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Inequalities for the incomplete gamma function

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Abstract

In this paper, we prersent some inequalities involving the incomplete gamma function.

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1 Introduction

The gamma function is defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt,$$

where $a > 0$.

The incomplete gamma function is defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt,$$

where $a, x > 0$. We let $\Gamma(a, 0) = \Gamma(a)$.

In 2006, Ismail and Laforgia [1] gave the inequalities as follows.

$$\Gamma(a, x)\Gamma(a, y) \leq \Gamma(a, x + y)\Gamma(a, 0) \tag{1}$$

where $0 < a < 1$ and $x, y > 0$. If $a > 1$, then the inequality (1) is reversed.

In 2010, Sulaiman [2] gave the inequalities as follows.

$$\Gamma(a, x)\Gamma(a, y) \geq \Gamma(a, xy)\Gamma(a, 1) \tag{2}$$

where $a > 0$ and $x, y > 1$. If $0 < y < 1$, then the inequality (2) is reversed.

In this paper, we present the generalizations for the these inequalities.

2 Results

Theorem 2.1. *Let $0 < a < 1$ and $x > 0$ and let $0 \leq c < y$. Then*

$$\Gamma(a,x)\Gamma(a,y) \leq \Gamma(a,x+y-c)\Gamma(a,c). \tag{3}$$

Proof. Let $g(t) = t^{a-1}e^{-t}$, $F(t) = \dfrac{\Gamma(a,t)}{\Gamma(a,c)}$ and $G(t) = F(t+y-c) - F(t)F(y)$ for all $t > 0$. Then, for any $t > 0$,

$$\begin{aligned} G'(t) &= F'(t+y-c) - F'(t)F(y) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(1 - \frac{g(t+y-c)}{F(y)g(t)}\right) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(1 - \frac{e^{c-y}}{F(y)}\left(1 + \frac{y-c}{t}\right)^{a-1}\right). \end{aligned}$$

We note that $\left(1 + \dfrac{y-c}{t}\right)^{a-1}$ is increasing in $t > 0$ since $a < 1$ and $y > c$.

Let $H(t) = 1 - \dfrac{e^{c-y}}{F(y)}\left(1 + \dfrac{y-c}{t}\right)^{a-1}$ for all $t > 0$. Then H is decreasing.

We note that $G(c) = F(y) - F(c)F(y) = 0$ and $\lim\limits_{t\to\infty} G(t) = 0$.

By Roll's theorem, there is a point $p \in (c,\infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c,p)$ and $H(t) < 0$ for all $t \in (p,\infty)$. Then $G'(t) > 0$ for all $t \in (c,p)$ and $G'(t) < 0$ for all $t \in (p,\infty)$. This implies that $G(x) \geq 0$. Then $F(x+y-c) \geq F(x)F(y)$. Hence, we obtain the inequality (3). □

We note on Theorem 2.1 that if $c = 0$ then we obtain the inequality (1).

Theorem 2.2. *Let $a > 1$ and $x > 0$ and let $0 \leq c < y$. Then*

$$\Gamma(a,x)\Gamma(a,y) \geq \Gamma(a,x+y-c)\Gamma(a,c). \tag{4}$$

Proof. Let $g(t) = t^{a-1}e^{-t}$, $F(t) = \dfrac{\Gamma(a,t)}{\Gamma(a,c)}$ and $G(t) = F(t)F(y) - F(t+y-c)$ for all $t > 0$. Then, for any $t > 0$,

$$\begin{aligned} G'(t) &= F'(t)F(y) - F'(t+y-c) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(\frac{g(t+y-c)}{F(y)g(t)} - 1\right) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(\frac{e^{c-y}}{F(y)}\left(1 + \frac{y-c}{t}\right)^{a-1} - 1\right). \end{aligned}$$

We note that $\left(1+\dfrac{y-c}{t}\right)^{a-1}$ is decreasing in $t>0$ since $a>1$ and $y>c$.
Let $H(t)=\dfrac{e^{c-y}}{F(y)}\left(1+\dfrac{y-c}{t}\right)^{a-1}-1$ for all $t>0$. Then H is decreasing.
We note that $G(c)=F(c)F(y)-F(y)=0$ and $\lim_{t\to\infty} G(t)=0$.

By Roll's theorem, there is a point $p\in(c,\infty)$ such that $G'(p)=0$. Then $H(p)=0$. Then $H(t)>0$ for all $t\in(c,p)$ and $H(t)<0$ for all $t\in(p,\infty)$. Then $G'(t)>0$ for all $t\in(c,p)$ and $G'(t)<0$ for all $t\in(p,\infty)$. This implies that $G(x)\geq 0$. Then $F(x)F(y)\geq F(x+y-c)$. Hence, we obtain the inequality (4). □

Theorem 2.3. *Let $a,c>0$ and $x,y>c$. Then*

$$\Gamma(a,x)\Gamma(a,y)\geq\Gamma(a,\frac{xy}{c})\Gamma(a,c). \tag{5}$$

Proof. Let $g(t)=t^{a-1}e^{-t}$, $F(t)=\dfrac{\Gamma(a,t)}{\Gamma(a,c)}$ and $G(t)=F(t)F(y)-F(\frac{ty}{c})$ for all $t>0$. Then, for any $t>0$,

$$\begin{aligned} G'(t)&=F'(t)F(y)-\frac{y}{c}F'(\frac{ty}{c})\\ &=\frac{g(t)F(y)}{\Gamma(a,c)}\left(\frac{yg(\frac{ty}{c})}{cF(y)g(t)}-1\right)\\ &=\frac{g(t)F(y)}{\Gamma(a,c)}\left(\frac{y^a}{c^aF(y)}e^{-t(y-c)/c}-1\right). \end{aligned}$$

We note that $e^{-t(y-c)/c}$ is decreasing in $t>0$ since $y>c$.

Let $H(t)=\dfrac{y^a}{c^aF(y)}e^{-t(y-c)/c}-1$ for all $t>0$. Then H is decreasing.

We note that $G(c)=F(c)F(y)-F(y)=0$ and $\lim_{t\to\infty} G(t)=0$.

By Roll's theorem, there is a point $p\in(c,\infty)$ such that $G'(p)=0$. Then $H(p)=0$. Then $H(t)>0$ for all $t\in(c,p)$ and $H(t)<0$ for all $t\in(p,\infty)$. Then $G'(t)>0$ for all $t\in(c,p)$ and $G'(t)<0$ for all $t\in(p,\infty)$. This implies that $G(x)\geq 0$. Then $F(x)F(y)\geq F(\frac{xy}{c})$. Hence, we obtain the inequality (5). □

We note on Theorem 2.3 that if $c=1$ then we obtain the inequality (2).

Theorem 2.4. *Let $a>0$ and $0<y<c<x$. Then*

$$\Gamma(a,x)\Gamma(a,y)\leq\Gamma(a,\frac{xy}{c})\Gamma(a,c). \tag{6}$$

Proof. Let $g(t) = t^{a-1}e^{-t}$, $F(t) = \dfrac{\Gamma(a,t)}{\Gamma(a,c)}$ and $G(t) = F(\frac{ty}{c}) - F(t)F(y)$ for all $t > 0$. Then, for any $t > 0$,

$$\begin{aligned} G'(t) &= \frac{y}{c}F'(\frac{ty}{c}) - F'(t)F(y) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(1 - \frac{yg(\frac{ty}{c})}{cF(y)g(t)}\right) \\ &= \frac{g(t)F(y)}{\Gamma(a,c)}\left(1 - \frac{y^a}{c^aF(y)}e^{-t(y-c)/c}\right). \end{aligned}$$

We note that $e^{-t(y-c)/c}$ is increasing in $t > 0$ since $y < c$.

Let $H(t) = 1 - \dfrac{y^a}{c^aF(y)}e^{-t(y-c)/c}$ for all $t > 0$. Then H is decreasing.

We note that $G(c) = F(c)F(y) - F(y) = 0$ and $\lim_{t\to\infty} G(t) = 0$.

By Roll's theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \geq 0$. Then $F(\frac{xy}{c}) \geq F(x)F(y)$. Hence, we obtain the inequality (6). $\square$

References

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