, e_{13}, e_{14}, e_{15},$ $e_{16}, e_{17}, e_{18}, e_{19}$. The radical is a semi direct product $\mathbb{R}^8 \rtimes \mathbb{R}^3$ with abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and 3-dimensional abelian complement spanned by e_9, e_{10}, e_{11}.

$A4.5ab(a = 1)$

Symmetries and Lie algebra

$e_1 = D_x$, $e_2 = D_y$, $e_3 = z$, $e_4 = e^w D_x$, $e_5 = e^w D_y$, $e_6 = e^{bw} D_z$, $e_7 = D_t$,
$e_8 = w D_t$, $e_9 = D_w$, $e_{10} = x D_x + y D_y$, $e_{11} = z D_z$, $e_{12} = t D_t$,
$e_{13} = x D_x - y D_y$, $e_{14} = y D_x$, $e_{15} = x D_y$.

$[e_1, e_{10}] = e_1, [e_1, e_{13}] = e_1, [e_1, e_{15}] = e_2, [e_2, e_{10}] = e_2, [e_2, e_{13}] = -e_2, [e_2, e_{14}] = e_1, [e_3, e_{11}] = e_3, [e_4, e_9] = -e_4, [e_4, e10] = e_4, [e_4, e_{13}] = e_4, [e_4, e_{15}] = e_5, [e_5, e_9] = -e_5, [e_5, e_{10}] = e_5, [e_5, e_{13}] = -e_5, [e_5, e_{14}] = e_4, [e_6, e_9] = -be_6, [e_6, e_{11}] = e_6, [e_7, e_{12}] = e_7, [e_8, e_9] = -e_7, [e_8, e_{12}] = e_8, [e_{13}, e_{14}] = -2e_{14}, [e_{13}, e_{15}] = 2e_{15}, [e_{14}, e_{15}] = -e_{13}$.

It is a 15-dimensional indecomposable Lie algebra with a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(2, \mathbb{R})$ and is spanned by e_{13}, e_{14}, e_{15}. The radical is a semi direct product $\mathbb{R}^8 \rtimes \mathbb{R}^4$ with abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and 4-dimensional abelian complement spanned by $e_9, e_{10}, e_{11}, e_{12}$.

$A4.5ab(a = -1)$
Symmetries and Lie algebra

$e_1 = D_x$, $e_2 = D_y$, $e_3 = D_z$, $e_4 = e^w D_x$, $e_5 = e^{-w} D_y$, $e_6 = e^{bw} D_z$,
$e_7 = D_t$, $e_8 = t D_t$, $e_9 = w D_t$, $e_{10} = 2 D_w + x D_x - y D_y$, $e_{11} = x D_x + y D_y$,
$e_{12} = z D_z$, $e_{13} = x D_x - y D_y$, $e_{14} = y e^w D_x$, $e_{15} = x e^{-w} D_y$.

$[e_1, e_{10}] = e1, [e_1, e_{11}] = e1, [e_1, e_{13}] = e_1, [e_1, e_{15}] = e_5, [e_2, e_{10}] = -e_2, [e_2, e_{11}] = e_2, [e_2, e_{13}] = -e_2, [e_2, e_{14}] = e_4, [e_3, e_{12}] = e_3, [e_4, e_{10}] = -e_4, [e_4, e_{11}] = e_4, [e_4, e_{13}] = e_4, [e_4, e_{15}] = e_2, [e_5, e_{10}] = e_5, [e_5, e_{11}] = e_5, [e_5, e_{13}] = -e_5, [e_5, e_{14}] = e_1, [e_6, e_{10}] = -2be_6, [e_6, e_{12}] = e_6, [e_7, e_9] = e_7, [e_8, e_9] = e_8, [e_8, e_{10}] = -2e_7, [e_{13}, e_{14}] = -2e_{14}, [e_{13}, e_{15}] = 2e_{15}, [e_{14}, e_{15}] = -e_{13}$.

It is a 15-dimensional indecomposable Lie algebra with a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(2, \mathbb{R})$ and is spanned by e_{13}, e_{14}, e_{15}. The radical is a semi direct product $\mathbb{R}^8 \rtimes \mathbb{R}^4$ with abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and 4-dimensional abelian complement spanned by $e_9, e_{10}, e_{11}, e_{12}$.

$A4.6ab[e_1, e_4] = ae_1, [e_2, e_4] = be_2 - e_3, [e_3, e_4] = e_2 + be_3$

Geodesics:

$$
\begin{aligned}
\ddot{x} &= (b\dot{x} + \dot{y})\dot{w} \\
\ddot{y} &= (b\dot{y} - \dot{x})\dot{w} \\
\ddot{z} &= a\dot{z}\dot{w} \\
\ddot{w} &= 0
\end{aligned}
\tag{6}
$$

Symmetries and Lie algebra

$$e_1 = D_t, \quad e_2 = D_x, \quad e_3 = D_y, \quad e_4 = D_z, e_5 = e^{aw} D_z, \quad e_6 = e^{bw}(\sin w \, D_x + \cos w \, D_y),$$

$$e_7 = e^{bw}(-\cos w \, D_x + \sin w \, D_y), \quad e_8 = w D_t, \quad e_9 = z D_z, \quad e_{10} = -y D_x + x D_y,$$

$$e_{11} = x D_x + y D_y, e_{12} = t D_t, e_{13} = D_w,$$

$[e_1, e_{12}] = e_1, [e_2, e_{10}] = e_3, [e_2, e_{11}] = e_2, [e_3, e_{10}] = -e_2, [e_3, e_{11}] = e_3, [e_4, e_9] = e_4, [e_5, e_9] = e_5, [e_5, e_{13}] = -a e_5, [e_6, e_{10}] = e_7, [e_6, e_{11}] = e_6, [e_6, e_{13}] = -b e_6 + e_7, [e_7, e_{10}] = -e_6, [e_7, e_{11}] = e_7, [e_7, e_{13}] = -b e_7 - e_6, [e_8, e_{12}] = e_8, [e_8, e_{13}] = -e_1.$

The symmetry algebra is a 13-dimensional indecomposable solvable algebra $\mathbb{R}^5 \rtimes \mathbb{R}^8$. Its nilradical is 8-dimensional abelian and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$. The complement to the nilradical is abelian and spanned by $e_9, e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.6ab(b = 0)$

Geodesics:
$$\begin{aligned} \ddot{x} &= \dot{y}\dot{w} \\ \ddot{y} &= -\dot{x}\dot{w} \\ \ddot{z} &= a\dot{z}\dot{w} \\ \ddot{w} &= 0 \end{aligned} \tag{7}$$

Symmetries and Lie algebra

$$e_1 = D_y, \quad e_2 = D_x, \quad e_3 = D_z, \quad e_4 = D_w, \quad e_5 = t D_t, \quad e_6 = D_t,$$

$$e_7 = z D_z, \quad e_8 = w D_t, \quad e_9 = x D_x + y D_y, \quad e_{10} = -y D_x + x D_y, \quad e_{11} = e^{aw} D_z,$$

$$e_{12} = \sin w \, D_x + \cos w \, D_y, \quad e_{13} = -\cos w \, D_x + \sin w \, D_y,$$

$$e_{14} = (y \cos w + x \sin w) D_x + (x \cos w - y \sin w) D_y,$$

$$e_{15} = -(x \cos w - y \sin w) D_x + (y \cos w + x \sin w) D_y$$

$[e_1, e_9] = e_1, [e_1, e_{10}] = -e_2, [e_1, e_{14}] = -e_{13}, [e_1, e_{15}] = e_{12}, [e_2, e_9] = e_2, [e_2, e_{10}] = e_1, [e_2, e_{14}] = e_{12}, [e_2, e_{15}] = e_{13}, [e_3, e_7] = e_3, [e_4, e_8] = e_6, [e_4, e_{11}] = a e_{11}, [e_4, e_{12}] = -e_{13}, [e_4, e_{13}] = e_{12}, [e_4, e_{14}] = -e_{15}, [e_4, e_{15}] = e_{14}, [e_5, e_6] = -e_6, [e_5, e_8] = -e_8, [e_7, e_{11}] = -e_{11}, [e_9, e_{12}] = -e_{12}, [e_9, e_{13}] = -e_{13}, [e_{10}, e_{12}] = -e_{13}, [e_{10}, e_{13}] = e_{12}, [e_{10}, e_{14}] = -2 e_{15}, [e_{10}, e_{15}] = 2 e_{14}, [e_{12}, e_{14}] = e_2, [e_{12}, e_{15}] = e_1, [e_{13}, e_{14}] =$

$-e_1, [e_{13}, e_{15}] = e_2, [e_{14}, e_{15}] = 2e_{10}.$

The symmetry algebra is a 15-dimensional indecomposable algebra that has a non-trivial Levi decomposition. The semi-simple part is $\mathfrak{sl}(2, \mathbb{R})$ and spanned by e_{13}, e_{14}, e_{15}. The nilradical is abelian and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and the complement to the nilradical is abelian and spanned by $e_9, e_{10}, e_{11}, e_{12}$.

A4.7 $[e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3.$

Geodesics:

$$
\begin{aligned}
\ddot{x} &= 2\dot{x}\dot{w} + z\dot{y}\dot{w} - (y + z)\dot{z}\dot{w} \\
\ddot{y} &= \dot{y}\dot{w} + \dot{z}\dot{w} \\
\ddot{z} &= \dot{z}\dot{w} \\
\ddot{w} &= 0
\end{aligned}
\tag{8}
$$

Symmetries and Lie algebra

$$
\begin{aligned}
&e_1 = D_x, \ e_2 = zD_y, \ e_3 = D_y + zD_x, \ e_4 = D_z - yD_x, \ e_5 = e^{2w}D_x, \\
&e_6 = e^w(D_y - zD_x), \ e_7 = e^w(D_z + (w-1)D_y + (y + z - wz)D_x), \ e_8 = D_t, \\
&e_9 = wD_t, \ e_{10} = tD_t, \ e_{11} = D_w, \ e_{12} = 2xD_x + yD_y + zD_z
\end{aligned}
$$

$[e_1, e_{10}] = 2e_1, [e_2, e_3] = -e_3, [e_2, e_5] = -e_5, [e_4, e_5] = e_3, [e_4, e_9] = 2e_9, [e_4, e_{11}] = e_{11}, [e_4, e_{12}] = e_{11} + e_{12}, [e_6, e_8] = -e_7, [e_6, e_{12}] = -e_{11}, [e_7, e_8] = -2e_1, [e_7, e_{10}] = e_7, [e_8, e_{10}] = e_8, [e_9, e_{10}] = 2e_9, [e_{10}, e_{11}] = -e_{11}, [e_{10}, e_{12}] = -e_{12}, [e_{11}, e_{12}] = 2e_9.$

The symmetry algebra is a 12-dimensional indecomposable solvable algebra $\mathbb{R}^3 \ltimes (N_7 \oplus \mathbb{R}^2)$. Its nilradical is 9-dimensional decomposable, a direct sum of a 7-dimensional nilpotent N_7 spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and $\mathbb{R}^2$ spanned by e_8, e_9. The complement to the nilradical is abelian spanned by e_{10}, e_{11}, e_{12}.

A4.8, 4.9b $[e_2, e_3] = e_1, [e_1, e_4] = (b + 1)e_1, [e_2, e_4] = e_2, [e_3, e_4] = be_3.$

Geodesics:

$$
\begin{aligned}
\ddot{x} &= \dot{y}\dot{z} - y\dot{z}\dot{w} + (b + 1)\dot{x}\dot{w} \\
\ddot{y} &= \dot{y}\dot{w} \\
\ddot{z} &= b\dot{z}\dot{w} \\
\ddot{w} &= 0
\end{aligned}
\tag{9}
$$

Symmetries and Lie algebra

$$e_1 = D_x, \quad e_2 = D_z, \quad e_3 = D_y + zD_x, \quad e_4 = e^{(b+1)w}D_x, e_5 = e^w D_y,$$
$$e_6 = e^{-w}(D_z + yD_x), \quad e_7 = tD_x, \quad e_8 = D_t, \quad e_9 = wD_t, \quad e_{10} = wD_x,$$
$$e_{11} = xD_x + yD_y, \quad e_{12} = xD_x + zD_z, e_{13} = D_w.$$

$[e_1, e_{10}] = e_1, [e_1, e_{11}] = e_{11}, [e_2, e_3] = e_1, [e_2, e_{11}] = e_2, [e_3, e_{10}] = e_3, [e_4, e_{10}] = e_4, [e_4, e_{11}] = e_4, [e_4, e_{12}] = -(b+1)e_4, [e_5, e_6] = e_4, [e_5, e_{10}] = e_5, [e_5, e_{12}] = -e_5, [e_6, e_{11}] = e_6, [e_6, e_{12}] = -be_6, [e_7, e_9] = e_7, [e_7, e_{12}] = -e_8, [e_8, e_9] = e_8.$

The algebra is a 12-dimensional indecomposable solvable. The nilradical is 8-dimensional decomposable, spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and isomorphic to $H \oplus H \oplus \mathbb{R}^2$. The complement is 4-dimensional abelian and spanned by $e_9, e_{10}, e_{11}, e_{12}$.

$A4.8(b = -1)$

Symmetries and Lie algebra

$$e_1 = D_x, \quad e_2 = D_z, \quad e_3 = D_y + zD_x, \quad e_4 = e^w D_y, \quad e_5 = e^{-w}(D_z + yD_x),$$
$$e_6 = -tD_x, \quad e_7 = wD_t,$$
$$e_8 = D_t, \quad e_9 = wD_x, \quad e_{10} = tD_t,$$
$$e_{11} = xD_x + yD_y, \quad e_{12} = xD_x + zD_z, e_{13} = D_w$$

$[e_1, e_{11}] = e_1, [e_1, e_{12}] = e_{12}, [e_2, e_3] = e_1, [e_2, e_{12}] = e_2, [e_3, e_{11}] = e_3, [e_4, e_5] = e_1, [e_4, e_{11}] = e_4, [e_4, e_{13}] = -e_4, [e_5, e_{12}] = e_5, [e_5, e_{13}] = e_{13}, [e_6, e_7] = e_9, [e_6, e_8] = e_1, [e_6, e_{10}] = -e_6, [e_6, e_{11}] = e_{11}, [e_6, e_{12}] = be_6, [e_7, e_{10}] = e_7, [e_7, e_{13}] = -e_8, [e_8, e_{10}] = e_8, [e_9, e_{11}] = e_9, [e_9, e_{12}] = e_9, [e_9, e_{13}] = -e_1.$

The algebra is a 13-dimensional indecomposable solvable. The nilradical is 9-dimensional indecomposable and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and the complement is 4-dimensional abelian and spanned by $e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.9b(b = 0)$

Symmetries and Lie algebra

$$e_1 = D_x, \quad e_2 = D_z, \quad e_3 = D_y + zD_x, \quad e_4 = e^w D_y, \quad e_5 = yD_x,$$
$$e_6 = zD_t, \quad e_7 = D_t, \quad e_8 = e^w D_x, \quad e_9 = wD_t, \quad e_{10} = tD_t,$$
$$e_{11} = xD_x + yD_y, \quad e_{12} = xD_x + zD_z, e_{13} = D_w$$

$[e_1, e_{11}] = e_1, [e_1, e_{12}] = e_{12}, [e_2, e_3] = e_1, [e_2, e_6] = e_7, [e_2, e_{12}] = e_2, [e_3, e_5] = e_1, [e_3, e_{11}] = e_3, [e_4, e_5] = e_8, [e_4, e_{11}] = e_4, [e_4, e_{13}] = -e_4, [e_5, e_{12}] = e_5, [e_6, e_{10}] =$

$e_6, [e_6, e_{12}] = -e_6, [e_7, e_{10}] = e_7, [e_8, e_{11}] = e_8, [e_8, e_{12}] = e_8, [e_8, e_{13}] = -e_8, [e_9, e_{10}] = e_9, [e_9, e_{13}] = -e_7.$

The algebra is a 13-dimensional indecomposable solvable. The nilradical is 9-dimensional decomposable $N_8 \oplus \mathbb{R}$ and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and the complement is 4-dimensional abelian and spanned by $e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.9b(b = -\frac{1}{2})$
Symmetries and Lie algebra

$$e_1 = D_x, \ \ e_2 = D_z, \ \ e_3 = D_y + zD_x, \ \ e_4 = e^{-\frac{w}{2}}(D_z + yD_x), \ \ e_5 = ze^{\frac{w}{2}}D_x,$$
$$e_6 = -e^w D_y, \ \ e_7 = e^{\frac{w}{2}}D_x, \ \ e_8 = D_t, \ \ e_9 = wD_t,$$
$$e_{10} = tD_t, \ \ e_{11} = xD_x + yD_y, \ \ e_{12} = xD_x + zD_z, e_{13} = D_w$$

$[e_1, e_{11}] = e_1, [e_1, e_{12}] = e_{12}, [e_2, e_3] = e_1, [e_2, e_5] = e_7, [e_2, e_{12}] = e_2, [e_3, e_{11}] = e_3, [e_4, e_5] = e_1, [e_4, e_6] = e_7, [e_4, e_{12}] = e_4, [e_4, e_{13}] = \frac{1}{2}e_4, [e_5, e_{11}] = e_5, [e_5, e_{13}] = -\frac{1}{2}e_5, [e_6, e_{11}] = e_6, [e_6, e_{13}] = -e_6, [e_7, e_{11}] = e_7, [e_7, e_{12}] = e_7, [e_7, e_{13}] = -\frac{1}{2}e_7, [e_8, e_{10}] = e_8, [e_9, e_{10}] = e_9, [e_9, e_{13}] = -e_8.$

The algebra is a 13-dimensional indecomposable solvable. The nilradical is 9-dimensional decomposable $N_7 \oplus \mathbb{R}^2$ and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and the complement is 4-dimensional abelian and spanned by $e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.10, 4.11a : [e_2, e_3] = e_1, [e_1, e_4] = 2ae_1, [e_2, e_4] = ae_2 - e_3, [e_3, e_4] = e_2 + ae_3$

Geodesics:

$$\begin{aligned}
\ddot{x} &= (a\dot{x} + \dot{y})\dot{w} \\
\ddot{y} &= (a\dot{y} - \dot{x})\dot{w} \\
\ddot{z} &= (2a\dot{z} + (ay - x)\dot{x} - (ax + y)\dot{y})\dot{w} \\
\ddot{w} &= 0
\end{aligned} \tag{10}$$

Symmetries and Lie algebra

$$e_1 = 2e^{aw}D_z, \ \ e_2 = e^{aw}(\sin w \, D_x + \cos w \, D_y + (x \cos w - y \sin w)D_z),$$
$$e_3 = e^{aw}(-\cos w \, D_x + \sin w D_y + (ay \cos w + ax \sin w)D_z),$$
$$e_4 = -2D_z, \ \ e_5 = D_x + yD_z, \ \ e_6 = D_y - xD_z, \ \ e_7 = D_t, \ \ e_8 = D_x = wD_t,$$
$$e_9 = xD_y - yD_x, \ \ e_{10} = tD_t, \ \ e_{11} = xD_x + yD_y + 2zD_z, \ \ e_{12} = D_w.$$

$[e_1, e_{11}] = 2e_1, [e_1, e_{12}] = -2ae_1, [e_2, e_3] = e_1, [e_2, e_{10}] = e_3, [e_2, e_{11}] = e_2, [e_2, e_{12}] = -ae_2 + e_3, [e_3, e_{10}] = -e_2, [e_3, e_{11}] = e_3, [e_3, e_{12}] = -ae_3 - e_2, [e_4, e_{11}] = 2e_4, [e_5, e_6] = e_4, [e_5, e_{10}] = e_6, [e_5, e_{11}] = e_5, [e_6, e_{10}] = -e_5, [e_6, e_{11}] = e_6, [e_7, e_9] = e_7, [e_8, e_9] = e_8, [e_8, e_{12}] = -e_7.$

The algebra is a 12-dimensional indecomposable solvable. The nilradical is 8-dimensional decomposable $H \oplus H \oplus \mathbb{R}^2$ and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ and the complement is 4-dimensional abelian and spanned by $e_9, e_{10}, e_{11}, e_{12}$.

$A4.10(a = 0)$

Symmetries and Lie algebra

$e_1 = 2D_z, \ \ e_2 = D_y - xD_z, \ \ e_3 = D_x + yD_z,$
$e_4 = \cos w \, D_x - \sin w D_y - (y \cos w \, + x \sin w)D_z,$
$e_5 = \sin w \, D_x + \cos w \, D_y + (x \cos w - y \sin w \,)D_z, \ \ e_6 = D_t, \ \ e_7 = 2tD_z, \ \ e_8 = wD_t,$
$e_9 = -2wD_z, \ \ e_{10} = tD_t, \ \ e_{11} = xD_y - yD_x, \ \ e_{12} = xD_x + yD_y + 2zD_z, \ \ e_{13} = D_w$

$[e_1, e_{12}] = 2e_1, [e_2, e_3] = ae_1, [e_2, e_{11}] = -e_3, [e_2, e_{12}] = e_2, [e_3, e_{11}] = e_2, [e_3, e_{12}] = e_3, [e_4, e_5] = e_1, [e_4, e_{11}] = e_5, [e_4, e_{12}] = e_4, [e_4, e_{13}] = e_5, [e_5, e_{11}] = -e_4, [e_5, e_{12}] = e_5, [e_5, e_{13}] = -e_4, [e_6, e_7] = e_1, [e_6, e_{10}] = e_6, [e_7, e_8] = e_9, [e_7, e_{10}] = -e_7, [e_7, e_{12}] = 2e_7, [e_8, e_{10}] = e_8, [e_8, e_{13}] = -e_6, [e_9, e_{12}] = 2e_9, [e_9, e_{13}] = e_1.$

The algebra is a 13-dimensional indecomposable solvable. The nilradical is 9-dimensional indecomposable and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and the complement is 4-dimensional abelian and spanned by $e_{10}, e_{11}, e_{12}, e_{13}$.

$A4.12 : [e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1$

Geodesics:
$$\begin{aligned}
\ddot{x} &= \dot{x}\dot{z} + \dot{y}\dot{w} \\
\ddot{y} &= \dot{y}\dot{z} - \dot{x}\dot{w} \\
\ddot{z} &= 0 \\
\ddot{w} &= 0
\end{aligned} \tag{11}$$

Symmetries and Lie algebra

$e_1 = D_t, \ \ e_2 = D_x, \ \ e_3 = D_y, \ \ e_4 = wD_t, \ \ e_5 = zD_t, \ \ e_6 = e^z(\sin w \, D_x - \cos w \, D_y),$

$e_7 = e^z(\cos w \, D_x + \sin w \, D_y), \ \ e_8 = -D_z, \ \ e_9 = tD_t, \ \ e_{10} = xD_x + yD_y$

$e_{11} = xD_y - yD_x, \ \ e_{12} = D_w$

$[e_1, e_9] = e_1, [e_2, e_{10}] = e_2, [e_2, e_{11}] = e_3, [e_3, e_{10}] = e_3, [e_3, e_{11}] = -e_2, [e_4, e_9] = e_4, [e_4, e_{12}] = -e_1, [e_5, e_8] = e_1, [e_5, e_9] = e_5, [e_6, e_8] = e_6, [e_6, e_{10}] = e_6, [e_6, e_{11}] = e_7, [e_6, e_{12}] = e_7, [e_7, e_8] = e_7, [e_7, e_{10}] = e_7, [e_7, e_{11}] = -e_6, [e_7, e_{12}] = -e_6.$

The symmetry algebra is solvable with nilradical $\mathbb{R}^7$ spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and abelian complement $\mathbb{R}^5$ spanned by $e_8, e_9, e_{10}, e_{11}, e_{12}$.

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