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Inequalities for the polygamma function

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Abstract

In this paper, we prersent some inequalities involving the polygamma function.

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1 Introduction

The Euler gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

where $x > 0$.

The polygamma function ψ of order $n \in \mathbb{N}$ is defined by

$$\Psi_n(x) = \frac{d^{n+1}}{dx^{n+1}} \ln \Gamma(x),$$

where $x > 0$.

The polygamma function may be represented as

$$\Psi_n(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}} dt,$$

where $n \in \mathbb{N}$ and $x > 0$.

In 2006, Laforgia and Natalini [1] proved that

$$\Psi_m(x)\Psi_n(x) \geq \Psi^2_{\frac{m+n}{2}}(x)$$

where $m, n, \frac{m+n}{2} \in \mathbb{N}$ and $x > 0$.

In 2011, Sulaiman [2] gave the inequalities as follows.

$$\Psi_m^{1/p}(x)\Psi_n^{1/q}(x) \geq \Psi_{\frac{m}{p}+\frac{n}{q}}(x) \tag{1}$$

where $m, n, \frac{m}{p} + \frac{n}{q} \in \mathbb{N}$, $x > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\Psi_n^{1/p}(px)\Psi_n^{1/q}(qx) \geq \Psi_n\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \tag{2}$$

where n is a positive odd integer, $x, y > 1$, $\frac{1}{x} + \frac{1}{y} \leq 1$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$-\Psi_n^{1/p}(px)\Psi_n^{1/q}(qx) \leq \Psi_n\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \tag{3}$$

where n is a positive even integer, $x, y > 1$, $\frac{1}{x} + \frac{1}{y} \leq 1$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper, we present the generalizations for the inequalities (1), (2) and (3).

2 Results

Theorem 2.1. *Let $n_1, n_2, ..., n_k \in \mathbb{N}$, $x > 0$ and $p_1, p_2, ...p_k > 1$ be such that $\sum_{i=1}^k \frac{n_i}{p_i} \in \mathbb{N}$ and $\sum_{i=1}^k \frac{1}{p_i} = 1$. Then*

$$\prod_{i=1}^k \Psi_{n_i}^{1/p_i}(x) \geq \Psi_{\sum_{i=1}^k \frac{n_i}{p_i}}(x).$$

Proof. By the generalized Hölder inequality,

$$\begin{aligned}
\Psi_{\sum_{i=1}^k \frac{n_i}{p_i}}(x) &= (-1)^{\sum_{i=1}^k \frac{n_i}{p_i}+1} \int_0^\infty \frac{t^{\sum_{i=1}^k \frac{n_i}{p_i}} e^{-xt}}{1-e^{-t}} dt \\
&= (-1)^{\sum_{i=1}^k \frac{n_i+1}{p_i}} \int_0^\infty \prod_{i=1}^k \frac{t^{\frac{n_i}{p_i}} e^{-\frac{x}{p_i}t}}{(1-e^{-t})^{1/p_i}} dt \\
&\leq (-1)^{\sum_{i=1}^k \frac{n_i+1}{p_i}} \prod_{i=1}^k \left(\int_0^\infty \frac{t^{n_i} e^{-xt}}{1-e^{-t}} dt\right)^{1/p_i} \\
&= \prod_{i=1}^k \Psi_{n_i}^{1/p_i}(x).
\end{aligned}$$

□

We note on Theorem 2.1 that if $k = 2$ then we obtain the inequality (1).

Theorem 2.2. *Let n be a positive odd integer, and let $x_1, x_2, ..., x_k > 1$ and $p_1, p_2, ...p_k > 1$ be such that $\sum_{i=1}^k x_i \leq \prod_{i=1}^k x_i$ and $\sum_{i=1}^k \frac{1}{p_i} = 1$. Then*

$$\prod_{i=1}^k \Psi_n^{1/p_i}(p_i x_i) \geq \Psi_n\left(\sum_{i=1}^k \frac{x_i^{p_i}}{p_i}\right).$$

Proof. We note that

$$\prod_{i=1}^k x_i \leq \sum_{i=1}^k \frac{x_i^{p_i}}{p_i}. \tag{4}$$

By the generalized Hölder inequality,

$$\begin{aligned}
\Psi_n\left(\sum_{i=1}^k \frac{x_i^{p_i}}{p_i}\right) &= \int_0^\infty \frac{t^n e^{-t\left(\sum_{i=1}^k \frac{x_i^{p_i}}{p_i}\right)}}{1-e^{-t}} dt \\
&\leq \int_0^\infty \frac{t^n e^{-t\prod_{i=1}^k x_i}}{1-e^{-t}} dt \\
&\leq \int_0^\infty \frac{t^n e^{-t\sum_{i=1}^k x_i}}{1-e^{-t}} dt \\
&= \int_0^\infty \prod_{i=1}^k \frac{t^{\frac{n}{p_i}} e^{-x_i t}}{(1-e^{-t})^{1/p_i}} dt \\
&\leq \prod_{i=1}^k \left(\int_0^\infty \frac{t^n e^{-p_i x_i t}}{1-e^{-t}} dt\right)^{1/p_i} \\
&= \prod_{i=1}^k \Psi_n^{1/p_i}(p_i x_i).
\end{aligned}$$

□

We note on Theorem 2.2 that if $k = 2$ then we obtain the inequality (2).

Theorem 2.3. *Let n be a positive even integer, and let $x_1, x_2, ..., x_k > 1$ and $p_1, p_2, ...p_k > 1$ be such that $\sum_{i=1}^k x_i \leq \prod_{i=1}^k x_i$ and $\sum_{i=1}^k \frac{1}{p_i} = 1$. Then*

$$(-1)^{k+1}\prod_{i=1}^k \Psi_n^{1/p_i}(p_i x_i) \leq \Psi_n\left(\sum_{i=1}^k \frac{x_i^{p_i}}{p_i}\right).$$

Proof. By the inequality (4) and by the generalized Hölder inequality,

$$\begin{aligned}
\Psi_n\left(\sum_{i=1}^{k}\frac{x_i^{p_i}}{p_i}\right) &= -\int_0^\infty \frac{t^n e^{-t\left(\sum_{i=1}^{k}\frac{x_i^{p_i}}{p_i}\right)}}{1-e^{-t}}dt \\
&\geq -\int_0^\infty \frac{t^n e^{-t\prod_{i=1}^{k}x_i}}{1-e^{-t}}dt \\
&\geq -\int_0^\infty \frac{t^n e^{-t\sum_{i=1}^{k}x_i}}{1-e^{-t}}dt \\
&= -\int_0^\infty \prod_{i=1}^{k}\frac{t^{\frac{n}{p_i}}e^{-x_i t}}{(1-e^{-t})^{1/p_i}}dt \\
&\geq -\prod_{i=1}^{k}\left(\int_0^\infty \frac{t^n e^{-p_i x_i t}}{1-e^{-t}}dt\right)^{1/p_i} \\
&= (-1)^{k+1}\prod_{i=1}^{k}\Psi_n^{1/p_i}(p_i x_i).
\end{aligned}$$

□

We note on Theorem 2.3 that if $k=2$ then we obtain the inequality (3).

References

[1] A. Laforgia and P. Natalini, Turan-type inequalites for some special functions, J. Ineq. Pure Appl. Math., 2006, **7**(1), Art. 32.

[2] W. T. Sulaiman, Turan inequalites for the digamma and polygamma functions, South Asian J. Math., 2011, **1**(2), 49–55.

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