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# Lie algebras with idempotent derivations

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**Abstract**

The structure of Lie algebras with idempotent derivations is studied. It is proved that a Lie algebra $L$ has an idempotent derivation $D$ if and only if $L = I \oplus K$, where $I$ is an abelian ideal which is the image of $D$, $K$ is a subalgebra of $L$ which is the kernel of $D$, and $D$ is identity on $I$.

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**Keywords:** Lie algebra, derivation, idempotent derivation

## 1 Introduction

It is well known that derivations are very important in the study of the structure of Lie algebras [1, 2, 3]. For example, a Lie algebra is nilpotent if it has an invertible derivation [4]. In this paper, we study Lie algebras with idempotent derivations. First we recall some definitions used in the paper. A Lie algebra $L$ [5] is a vector space over a field $F$ which with a bilinear skew-symmetric multiplication satisfying Jacobi identity, that is, for all $x, y, z \in L$,

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]. \quad (1)$$

If a linear map $D \in End(L)$ satisfies for all $x, y \in L$,

$$D([x,y]) = [D(x), y] + [x, D(y)], \tag{2}$$

then $D$ is a derivation of $L$. The set of all derivations of $L$, is denoted by $Der(L)$, is a linear Lie algebra. For $D \in Der(L)$, if $D^2 = D$, then $D$ is called an idempotent derivation.

In the following we suppose $L$ is a finite dimensional Lie algebra over the complex field.

# 2 Main results

**Lemma 1** *Let $L$ be a Lie algebra and $D$ be an idempotent derivation. Then for all $x, y \in L$,*

*1) $[Dx, Dy] = 0$; 2) $D([D(x), y]) = [D(x), y]$.*

**Proof** By identity (2), for all $x, y \in L$, we have

$$D([x,y]) = D^2([x,y]) = [D(x), y] + 2[D(x), D(y)] + [x, D(y)]$$
$$= D([x,y]) + 2[D(x), D(y)].$$

Thanks to $chF = 0$, $[Dx, Dy] = 0$, and

$$D([D(x), y]) = [D^2(x), y] + [D(x), D(y)] = [D(x), y].$$

The proof is completed.

**Lemma 2** *Let $L$ be a Lie algebra, $D$ be an idempotent derivation. Then the image of $D$ on $L$, is denoted by $I = D(L)$, is an abelian ideal of $L$, and the kernel of $D$, is denoted by $K = KerD$ is a subalgebra of $L$.*

**Proof** By Lemma 1, we know that for all $x, y \in L$, $[D(x), D(y)] = 0$, so $[D(L), D(L)] = 0$. It implies that $I$ is an abelian subalgebra of $L$.

Since for all $x, y \in L$, $[D(x), y] = D([x,y]) - [x, D(y)] = D([x,y]) + [D(y), x] = D([x,y]) + D([D(y), x])$, it follows that $[D(L), L] \subset D(L)$, that is, $I$ is an abelian ideal of $L$.

For all $x, y \in K$, by identity (2), $D([x,y]) = [D(x), y] + [x, D(y)] = 0$. It shows that $K = KerD$ is a subalgebra of $L$. The result follows.

**Theorem 1** *Let $L$ be a Lie algebra, and $D$ be an idempotent derivation. Then $L$ has the semi-direct decomposition: $L = I \oplus K$, where $I = D(L)$ is an abelian ideal, and $K = KerD$ is a subalgebra of $L$.*

**Proof** The result follows from Lemma 1 and Lemma 2.

**Theorem 2** *Let $L$ be a Lie algebra, $D$ be an idempotent derivation. Then there exists a basis $\{x_1, \cdots, x_r, y_1, \cdots, y_s\}$ of $L$ such that*

$$\begin{cases} D(x_i) = x_i, 1 \le i \le r, \\ D(y_j) = 0, 1 \le j \le s. \end{cases}$$

*And the multiplication of $L$ is*

$$[x_i, x_j] = 0, 1 \leq i, j \leq r,$$

$$[x_i, y_k] = \sum_{t=1}^{r} a_t^{ik} x_t,\ a_t^{ik} \in F,\ 1 \leq i \leq r,\ 1 \leq k \leq s,$$

$$[y_k, y_l] = \sum_{t=1}^{s} b_t^{kl} y_t,\ b_t^{kl} \in F,\ 1 \leq k,\ l \leq s.$$

**Proof** Since $\dim L < \infty$ and $D^2 = D$, from the property of linear algebra, $D$ is triangled and the eigenvalue of $D$ are 1 and zero. Therefore, there is a basis $\{x_1, \cdots, x_r, y_1, \cdots, y_s\}$ of $L$ such that $D(x_i) = x_i, 1 \leq i \leq r$, and $D(y_j) = 0$, $1 \leq j \leq s$. Thanks to Lemma 1 and Lemma 2, $I = \sum_{i=1}^{r} F x_i$ is an abelian ideal and $K = \sum_{j=1}^{r} F y_j$ is a subalgebra. Therefore, the multiplication of $L$ in the basis $\{x_1, \cdots, x_r, y_1, \cdots, y_s\}$ is $[x_i, x_j] = 0, 1 \leq i, j \leq r$, $[x_i, y_k] = \sum_{t=1}^{r} a_t^{ik} x_t$, $1 \leq i \leq r$, $1 \leq k \leq s$, $[y_k, y_l] = \sum_{t=1}^{s} b_t^{kl} y_t$, $1 \leq k, l \leq s$, where $a^{ik}, b^{kl} \in F$, $1 \leq i \leq r$, $1 \leq k, l \leq s$. The result follows.

**Theorem 3** *Let $L$ be a Lie algebra. Then there exists an idempotent derivation on $L$ if and only if $L = I \oplus K$, where $I$ is an abelian ideal, and $K$ is a subalgebra.*

**Proof** If there exists an idempotent derivation $D$ on $L$, by Theorem 1 $L = I \oplus K$, where $I = D(L)$ is an abelian ideal and $K = KerD$ is a subalgebra.

Conversely, define linear map $D : L \to L$ by

$$D(z) = \begin{cases} z, & \text{for all } z \in I, \\ 0, & \text{for all } z \in K. \end{cases}$$

Obviously, $D$ satisfies $D^2 = D$, and for all $x_1, x_2 \in I, y_1, y_2 \in K$,

$$D([y_1, y_2]) = 0 = [D(y_1), y_2] + [y_1, D(y_2)],$$

$$D([x_1, x_2]) = [x_1, x_2] = 0, \quad [D(x_1), x_2] + [x_1, D(x_2)] = 0,$$

$$D([x_1, y_2]) = [x_1, y_2] = [D(x_1), y_2] = [D(x_1), y_2] + [x_1, D(y_2)].$$

It shows that $D$ is a derivation of Lie algebra $L$.

**Remark** Let $(A, [,])$ be a finite dimensional Lie algebra over the complex field, $(M, \rho)$ be an $A$-module. Then $L = A \oplus M$ is a Lie algebra in the multiplication $[,]_L$: for all $x, y \in A$ and $s, t \in M$,

$$[x, y]_L = [x, y],\ [s, t]_L = 0, [x, s]_L = \rho(x)(s).$$

Then $M$ is an abelian ideal of $L$ and $A$ is a subalgebra. By Theorem 3 the linear map $D : L \to L$, defined by $D(x) = 0, D(s) = s$ for all $x \in A, s \in M$, is an idempotent derivation of $L$.

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