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Iterating Linear Causal Recurrence Relations

Mircea Ion Cîrnu[1]

Faculty of Applied Sciences, Polytechnic University, Bucharest, Romania

Abstract

The iteration formula of the linear causal recurrence relations with variable coefficients in a ring is given. Using this formula, the initial value problem for such relations is solved. Particular cases and applications to combinatorics, especially to sequences of numbers and polynomials, and to quicksort theory are given. The paper continues the author's concerns related to the study of linear and non-linear recurrence relations, published in a series of articles cited in References.

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1 Introduction

We give here a formula for the iterations of the causal linear recurrence relations (difference equations) with variable coefficients in a commutative ring with unity. Particularly, the closed formula for the solution of this recurrence relation is given. Some relations of the coefficients appearing in these formulas are given. The obtained results contain as particular cases those given by H. W. Gould and Jocelyn Quaintance for Bell numbers and polynomials, variant sequences, floor and roof functions. Several applications to generalized Fibonacci sequences in rings, particularly to Fibonacci numbers and polynomials, Pell and Jacobstahl numbers, are presented. The closed form and the iteration formula for Horadam sequences in rings are also obtained, they being the solutions of the second order linear recurrence relations, with arbitrary initial values. Particular cases of Lucas numbers and polynomials, Pell-Lucas and Jacobstahl-Lucas numbers, and Chebyshev polynomials are considered. As a last application of the iteration formula, a new derivation of the form of the general term of the quicksort recurrence relation is given.

[1] *Email address:* cirnumircea@yahoo.com

2 Iteration Formula for the Linear Causal Recurrence Relations with Variable Coefficients in Rings

Let R be a commutative ring with multiplicative identity 1, and $a_n(k) \in R$, for $k = 0, 1, \ldots, n$.

Theorem 2.1. *If the elements $f(n) \in R$ satisfy the linear causal recurrence relation with variable coefficients*

$$f(n+1) = \sum_{k=0}^{n} a_n(k) f(k), \ n = 0, 1, 2, \ldots, \tag{1}$$

then their iterations are given, for $n = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, by formula

$$f(n+r) = \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_j^r(n) a_{n+j}(k) f(k), \tag{2}$$

the coefficients $A_j^r(n) \in R$ satisfying the linear recurrence relation

$$A_j^r(n) = \sum_{i=j+1}^{r-1} A_j^i(n) a_{n+r-1}(n+i), \ j = 0, 1, \ldots, r-2, \tag{3}$$

and the condition

$$A_{r-1}^r(n) = 1. \tag{4}$$

Proof. We prove by induction after r. From relation (1) it follows that (2) is true for $r = 1$, if $A_0^1(n) = 1$. According to relation (1), we have

$$f(n+2) = \sum_{k=0}^{n+1} a_{n+1}(k) f(k) = \sum_{k=0}^{n} a_{n+1}(k) f(k) + a_{n+1}(n+1) f(n+1) =$$

$$= \sum_{k=0}^{n} a_{n+1}(k) f(k) + a_{n+1}(n+1) \sum_{k=0}^{n} a_n(k) f(k),$$

hence (2) is true for $r = 2$, if $A_0^2(n) = a_{n+1}(n+1)$ and $A_1^2(n) = 1$, according to (3) and (4). We suppose that (2), (3) and (4) are true for every natural number less or equal than a given natural number r. Then, using again the relation (1), it follows that

$$f(n+r+1) = \sum_{k=0}^{n+r} a_{n+r}(k) f(k) = \sum_{k=0}^{n} a_{n+r}(k) f(k) + \sum_{i=1}^{r} a_{n+r}(n+i) f(n+i) =$$

$$= \sum_{k=0}^{n} a_{n+r}(k)f(k) + \sum_{i=1}^{r}\sum_{k=0}^{n}\sum_{j=0}^{i-1} a_{n+r}(n+i)A_j^i(n)a_{n+j}(k)f(k) =$$

$$= \sum_{k=0}^{n} a_{n+r}(k)f(k) + \sum_{k=0}^{n}\sum_{j=0}^{r-1}\sum_{i=j+1}^{r} a_{n+r}(n+i)A_j^i(n)a_{n+j}(k)f(k) =$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{r} A_j^{r+1}(n)a_{n+j}(k)f(k),$$

where, for $j = 0, 1, \ldots, r-1$, we denote

$$A_j^{r+1}(n) = \sum_{i=j+1}^{r} A_j^i(n)a_{n+r}(n+i), \tag{5}$$

and $A_r^{r+1}(n) = 1,\ n = 0, 1, 2, \ldots$. Therefore (2), (3) and (4) are true for $r+1$. According to the induction axiom, the relations (2), (3) and (4) are true for every natural number r.

□

Example 2.2. *We have* $A_0^1(n) = 1$; $A_0^2(n) = a_{n+1}(n+1),\ A_1^2(n) = 1$; $A_0^3(n) = a_{n+2}(n+1) + a_{n+2}(n+2)a_{n+1}(n+1)$, $A_1^3(n) = a_{n+2}(n+2)$, $A_2^3(n) = 1$; $A_0^4(n) = a_{n+3}(n+1) + a_{n+3}(n+2)a_{n+2}(n+1) + a_{n+3}(n+3)a_{n+2}(n+1) + a_{n+3}(n+3)a_{n+2}(n+2)a_{n+1}(n+1)$, $A_1^4(n) = a_{n+3}(n+2) + a_{n+3}(n+3)a_{n+2}(n+2)$, $A_2^4(n) = a_{n+3}(n+3)$, $A_3^4(n) = 1$.

3 Shift Properties for Coefficients

Theorem 3.1. *For* $r, m = 1, 2, \ldots$ *and* $j = 0, 1, \ldots, r-1$, *we have the following formula*

$$A_{j+m}^{r+m}(n) = A_j^r(n+m),\ n = 0, 1, 2, \ldots . \tag{6}$$

Proof. First we prove relation (6) for $m = 1$, hence the equality

$$A_{j+1}^{r+1}(n) = A_j^r(n+1),\ n = 0, 1, 2, \ldots,\ r = 1, 2, \ldots,\ j = 0, 1, \ldots, r-1, \tag{7}$$

by induction on r. The equality (7) is obvious for $r = 1$ and $j = 0$. We suppose that (7) is true for a natural number r, when $j = 0, 1, \ldots, r-1$ and n is arbitrary. In this case, using (5) we have

$$A_j^{r+1}(n+1) = \sum_{i=j+1}^{r} A_j^i(n+1)a_{n+r+1}(n+i+1) = \sum_{i=j+1}^{r} A_{j+1}^{i+1}(n)a_{n+r+1}(n+i+1) =$$

$$= \sum_{i=j+2}^{r+1} A_{j+1}^{i}(n) a_{n+r+1}(n+i) = A_{j+1}^{r+2}(n),\ j = 0, 1, \ldots, r-1.$$

This relation, also true for $j = r$, is exactly the relation (7) for r substituted with $r+1$. According to induction axiom, the relation (7) is true in the specified conditions.
Now we show the relation (6) by induction after m. According to (7), it is true for $m = 1$. We suppose relation (6) true for a natural number m. Then, using (7) we have

$$A_{j+m+1}^{r+m+1}(n) = A_{j+1}^{r+1}(n+m) = A_j^r(n+m+1).$$

According to the induction axiom, the relation (6) is true, in the specified conditions, for every natural number m. □

Definition 3.2. *For $r = 1, 2, \ldots$ and $j = 0, 1, \ldots, r-1$, we denote*

$$A_j^r = A_j^r(0). \tag{8}$$

With this notation, taking $n = 0$ and substituting m with n, formula (6) becomes

$$A_j^r(n) = A_{j+n}^{r+n},\ r = 1, 2, \ldots,\ j = 0, 1, \ldots, r-1,\ n = 0, 1, 2, \ldots, \tag{9}$$

therefore the coefficients $A_j^r(n)$ can be obtained from the coefficients A_j^r.

Example 3.3. *We have $A_{r-1}^r = 1,\ r = 1, 2, \ldots,\ A_0^1 = 1;\ A_0^2 = a_1(1), A_1^2 = 1;\ A_0^3 = a_2(1) + a_2(2)a_1(1), A_1^3 = a_2(2), A_2^3 = 1;\ A_0^4 = a_3(1) + a_3(2)a_1(1) + a_3(3)a_2(1) + a_3(3)a_2(2)a_1(1),\ A_1^4 = a_3(2) + a_3(3)a_2(2),\ A_2^4 = a_3(3),\ A_3^4 = 1.$*

4 Solving Linear Causal Recurrence Relations

First application of the iteration theorem 2.1 is to determine the general form of the solutions of the linear causal recurrence relation (1). This will be given in the following theorem.

Theorem 4.1. *The sequence $f(n)$ is solution of the linear causal recurrence relation (1) if and only if has the form*

$$f(n) = \sum_{j=0}^{n-1} A_j^n a_j(0) f(0),\ n = 1, 2, \ldots, \tag{10}$$

the coefficients A_j^n being given by the recurrence formula

$$A_j^n = \sum_{i=j+1}^{n-1} A_j^i a_{n-1}(i),\ j = 0, 1, \ldots, n-2, \tag{11}$$

and

$$A_{n-1}^{n} = 1. \tag{12}$$

Proof. Taking $n = 0$ in the formulas (2),(3) and (4), we obtain the formulas (10), (11) and (12), if the variable r is substituted with n. Reciprocally, if $f(n)$ is given by formula (10), then using (11) and (12) we have

$$\sum_{k=0}^{n} a_n(k)f(k) = a_n(0)f(0) + \sum_{k=1}^{n} a_n(k)\sum_{j=0}^{k-1} A_j^k a_j(0)f(0) =$$

$$= a_n(0)f(0) + \sum_{j=0}^{n-1}\sum_{k=j+1}^{n} A_j^k a_n(k)a_j(0)f(0) =$$

$$= a_n(0)f(0) + \sum_{j=0}^{n-1} A_j^{n+1} a_j(0)f(0) =$$

$$= \sum_{j=0}^{n} A_j^{n+1} a_j(0)f(0) = f(n+1),\ n = 1, 2, \ldots,$$

therefore the sequence $f(n)$ is solution of the relation (1). □

Remark. From the theorem 4.1 follows the existence and unicity of the solution $f(n)$ of the recurrence relation (1), when its first term $f(0)$ is given.

5 Two Particular Cases

We briefly present the two cases in which the coefficients $a_n(k)$ do not depend on one variable, solved by obviously telescoping procedures.
1) If the relation (1) has the *stationary* form

$$f(n+1) = \sum_{k=0}^{n} a(k)f(k),\ n = 1, 2, \ldots,$$

then its general term is

$$f(n) = (1 + a(n-1))(1 + a(n-2)) \cdots (1 + a(1))a(0)f(0).$$

2) If the relation (1) has the form

$$f(n+1) = a_n \sum_{k=0}^{n} f(k),\ n = 1, 2, \ldots,$$

then its general term is

$$f(n) = a_{n-1}(1 + a_{n-2})(1 + a_{n-3}) \cdots (1 + a_0)f(0).$$

6 General Term of a Linear Causal Recurrence Relation Expressed Relative to Some Initial Terms

Another application of the theorem 2.1 is to determine the general term of the equation (1) relative to a finite number of initial terms.

Theorem 6.1. *If the initial terms* $f(0),\ f(1) = a_0(0)f(0),\ \ldots\ ,\ f(m) = \sum_{j=0}^{m-1} A_j^m a_j(0)f(0)$, *are given for a fixed* $m = 0, 1, 2, \ldots$, *then the general term of the linear causal recurrence relation* (1) *is given by the formula*

$$f(n) = \sum_{k=0}^{m} \sum_{j=m}^{n-1} A_j^n a_j(k) f(k),\ n > m\ . \tag{13}$$

Proof. Substituting n with m and r with n, the formula (2) becomes

$$f(n+m) = \sum_{k=0}^{m} \sum_{j=m}^{n-1} A_j^n(m) a_{j+m}(k) f(k)\ , n \geq 1. \tag{14}$$

According to (9), equation (14) gives

$$f(n+m) = \sum_{k=0}^{m} \sum_{j=0}^{n-1} A_{j+m}^{n+m} a_{j+m}(k) f(k) = \sum_{k=0}^{m} \sum_{j=m}^{n+m-1} A_j^{n+m} a_j(k) f(k)\ . \tag{15}$$

Substituting n with $n - m$, for $n > m$, equation (15) reduces to (13). □

Example 6.2. *For* $m = 1, 2$, *we obtain from* (13) *following formulas*

$$f(n) = \sum_{j=1}^{n-1} A_j^n a_j(0) f(0) + \sum_{j=1}^{n-1} A_j^n a_j(1) f(1),\ n \geq 2, \tag{16}$$

$$f(n) = \sum_{j=2}^{n-1} A_j^n a_j(0) f(0) + \sum_{j=2}^{n-1} A_j^n a_j(1) f(1) + \sum_{j=2}^{n-1} A_j^n a_j(2) f(2),\ n \geq 3. \tag{17}$$

7 Closed Formulas for Coefficients

Theorem 7.1. *For* $n = 2, 3, \ldots$ *and* $m = 0, 1, \ldots, n-2$, *we have*

$$A_m^n = \sum_{j=m+1}^{n-1} A_j^n a_j(m+1)\ . \tag{18}$$

Proof. Applying (13) for m and $m+1$, one obtains for $n \geq m+2$,

$$f(n) = \sum_{k=0}^{m} \sum_{j=m}^{n-1} A_j^n a_j(k) f(k) = \sum_{k=0}^{m+1} \sum_{j=m+1}^{n-1} A_j^n a_j(k) f(k) \ ,$$

hence

$$\sum_{k=0}^{m} A_m^n a_m(k) f(k) + \sum_{k=0}^{m} \sum_{j=m+1}^{n-1} A_j^n a_j(k) f(k) =$$

$$\sum_{k=0}^{m} \sum_{j=m+1}^{n-1} A_j^n a_j(k) f(k) + \sum_{j=m+1}^{n-1} A_j^n a_j(m+1) f(m+1) \ ,$$

therefore

$$A_m^n \sum_{k=0}^{m} a_m^n(k) f(k) = \sum_{j=m+1}^{n-1} A_j^n a_j(m+1) f(m+1). \tag{19}$$

Using (1) for $n = m$, relation (19) becomes

$$A_m^n f(m+1) = \sum_{j=m+1}^{n-1} A_j^n a_j(m+1) f(m+1),$$

from which one obtains the formula (18). □

Example 7.2. *For $m = n-2, n-3, n-4$, from formula* (18) *one obtains* $A_{n-2}^n = a_{n-1}(n-1)$, $A_{n-3}^n = a_{n-1}(n-1)a_{n-2}(n-2) + a_{n-1}(n-2)$, $A_{n-4}^n = a_{n-1}(n-1)a_{n-2}(n-2)a_{n-3}(n-3) + a_{n-1}(n-2)a_{n-3}(n-3) + a_{n-1}(n-1)a_{n-2}(n-3) + a_{n-1}(n-3)$. *For $m = 0, 1$, formula* (18) *becomes*

$$A_0^n = \sum_{j=1}^{n-1} A_j^n a_j(1) = \sum_{j=1}^{n-1} A_0^j a_{n-1}(j), \ n \geq 2, \tag{20}$$

$$A_1^n = \sum_{j=2}^{n-1} A_j^n a_j(2) = \sum_{j=2}^{n-1} A_1^j a_{n-1}(j), \ n \geq 3. \tag{21}$$

8 A New Closed Formula for Solutions

In accordance with relations (9) and (18), we define $A_{-1}^0 = A_0^1(-1) = 1$ and

$$A_{-1}^n = A_0^{n+1}(-1) = \sum_{j=0}^{n-1} A_j^n a_j(0), \ n = 1, 2, \ldots. \tag{22}$$

Theorem 8.1. *The causal linear recurrence equation* (1) *has the solutions*

$$f(n) = A_{-1}^n f(0), \ n = 0, 1, \ldots \ . \tag{23}$$

Proof. Formula (23) results from (10) and (22). □

9 Particular Cases

We present here some particular cases contained especially in the work of H. W. Gould and Jocelyn Quaintance.

9.1 Bell Numbers

It is well known that the real numbers that satisfy the relation (1) with the coefficients $a_n(k) = \binom{n}{k}$ are *the Bell numbers* $f(n) = B(n)$. Aspects of iterating linear recurrence relations presented in a general setting in our paper and other related results were given in the ring of real numbers for Bell numbers in the works [7], [8]. See also [9].

9.2 Bell Polynomials

In the ring of polynomials and for the coefficients $a_n(t) = \binom{n}{k}t$, the solutions of the recurrence equation (1) are named *Bell polynomials*. Aspects of the iteration theory in this situation were given in [7].

9.3 Variant Sequences

Gould and Quaintance studied in [9] the so-called *variant sequences*, namely solutions in the ring of real numbers of the recurrence relation (1), with coefficients $a_n(k) = \binom{n}{k}a^k b^{n-k}$, where a and b are given real numbers. For $a = b = 1$ the Bell numbers are obtained and for $a = 1$ and $b = -1$, are obtained *the Uppuluri-Carpenter numbers*, given in the paper [15].

9.4 Floor and Roof Functions

In [10] other particular situations are given in which some recurrence relations are iterated.

10 Applications

10.1 Application to Generalized Fibonacci Sequences

For $p, q \in R$, let $F_n^{(p,q)} \in R$, $n = 0, 1, 2, \ldots$, be *the generalized Fibonacci sequence*, that has the initial values $F_0^{(p,q)} = 0$, $F_1^{(p,q)} = 1$ and satisfies the second order linear recurrence relation

$$F_{n+1}^{(p,q)} = pF_n^{(p,q)} + qF_{n-1}^{(p,q)},\ n = 1, 2, \ldots. \tag{24}$$

Example 10.1. $F_2^{(p,q)} = p,\ F_3^{(p,q)} = p^2 + q,\ F_4^{(p,q)} = p^3 + 2pq,\ F_5^{(p,q)} = p^4 + 3p^2q + q^2$.

Theorem 10.2. *The generalized Fibonacci sequences satisfy the iteration formula*

$$F_{n+r}^{(p,q)} = F_r^{(p,q)} F_{n+1}^{(p,q)} + qF_{r-1}^{(p,q)} F_n^{(p,q)},\ n = 0, 1, \ldots, r = 1, 2, \ldots . \tag{25}$$

Proof. We denote $f(n) = F_{n+1}^{(p,q)},\ n = 0, 1, 2, \ldots$. Taking $a_0(0) = a_1(1) = p,\ a_1(0) = q$, $a_n(k) = 0$ for $k = 0, 1, \ldots, n-2$, and $a_n(n-1) = q,\ a_n(n) = p$, for $n = 1, 2, \ldots$, it follows that the sequence $f(n)$ satisfies recurrence relation (1), hence its iterates are given by relation (2), from which it follows

$$F_{n+r}^{(p,q)} = f(n+r-1) = \sum_{k=0}^{n-1} \sum_{j=0}^{r-1} A_j^r(n-1) a_{n-1+j}(k) F_{k+1}^{(p,q)} =$$

$$= \sum_{j=0}^{r-1} A_j^r(n-1) a_{n-1+j}(n-1) F_n^{(p,q)} + \sum_{j=0}^{r-1} A_j^r(n-1) a_{n-1+j}(n-2) F_{n-1}^{(p,q)} =$$

$$= pA_0^r(n-1) F_n^{(p,q)} + qA_1^r(n-1) F_n^{(p,q)} + qA_0^r(n-1) F_{n-1}^{(p,q)} =$$

$$= A_0^r(n-1) F_{n+1}^{(p,q)} + qA_1^r(n-1) F_n^{(p,q)} . \tag{26}$$

For $j = 0, 1, \ldots$ and $n \geq j+2$, from (11) it follows

$$A_j^{n+1} = \sum_{k=j+1}^{n} A_j^k a_n(k) = pA_j^n + qA_j^{n-1}. \tag{27}$$

Because $A_j^{j+1} = 1$ and $A_j^{j+2} = a_{j+1}(j+1) = p$, by the uniqueness of the sequence of generalized Fibonacci numbers, one obtains $A_j^n = F_{n-j}^{(p,q)},\ n = 1, 2, \ldots$ and $j = 0, 1, \ldots, n-1$. Particularly, using (9), we have $A_0^r(n-1) = A_{n-1}^{r+n-1} = F_r^{(p,q)}$ and $A_1^r(n-1) = A_n^{r+n-1} = F_{r-1}^{(p,q)}$, therefore the relation (26) reduces to formula (25). □

Corollary. *The generalized Fibonacci sequences satisfy for* $n = 1, 2, \ldots$ *the formula*

$$F_{2n}^{(p,q)} = \left[F_{n+1}^{(p,q)} + qF_{n-1}^{(p,q)} \right] F_n^{(p,q)} , \tag{28}$$

from which it follows that the generalized Fibonacci number $F_{2n}^{(p,q)}$ *is divisible by* $F_n^{(p,q)}$ *and*

$$F_{2n+1}^{(p,q)} = \left(F_{n+1}^{(p,q)} \right)^2 + q \left(F_n^{(p,q)} \right)^2 . \tag{29}$$

Proof. Both formulas result from (25) for $r = n$, respectively for $r = n+1$.

Particular cases.

1. Fibonacci Numbers. In the ring of natural numbers, for $p = q = 1$, the sequence $F_n^{(1,1)} = F_n$ is *the sequence of Fibonacci numbers* that satisfy the recurrence relation $F_{n+1} = F_n + F_{n-1}$ and have initial values $F_0 = 0,\ F_1 = 1$. Then (25) reduces to the well known iteration formula

$$F_{n+r} = F_r F_{n+1} + F_{r-1} F_n,\ n = 0, 1, \ldots,\ r = 1, 2, \ldots\ . \tag{30}$$

From (28) and (29), one obtains the formulas

$$F_{2n} = (F_{n+1} + F_{n-1})\, F_n\ , \tag{31}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2\ . \tag{32}$$

Remark. Similar formulas, derived by (28) and (29), also occur in the other cases presented below, but are omitted.

2. Fibonacci Polynomials. In the ring of polynomials, for $p = x$ and $q = 1$, the polynomials $F_n^{(x,1)}(x) = F_n(x)$ are the usually *Fibonacci polynomials* that satisfy the recurrence relation $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ and have initial values $F_0(x) = 0$ and $F_1(x) = 1$. Then (25) reduces to the iteration formula

$$F_{n+r}(x) = F_r(x) F_{n+1}(x) + F_{r-1}(x) F_n(x),\ n = 0, 1, \ldots,\ r = 1, 2, \ldots\ . \tag{33}$$

3. Pell Numbers. In the ring of natural numbers, for $p = 2$ and $q = 1$, the numbers $F_n^{(2,1)} = P_n$ are *the Pell numbers.* They satisfy the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$ and have initial values $P_0 = 0$, $P_1 = 1$. Then (25) reduces to the iteration formula

$$P_{n+r} = P_r P_{n+1} + P_{r-1} P_n,\ n = 0, 1, \ldots,\ r = 1, 2, \ldots . \tag{34}$$

4. Jacobstahl Numbers. In the ring of natural numbers, for $p = 1$ and $q = 2$, the numbers $F_n^{(2,1)} = J_n$ are *the Jacobstahl numbers.* They satisfy the recurrence relation $J_{n+1} = J_n + 2J_{n-1}$ and have initial values $J_0 = 0$ and $J_1 = 1$. Then (25) reduces to the iteration formula

$$J_{n+r} = J_r J_{n+1} + 2J_{r-1} J_n,\ n = 0, 1, \ldots,\ r = 1, 2, \ldots\ . \tag{35}$$

10.2 Second Order Linear Recurrence Relations with Arbitrary Initial Values

For p and q integer numbers, let $S_n^{(p,q)} \in R,\ n = 0, 1, 2, \ldots$, be the *Horadam sequence*, see [13], that has the initial values $S_0^{(p,q)}$ and $S_1^{(p,q)}$, and satisfies the second order linear recurrence relation

$$S_{n+1}^{(p,q)} = pS_n^{(p,q)} + qS_{n-1}^{(p,q)},\ n = 1, 2, \dots, \tag{36}$$

Remark. The generalized Fibonacci numbers $S_n^{(p,q)} = F_n^{(p,q)}$ are solutions of equation (36) with initial values $S_0^{(p,q)} = 0$ and $S_1^{(p,q)} = 1$.

Theorem 10.3. *The solutions of the equation* (34) *with initial values* $S_0^{(p,q)}$ *and* $S_1^{(p,q)}$ *are given by the relation*

$$S_n^{(p,q)} = S_1^{(p,q)} F_n^{(p,q)} + qS_0^{(p,q)} F_{n-1}^{(p,q)},\ n = 1, 2, \dots, \tag{37}$$

and satisfy the iteration formula

$$S_{n+r}^{(p,q)} = F_r^{(p,q)} S_{n+1}^{(p,q)} + qF_{r-1}^{(p,q)} S_n^{(p,q)},\ n = 0, 1, \dots,\ r = 1, 2, \dots\ , \tag{38}$$

where $F_n^{(p,q)}$ *are generalized Fibonacci numbers.*

Proof. If $S_n^{(p,q)}$ is given by relation (37), using (24) we have

$$S_{n+1}^{(p,q)} = S_1^{(p,q)} F_{n+1}^{(p,q)} + qS_0^{(p,q)} F_n^{(p,q)} =$$

$$= S_1^{(p,q)} [pF_n^{(p,q)} + qF_{n-1}^{(p,q)}] + qS_0^{(p,q)} [pF_{n-1}^{(p,q)} + qF_{n-2}^{(p,q)}] =$$

$$= p[S_1^{(p,q)} F_n^{(p,q)} + qS_0^{(p,q)} F_{n-1}^{(p,q)}] + q[S_1^{(p,q)} F_{n-1}^{(p,q)} + qS_0^{(p,q)} F_{n-2}^{)p,q)}] = pS_n^{(p,q)} + qS_{n-1}^{(p,q)},$$

hence $S_n^{(p,q)}$ verify the equation (36). Using (37) and (25), we have

$$S_{n+r}^{(p,q)} = S_1^{(p,q)} F_{n+r}^{(p,q)} + qS_0^{(p,q)} F_{n+r-1}^{(p,q)} =$$

$$= S_1^{(p,q)} [F_r^{(p,q)} F_{n+1}^{(p,q)} + qF_{r-1}^{(p,q)} F_n^{(p,q)}] + qS_0^{(p,q)} [F_r^{(p,q)} F_n^{(p,q)} + qF_{r-1}^{(p,q)} F_{n-1}^{(p,q)}] =$$

$$= F_r^{(p,q)} [S_1^{(p,q)} F_{n+1}^{(p,q)} + qS_0^{(p,q)} F_n^{(p,q)}] + qF_{r-1}^{(p,q)} [S_1^{(p,q)} F_n^{(p,q)} + qS_0^{(p,q)} F_{n-1}^{(p,q)}] =$$

$$= F_r^{(p,q)} S_{n+1}^{(p,q)} + qF_{r-1}^{(p,q)} S_n^{(p,q)}\ .$$

□

Particular Cases

1. Lucas Numbers $S_n^{(1,1)} = L_n$ satisfy the recurrence relation $L_{n+1} = L_n + L_{n-1}$ and have the initial values $L_0 = 2,\ L_1 = 1$. For these numbers, (38) becomes

$$L_{n+r} = F_r L_{n+1} + F_{r-1} L_n,\ n = 0, 1, \dots,\ r = 1, 2, \dots\ . \tag{39}$$

2. Lucas Polynomials $S_n^{(x,1)}(x) = L_n(x)$ satisfy the recurrence relation $L_{n+1}(x) = xL_n(x) + L_{n-1}(x)$ and have the initial values $L_0(x) = 2$ and $L_1(x) = 1$. The relation (38) becomes

$$L_{n+r}(x) = F_r(x) L_{n+1}(x) + F_{r-1}(x) L_n(x),\ n = 0, 1, \dots,\ r = 1, 2, \dots\ . \tag{40}$$

3. Pell-Lucas Numbers $S_n^{(2,1)} = (PL)_n$ satisfy the recurrence relation $(PL)_{n+1} = 2(PL)_n + (PL)_{n-1}$ and have the initial values $(PL)_0 = 2$ and $(PL)_1 = 2$. The relation (38) becomes

$$(PL)_{n+r} = P_r(PL)_{n+1} + P_{r-1}(PL)_n,\ n = 0,1,\ldots,\ r-1,2,\ldots\ . \qquad (41)$$

4. Jacobstahl-Lucas Numbers $S_n^{(1,2)} = (JL)_n$ satisfy the recurrence relation $(JL)_{n+1} = (JL)_n + 2(JL)_{n-1}$ and have the initial values $(JL)_0 = 2$ and $(JL)_1 = 1$. The relation (38) becomes

$$(JL)_{n+r} = J_r(JL)_{n+1} + 2J_{r-1}(JL)_n,\ n = 0,1,\ldots,\ r = 1,2,\ldots\ . \qquad (42)$$

5. Chebyshev polynomials of first kind. $T_n(x) = S_n(2x,-1;1,x) = cos(n\,arccos(x))$ satisfy the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. For these polynomials we have

$$T_n(x) = xF_n(2x,-1)(x) - F_{n-1}(2x,-1)(x), \qquad (43)$$

and

$$T_{n+r}(x) = F_r(2x,-1)(x)T_{n+1}(x) - F_{r-1}(2x,-1)(x)T_n(x), \qquad (44)$$

for $n, r = 1, 2, \ldots$. Here $F_n(2x,-1)(x)$ are the generalized Fibonacci polynomials that satisfy the recurrence relation $F_{n+1}(2x,-1)(x) = 2xF_n(2x,-1)(x) - F_{n-1}(2x,-1)(x)$ and the initial conditions $F_0(2x,-1)(x) = 0$ and $F_1(2x,-1)(x) = 1$.

6. Chebyshev polynomials of second kind. $U_n(x) = S_n(2x,-1;1,2x)$ satisfy the same recurrence relation $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ and the initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. We have

$$U_n(x) = 2xF_n(2x,-1)(x) - F_{n-1}(2x,-1)(x), \qquad (45)$$

and

$$U_{n+r}(x) = F_r(2x,-1)(x)U_{n+1}(x) - F_{r-1}(2x,-1)(x)U_n(x). \qquad (46)$$

10.3 Quicksort Theory

The quicksort algorithm was developed by C.A.R. Hoare [12], (see [11] and [14]), by employing a divide and conquer strategy to divide a list into sub-lists. The average number of comparison over all permutations of the input sequence can be estimated accurately by solving the quicksort recurrence relation

$$C_{n+1} = n + \frac{2}{n+1}\sum_{k=0}^{n} C_k,\ n = 0,1,\ldots. \qquad (47)$$

with the initial value $C_0 = 0$. Taking $f(0) = 1$, $f(n) = C_n$, $n \geq 1$, and $a_n(0) = n$, $a_n(k) = \frac{2}{n+1}$, $k \geq 1, n = 0, 1, 2, \ldots$, it follows that the quicksort equation has the form (1). Using (10), we have

$$f(n) = \sum_{j=0}^{n-1} A_j^n a_j(0) f(0) = \sum_{j=1}^{n-1} j A_j^n. \tag{48}$$

Using (18), we obtain for $n = 3, 4, \ldots$ and $m = 0, 1, \ldots, n-3$,

$$A_m^n = \sum_{j=m+1}^{n-1} A_j^n a_j(m+1) = \sum_{j=m+1}^{n-1} \frac{2}{j+1} A_j^n =$$

$$= \frac{2}{m+2} A_{m+1}^n + \sum_{j=m+2}^{n-1} \frac{2}{j+1} A_j^n = \frac{2}{m+2} A_{m+1}^n + A_{m+1}^n = \frac{m+4}{m+2} A_{m+1}^n,$$

because $A_{m+1}^n = \sum_{j=m+2}^{n-1} \frac{2}{j+1} A_j^n$. Therefore, for $m \leq n-3$, we obtain

$$A_m^n = \frac{m+4}{m+2} A_{m+1}^n = \frac{(m+4)(m+5)}{(m+2)(m+3)} A_{m+2}^n = \frac{(m+4)(m+5)(m+6)}{(m+2)(m+3)(m+4)} A_{m+3}^n =$$

$$= \cdots = \frac{(m+4)(m+5)(m+6)\cdots(n-1)n(n+1)}{(m+2)(m+3)(m+4)\cdots(n-3)(n-2)(n-1)} A_{n-2}^n =$$

$$= \frac{n(n+1)}{(m+2)(m+3)} a_{n-1}(n-1) = \frac{n(n+1)}{(m+2)(m+3)} \frac{2}{n} = \frac{2(n+1)}{(m+2)(m+3)}\ . \tag{49}$$

Taking into account the above obtained relations and also the values $A_{n-1}^n = 1$ and $A_{n-2}^n = a_{n-1}(n-1) = \frac{2}{n}$, one obtains

$$f(n) = \sum_{j=1}^{n-3} j A_j^n + (n-2) A_{n-2}^n + (n-1) A_{n-1}^n =$$

$$= \sum_{j=1}^{n-3} \frac{2(n+1)j}{(j+2)(j+3)} + (n-2)\frac{2}{n} + n - 1 = 2(n+1) \sum_{j=1}^{n-3} \Big(\frac{3}{j+3} - \frac{2}{j+2}\Big) +$$

$$\frac{n^2+n-4}{n} = 2(n+1)\Big(\sum_{j=4}^{n} \frac{1}{j} + \frac{2}{n} - \frac{2}{3}\Big) + \frac{n^2+n-4}{n} = 2(n+1) \sum_{j=1}^{n} \frac{1}{j} - 4n\ . \tag{50}$$

Remark. The above obtained formula (50) is well-known, being usually proved by generating function method. We showed above how this formula can be deduced from the iteration theory of recurrence relations, but a simple and elementary proof by telescoping method can be given. So, from the quicksort recurrence relation (47) it follows

$$jC_j - (j-1)C_{j-1} = j(j-1) + 2\sum_{k=0}^{j-1} C_k - (j-1)(j-2) - 2\sum_{k=0}^{j-1} C_k = 2(j-1)+$$

$+2C_{j-1}$, hence $jC_j - (j+1)C_{j-1} = 2(j-1)$, or $\frac{1}{j+1}C_j - \frac{1}{j}C_{j-1} = \frac{2(j-1)}{j(j+1)}$. Writing the last relation for $j = 2, 3, \ldots, n$ and adding the obtained relations, it follows that $\frac{1}{n+1}C_n = 2\sum_{j=2}^{n} \frac{j-1}{j(j+1)} = 2\sum_{j=2}^{n}(\frac{2}{j+1} - \frac{1}{j}) = 2\sum_{j=3}^{n}\frac{1}{j} - 1 + \frac{4}{n+1} = 2\sum_{j=1}^{n}\frac{1}{j} - \frac{4n}{n+1}$, hence one obtains the formula (50).

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