Mathematica Aeterna, Vol. 4, 2014, no. 5, 437 - 444

Laplace Transformation of the Distribution of the Time of System Sojourns within a Band

T.I.Nasirova

Department of Applied Mathematics and Cybernetics
Baku State University
Z.Khalilov st.23, AZ-1148, Baku, Azerbaijan
nasirova.tamilla@mail.ru

U.Y.Kerimova

ANSA Cybernetic Institute
B. Vahabzade st. 9 , AZ-1141, Baku, Azerbaijan
ulviyye_kerimova@yahoo.com

B.G.Shamilova

Department of Applied Mathematics and Cybernetics
Baku State University
Z.Khalilov st.23, AZ-1148, Baku, Azerbaijan
bahar.shamilova@rambler.ru

Abstract

One of the important problems in probability theory is finding the distribution of the time of the sojourn of a system (a process) within a specified band. With this purpose, we will investigate the semi-Markov random processes with positive tendency, negative jumps and delaying boundary at zero in this article.The Laplace transformation of the distribution of the time of the system sojourn within a given band found.

Mathematics Subject Classification: 60A10, 60J25, 60G50.

Keywords: Laplace transformation, semi-Markov random process, process with positive tendency and negative jumps.

1 Introduction

Many researchers have been engaged in solving this kind of problems. In [4] a stepped jump process of a semi-Markov walk with two delay screens at zero and at a is constructed. The Laplace transformation of the distribution of the time of the system sojourn within a given band and its first and second moments are found. The Laplace transformation of the distribution of the duration of the sojourn of a process with independent increments within a given band was found in [3]. The distribution of the duration of the sojourn of a random walk with a discrete distribution within a given band was found in [1]. An asymptotic expansion of the distribution of the duration of a sojourn of a random walk with normal distribution within a given band was found in [6].

2 Problem statement

Let's assume that in probability space $\{\Omega, F, P(\cdot)\}$ is given the sequence of independent, equally distributed and independent themselves positive random variables ξ_k and ζ_k, $k=\overline{1,\infty}$. Using these random variables we will derive the following semi-Markov random process:

$$X_1(t) = z + t - \sum_{i=1}^{k-1} \zeta_i \,, \qquad \text{if} \qquad \sum_{i=1}^{k-1} \xi_i \le t < \sum_{i=1}^{k} \xi_i \,,\ k=\overline{1,\infty},$$

$X_1(t)$ is called semi-Markov random processes with positive tendency and negative jumps.

General form of process semi-Markov random walk with delaying boundary is given by A.A.Borovkov [2].

If process $X_1(t)$ is some process without boundary, then process $X(t)$ with delaying boundary at zero is defined following:

$$X(t) = X_1(t) - \inf_{0\le S\le t}(0, X_1(s)) \qquad \text{or} \qquad X(t) = X_1(t) - \min(0, \inf_{0\le S\le t} X_1(s)).$$

Idea of construction of the process semi-Markov random walk is following:

Let $X_1(0) = z \ge 0$. Process $X(t)$ is equally to process $X_1(t)$ until, the process $X_1(t)$ is positive.

Let $X_1(t) \le 0$; then $X(t)$ is equally to zero until, the process $X_1(t)$ will not have positive jump. In moment of jump of the process $X_1(t)$, process $X(t)$ will be have jump, such is equally to jump of the process $X_1(t)$.

The obtained process is called a process of a semi-Markov random walk with positive tendency, negative jumps and delaying boundary at zero.

Introduce a random variable τ denoting the duration of the sojourn of the process $X(t)$ within a band (a,b), where $b < a$, $a > 0, b > 0$.

$$\{\tau < t\} = \left\{\sup_{0\le s\le t} X(s) < a,\ \inf_{0\le s\le t} X(s) > b\right\}$$

The aim of the present study was to find an explicit form of the Laplace transformation of the conditional distributions of the random variable τ .

$$P\{\tau > t \mid X(0,\omega) = z\} = P\left\{\sup_{0\le s\le t} \mathrm{X(s)} < \mathrm{a}\ ;\ \inf_{0\le s\le t} \mathrm{X(s)} > \mathrm{b}\ \mid \mathrm{X(0)}\ = \mathrm{z}\right\}.$$

We denote

$$K(t,a,b \mid z) = P\left\{\sup_{0\le s\le t} \mathrm{X(s)} < a\ ;\ \inf_{0\le s\le t} \mathrm{X(s)} > b\ \mid \mathrm{X(0)}\ = z\right\},\ \mathrm{z} \ge 0,$$

$$\breve{K}(\theta,a,b \mid z) = \int_{t=0}^{\infty} e^{-\theta t} K(t,a,b \mid z)dt, \quad \theta > 0.$$

3 Main Results

Theorem. It holds

$$\tilde{K}(\theta,a,b \mid z) = \int_{t=0}^{a-z} \mathrm{e}^{-\theta\,\mathrm{t}}\ P\{\xi_1 > t\}\mathrm{dt} +$$

$$+ \int_{y=b}^{z} \tilde{K}(\theta,a,b \mid y) \int_{t=0}^{a-z} e^{-\theta\,\mathrm{t}}\ d_t\ \mathrm{P}\{\xi_1 < t\} d_y\ P\{\ \zeta_1 < z + t - y\} +$$

$$+ \int_{y=z}^{a} \tilde{K}(\theta,a,b \mid y) \int_{t=y-z}^{a-z} e^{-\theta\,\mathrm{t}}\ d_t\ \mathrm{P}\{\xi_1 < t\} d_y\ P\{\ \zeta_1 < z + t - y\}.$$

Proof. By the formula of the composite probability, we have

$$K(t,a,b \mid z) = P\left\{\inf_{0\le s\le t} \mathrm{X(s)} > b\,;\sup_{0\le s\le t} \mathrm{X(s)} < a\ ;\ \xi_1 > \mathrm{t}\ \mid \mathrm{X(0)}\ = z\right\} +$$

$$+ P\left\{\inf_{0\le s\le t} \mathrm{X(s)} > b\,;\sup_{0\le s\le t} \mathrm{X(s)} < a\ ;\ \xi_1 < \mathrm{t}\ \mid \mathrm{X(0)}\ = z\right\} = P\{z + t < a\}P\{\xi_1 > t\} +$$

$$+ \int_{s=0}^{t} \int_{y=b}^{a} P\{\xi_1 \in ds; z + s < a\,; \mathrm{z} + \mathrm{s} - \zeta_1 > b\,; \mathrm{z} + \mathrm{s} - \zeta_1 \in dy\ \} \times$$

$$\times P\left\{\inf_{0\le u\le t-s} \mathrm{X(u)} > b\,;\ \sup_{0\le u\le t-s} \mathrm{X(u)} > a \mid \mathrm{X(0)} = \mathrm{y}\right\}.$$

Thus we have

$$K(t,a,b\,|\,z) = P\{t < a - z\}P\{\xi_1 > t\} +$$

$$+\int_{s=0}^{t}\int_{y=b}^{a} P\{\xi_1 \in ds\}\mathrm{P}\{z + s < a\}\mathrm{d}_y P\{\mathrm{b} < \mathrm{z} + \mathrm{s} - \zeta_1 < y\}\times K(t - s, a, b\,|\,X(0) = y). \quad (1)$$

Obviously

$$P\{b < z + s - \zeta_1 < y\} = P\{b - z - s < -\zeta_1 < y - z - s\} = P\{z + s - b > \zeta_1 > z + s - y\} =$$
$$= P\{z + s - y < \zeta_1 < z + s - b\} = P\{\zeta_1 < z + s - b\} - P\{\zeta_1 < z + s - y\}.$$

If we apply the Laplace transformation with respect to t to the both hand sides of equation (1) we'll get the following integral equation

$$\tilde{K}(\theta,a,b\,|\,z) = \int_{t=0}^{a-z} \mathrm{e}^{-\theta t} P\{\xi_1 > t\}\mathrm{dt} +$$

$$+\int_{y=b}^{a}\tilde{K}(\theta,a,b\,|\,y)\int_{t=0}^{\infty} e^{-\theta t}\,\mathrm{P}\{\mathrm{z} + \mathrm{t} < \mathrm{a}\}d_t\,\mathrm{P}\{\xi_1 < t\}d_y P\{z + t - \zeta_1 < y\} =$$

$$= \int_{t=0}^{a-z} \mathrm{e}^{-\theta t} P\{\xi_1 > t\}\mathrm{dt} + \int_{y=b}^{a}\tilde{K}(\theta,a,b\,|\,y)\int_{t=0}^{a-z} e^{-\theta t}\,d_t\,\mathrm{P}\{\xi_1 < t\}d_y\,P\{\zeta_1 > z + t - y\}.$$

It is evident $z + t - y > 0 \Rightarrow t > y - z$.

Then we have

$$\tilde{K}(\theta,a,b\,|\,z) = \int_{t=0}^{a-z} \mathrm{e}^{-\theta t} P\{\xi_1 > t\}\mathrm{dt} +$$

$$+\int_{y=b}^{a}\tilde{K}(\theta,a,b\,|\,y)\int_{t=\max(0,y-z)}^{a-z} e^{-\theta t}\,d_t\,\mathrm{P}\{\xi_1 < t\}d_y\,P\{\zeta_1 > z + t - y\}.$$

If take into account

$$\max\{0, y - z\} = \begin{cases} 0, & \text{if } \mathrm{y} < \mathrm{z}, \\ \mathrm{y - z}, & \text{if } \mathrm{y} > \mathrm{z}, \end{cases}$$

that is, why we get an integral equation for $\tilde{K}(\theta,a,b\,|\,z)$

$$\tilde{K}(\theta,a,b\,|\,z) = \int_{t=0}^{a-z} \mathrm{e}^{-\theta t} P\{\xi_1 > t\}\mathrm{dt} +$$

$$+\int_{y=b}^{z}\tilde{K}(\theta,a,b\,|\,y)\int_{t=0}^{a-z} e^{-\theta t}\,d_t\,\mathrm{P}\{\xi_1 < t\}d_y\,P\{\zeta_1 < z + t - y\} + \quad (2)$$

$$+\int_{y=z}^{a}\tilde{K}(\theta,a,b\,|\,y)\int_{t=y-z}^{a-z} e^{-\theta t}\,d_t\,\mathrm{P}\{\xi_1 < t\}d_y\,P\{\zeta_1 < z + t - y\}.$$

Theorem is proofed.

This integral equation can be solved by the method of successive approximations, yet the resulting solution is unfit for applications. We will solve this integral equation in special case.

Corollary. In the case where the random variables ξ_1 and ζ_1 have an exponential distribution with the parameters μ and λ respectively,

$$\begin{cases} \mathrm{P}\{\xi_1(\omega) < t\} = \left[1 - e^{-\mu t}\right]\varepsilon(t), & \mu > 0, \\ \mathrm{P}\{\zeta_1(\omega) < t\} = \left[1 - e^{-\lambda t}\right]\varepsilon(t), & \lambda > 0, \end{cases}$$

where

$$\varepsilon(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0, \end{cases}$$

equation (2) will be as follows:

$$\begin{aligned} \tilde{K}(\theta, a, b \mid z) = {} & \frac{1 - e^{-(\lambda+\theta)(a-z)}}{\lambda + \theta} - \\ & - \frac{\lambda\mu}{\lambda + \mu + \theta} e^{-(\lambda+\mu+\theta)a} \mathrm{e}^{(\lambda+\theta)z} \int_{y=b}^{a} e^{\mu y} \tilde{K}(\theta, a, b \mid y)\, \mathrm{dy} + \\ & + \frac{\lambda\mu}{\lambda + \mu + \theta} e^{-\mu z} \int_{y=b}^{z} e^{\mu y} \tilde{K}(\theta, a, b \mid y)\, \mathrm{dy} + \\ & + \frac{\lambda\mu}{\lambda + \mu + \theta} \mathrm{e}^{(\lambda+\theta)z} \int_{y=z}^{a} e^{-(\lambda+\theta)y} \tilde{K}(\theta, a, b \mid y)\, \mathrm{dy}. \end{aligned} \tag{3}$$

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (3) by $e^{\mu z}$ and derive on z. Then we will multiply both sides of last equation by $e^{-(\lambda+\mu+\theta)z}$ and derive on z .Then we have following differential equation**:**

$$\tilde{K}''(\theta,a,b|z) - (\lambda - \mu + \theta)\, \tilde{K}'(\theta,a,b|z) - \mu\theta \tilde{K}(\theta,a,b|z) = -\mu. \tag{4}$$

The general solution of this differential equation will be

$$\tilde{K}(\theta,a,b|z) = c_1(\theta) e^{k_1(\theta)z} + c_2(\theta) e^{k_2(\theta)z} + \tilde{K}_{sp}(\theta,a,b|z), \tag{5}$$

where $k_i(\theta)$, $i = 1,2$ -are the roots of characteristic equation of (4)

$$k^2(\theta) - (\lambda - \mu + \theta) k(\theta) - \theta\mu = 0$$

$$\tilde{K}_{sp}(\theta,a,b|z) = -\frac{\mu}{k_2(\theta)\, k_1(\theta)}.$$

$\tilde{K}_{sp}(\theta,a,b|z)$ is the special solution of the equation (4).

Then the general solution of this differential equation will be

$$\begin{aligned} \tilde{K}(\theta,a,b|z) &= c_1(\theta) e^{k_1(\theta)z} + c_2(\theta) e^{k_2(\theta)z} - \frac{\mu}{k_2(\theta) - k_1(\theta)} = \\ &= c_1(\theta) e^{k_1(\theta)z} + c_2(\theta) e^{k_2(\theta)z} + \frac{1}{\theta}. \end{aligned} \tag{6}$$

By finding $c_1(\theta)$ and $c_2(\theta)$ from equation (2) we will get the following system of algebraic equations:

$$\begin{cases} \tilde{K}(\theta,a,b|0) = \dfrac{1-e^{-(\lambda+\theta)a}}{\lambda+\theta} - \dfrac{\lambda\mu}{\lambda+\mu+\theta} e^{-(\lambda+\mu+\theta)a} \int\limits_{y=b}^{a} e^{\mu y} \tilde{K}(\theta,a,b|y)dy + \\ \qquad + \dfrac{\lambda\mu}{\lambda+\mu+\theta} \int\limits_{y=b}^{0} e^{\mu y} \tilde{K}(\theta,a,b|y)dy + \\ \qquad + \dfrac{\lambda\mu}{\lambda+\mu+\theta} \int\limits_{y=0}^{a} e^{-(\lambda+\theta)y} \tilde{K}(\theta,a,b|y)dy, \\ \mu\tilde{K}(\theta,a,b|0) + \tilde{K}'(\theta,a,b|0) = \dfrac{\mu}{\lambda+\theta} - \lambda\mu e^{-(\lambda+\mu+\theta)a} \int\limits_{y=b}^{a} e^{\mu y} \tilde{K}(\theta,a,b|y)dy + \\ \qquad + \lambda\mu \int\limits_{y=0}^{a} e^{-(\lambda+\theta)y} \tilde{K}(\theta,a,b|y)dy. \end{cases} \tag{7}$$

By exploitation of equation (6), equation (7) becomes

$$\begin{cases} c_1(\theta) + c_2(\theta) + \dfrac{1}{\theta} = \dfrac{1-e^{-(\lambda+\theta)a}}{\lambda+\theta} - \\ - \dfrac{\lambda\mu}{\lambda+\mu+\theta} e^{-(\lambda+\mu+\theta)a} \int\limits_{y=b}^{a} e^{\mu y} \left[c_1(\theta)e^{k_1(\theta)y} + c_2(\theta)e^{k_2(\theta)y} + \dfrac{1}{\theta} \right] dy + \\ + \dfrac{\lambda\mu}{\lambda+\mu+\theta} \int\limits_{y=b}^{0} e^{\mu y} \left[c_1(\theta)e^{k_1(\theta)y} + c_2(\theta)e^{k_2(\theta)y} + \dfrac{1}{\theta} \right] dy + \\ + \dfrac{\lambda\mu}{\lambda+\mu+\theta} \int\limits_{y=0}^{a} e^{-(\lambda+\theta)y} \left[c_1(\theta)e^{k_1(\theta)y} + c_2(\theta)e^{k_2(\theta)y} + \dfrac{1}{\theta} \right] dy, \\ \\ \mu\left[c_1(\theta) + c_2(\theta) + \dfrac{1}{\theta} \right] + \left[c_1(\theta)k_1(\theta) + c_2(\theta)k_2(\theta) \right] = \dfrac{\mu}{\lambda+\theta} - \\ - \lambda\mu e^{-(\lambda+\mu+\theta)a} \int\limits_{y=b}^{a} e^{\mu y} \left[c_1(\theta)e^{k_1(\theta)y} + c_2(\theta)e^{k_2(\theta)y} + \dfrac{1}{\theta} \right] dy + \\ + \lambda\mu \int\limits_{y=0}^{a} e^{-(\lambda+\theta)y} \left[c_1(\theta)e^{k_1(\theta)y} + c_2(\theta)e^{k_2(\theta)y} + \dfrac{1}{\theta} \right] dy. \end{cases} \tag{8}$$

Now we proof linear dependence of this algebraic system.
If to consider the following substitutions:

$$\lambda + \theta - k_1(\theta) = \mu + k_2(\theta)$$
$$[\mu + k_1(\theta)][\mu + k_2(\theta)] = \lambda\mu$$
$$[\lambda + \theta - k_2(\theta)][\mu + k_2(\theta)] = \lambda\mu$$

equation (8) becomes:

$$\begin{cases} \sum_{i=1}^{2}\left[(\lambda+\mu+\theta)e^{-(\lambda+\theta-k_i(\theta))a}-(\mu+k_i(\theta))e^{-(\lambda+\mu+\theta)a}e^{(\mu+k_i(\theta))b}+(\mu+k_i(\theta))e^{(\mu+k_i(\theta))b}\right]\times \\ \times \mathrm{C_i}(\theta)=-\dfrac{\lambda(\lambda+\mu+\theta)}{\theta(\lambda+\theta)}e^{-(\lambda+\theta)a}+\dfrac{\lambda}{\theta}e^{-(\lambda+\mu+\theta)a}e^{\mu\,\mathrm{b}}-\dfrac{\lambda}{\theta}e^{\mu\,\mathrm{b}}, \\ \\ \sum_{i=1}^{2}\left[(\lambda+\mu+\theta)e^{-(\lambda+\theta-k_i(\theta))a}-(\mu+k_i(\theta))e^{-(\lambda+\mu+\theta)a}e^{(\mu+k_i(\theta))b}+(\mu+k_i(\theta))e^{(\mu+k_i(\theta))b}\right]\times \\ \times \mathrm{C_i}(\theta)=-\dfrac{\lambda(\lambda+\mu+\theta)}{\theta(\lambda+\theta)}e^{-(\lambda+\theta)a}+\dfrac{\lambda}{\theta}e^{-(\lambda+\mu+\theta)a}e^{\mu\,\mathrm{b}}-\dfrac{\lambda}{\theta}e^{\mu\,\mathrm{b}}. \end{cases} \tag{9}$$

Thus, (9) is a linear dependence equations system, as

$$C_2(\theta)=0$$

Then we have

$$C_1(\theta)=\frac{\lambda}{\theta(\lambda+\theta)}\times$$
$$\times\frac{-(\lambda+\mu+\theta)e^{-(\lambda+\theta)a}+(\lambda+\theta)e^{-(\lambda+\mu+\theta)a}e^{\mu\,\mathrm{b}}-(\lambda+\theta)e^{\mu\,\mathrm{b}}}{\left[(\lambda+\mu+\theta)e^{-(\lambda+\theta-k_i(\theta))a}-(\mu+k_i(\theta))e^{-(\lambda+\mu+\theta)a}e^{(\mu+k_i(\theta))b}+(\mu+k_i(\theta))e^{(\mu+k_i(\theta))b}\right]}. \tag{10}$$

Then the general solution of differential equation (4) will be as follows

$$\tilde{K}(\theta,a,b\,|z)=\frac{1}{\theta}+\frac{\lambda\,e^{k_1(\theta)z}}{\theta(\lambda+\theta)}\times$$
$$\times\frac{-(\lambda+\mu+\theta)e^{-(\lambda+\theta)a}+(\lambda+\theta)e^{-(\lambda+\mu+\theta)a}e^{\mu\,\mathrm{b}}-(\lambda+\theta)e^{\mu\,\mathrm{b}}}{\left[(\lambda+\mu+\theta)e^{-(\lambda+\theta-k_i(\theta))a}-(\mu+k_i(\theta))e^{-(\lambda+\mu+\theta)a}e^{(\mu+k_i(\theta))b}+(\mu+k_i(\theta))e^{(\mu+k_i(\theta))b}\right]}. \tag{11}$$

This expression is the Laplace transform of the distribution of the duration of the sojourn of a system within a band.

CONCLUSIONS

In the present study, the explicit form of the Laplace transformation of the distribution of the duration oft he sojourn of a system within a band was found. The results of the paper may be used in the store control problems with two level's for finding conditional, unconditional distributions of the resource.

REFERENCES

[1] A.A. Borovkov On a walk within a band with delay bounds,(Russian) *Math.Zam.17(*1975).

[2] A.A. Borovkov Veroyatnostnie processi v teorii massovoqo obslujivaniya (Probabilistic processes in queuing theory) Moscow: Nauka (1972).

[3] I.I.Gikhman, A.V.Skorokhod *Teoriya sluchainykh protsessov* (Theory of random processes), Moscow: Nauka, (1973), vol. 2, pp. 450–453.

[4] T. I. Nasirova, R. I. Sadikova Laplace transformation of the distribution of the time of system sojourns within a band . Automatic Control and Computer Sciences (2009), volume 43, Issue 4, pp 190-194.

[5] T.I Nasirova Processes of a semi-Markov walk, Baku: Elm, (1984), pp. 50–51.

[6] V.I. Lotov On random walks within a band, *Teoriya veroyatnostei i ee primenenie*, (1991), vol. 6, issue 1, pp. 160–165.