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Gravity as an Ultra-Low Frequency Scattering Effect

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Abstract

Under the weak gravitational field approximation, and, for the 00-component of the stress-energy tensor, Einstein's field equations reduce to Newton's model for a gravitational field which is compounded in the Poisson equation for the gravitational potential. By considering Newton's field equation to be the result of an ultra-low frequency scattering effect (in the limiting case), this work introduce a model where a gravitational field is taken to be the result of an ultra-low frequency gravitational wave (propagating in a near-Cartesian Minkowski space) scattering from an object (a mass). Further, by using a scalar wave-field model for the propagation of an electromagnetic wave, the effect of a light wave with wavelength λ scattering from a gravitational potential is investigated from which a scaling relation for the intensity of the resulting diffraction pattern is derived. It is shown that the intensity associated with the diffraction pattern is proportional to λ^{-6} which may provide an explanation as to why Einstein rings observed in the optical spectrum appear to be blue.

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1 Introduction

Just as an accelerated charge particle radiates electromagnetic waves, so gravitational waves are generated by accelerating masses which, like electromagnetic waves, propagate with an upper limit of the velocity of light speed. Under the 'weak field approximation' which linearises Einstein's field equations, where it is assumed that a 'flat-space' is weakly perturbed by a gravitational wave, the wave function is determined by a solution to the wave equation. In this context, although there are some important differences between electromagnetic and gravitational wave propagation (which are briefly discussed in this paper) there are also many similarities. We assume that one of these similarities is that, like electromagnetic waves, gravitational waves can be scattered by massive objects (composed of 'matter waves'), a scattering effect which is induced by a change in the characteristic velocity of the waves (a velocity that is strictly less than light speed).

This paper briefly reviews the Newtonian and Einsteinian field equations and then develops a case in which the Newtonian field equation may be considered to be a special case in regard to the scattering of ultra-low frequency (scalar) waves. The purpose of this is to investigate a possible association (on a phenomenological basis) between a Newtonian gravitational field and the scattering of ultra-low frequency (scalar) waves. In this context, we consider the 'Diffraction' of a scalar electromagnetic wave-field from an ultra-low frequency scalar wave-field based on Helmholtz scattering theory. A scaling relationship is derived which may provide an explanation as to why Einstein rings (e.g. [1], [2]) appear to be blue in the visible spectrum [3], thereby providing an experimental verification for the approach and the model considered.

2 Newtonian and Einsteinian Field Equations: A Brief Overview

2.1 Newton's Field Equation

For a single body with a point mass m, a Newtonian gravitational field is given by

$$\mathbf{g}(\mathbf{r}) = -\hat{\mathbf{n}}\frac{Gm}{r^2},\ \hat{\mathbf{n}} \equiv \frac{\mathbf{r}}{r},\ r \equiv \mid \mathbf{r} \mid$$

where G is the Gravitational Constant ($= 6.6738 \times 10^{-11}$ m^3 kg^{-1} s^{-2}) and $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ is the three-dimensional vector so that $r \equiv \mid \mathbf{r} \mid = \sqrt{x^2 + y^2 + z^2}$. This field generates a force $\mathbf{F}$ that is experienced by some other point mass m' according to Newton's second law of motion, i.e.

$$\mathbf{F}(\mathbf{r}) = m'\mathbf{g}(\mathbf{r})$$

Because gravitational fields are conservative (i.e. the work done by gravity from one position to another is path-independent) we can consider the gravitational field to have a potential $\phi(\mathbf{r})$ defined by the equation

$$\nabla\phi(\mathbf{r}) = \mathbf{g}(\mathbf{r})$$

where

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

from which it follows that

$$\phi(\mathbf{r}) = G\frac{m}{r}$$

Further, if we consider a body to have mass density $\rho(\mathbf{r})$ (inhomogeneous mass per unit volume with units of kg m^{-3}) over a finite region of space $\mathbf{r} \in \mathbb{R}^3$, then the above equation can be considered to be a (Green's function) solution to Poisson's equation

$$\nabla^2\phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r}),\ \mathbf{r} \in \mathbb{R}^3 \mapsto \rho(\mathbf{r}) \tag{1}$$

where ∇^2 is the Laplcian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

given that, for a point mass defined by the Dirac delta function, i.e. $\rho(\mathbf{r}) = m\delta^3(\mathbf{r})$,

$$\phi(\mathbf{r}) = 4\pi G\frac{1}{4\pi r} \otimes_{\mathbf{r}} \rho(\mathbf{r}) \equiv G \int\limits_{\mathbf{r}' \in \mathbb{R}^3} \frac{\rho(\mathbf{r}')}{\mid \mathbf{r} - \mathbf{r}' \mid} d^3\mathbf{r}' = Gm \int\limits_{\mathbf{r}' \in \mathbb{R}^3} \frac{\delta(\mathbf{r}')}{\mid \mathbf{r} - \mathbf{r}' \mid} d^3\mathbf{r}' = \frac{Gm}{r}$$

Equation (1) is the Newtonian (gravitational) field equation for a mass density source function $\rho(\mathbf{r})$ and scalar gravitational potential $\phi(\mathbf{r})$ which has units of energy per unit mass and dimensions of $[\text{L}]^2[\text{T}]^{-2}$ where L and T are taken to denote length and time, respectively.

2.2 Einstein's Field Equations

Einstein's field equations (of General Relativity) are given by [4]

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4} T_{\mu\nu} \tag{2}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor, G is Newton's Gravitational Constant and c_0 is the speed of light, the summation over the indices μ and ν being implied with both indices ranging over $0, 1, 2, 3$ corresponding to $c_0 t, x, y, z$, respectively. Comparing Equation (2) with Equation (1) we see that the source function is an energy-momentum tensor (which includes both mass-densities and currents) and the gravitational potential is replaced with a metric tensor.

When the gravitational field is weak and the sources are moving very slowly compared to light-speed, General Relativity reduces to the Newtonian theory of gravity, and the metric tensor can be expanded in terms of the gravitational potential. Equation (2) describes wave behaviour which can be examined analytically by considering a flat (Minkowski) space perturbed by a wave amplitude metric denoted by $h^{\alpha\beta}$. This is the case when a source taken to be generating gravitational waves (an accelerating mass) is far from the point in space-time at which the gravitational waves are observed - the gravitational field is very weak and the space-time approximates to that of a Minkowski space. Since space-time is flat in the absence of a gravitational field, a weak gravitational field can be defined as one in which space-time is 'nearly' flat which is the basis for the approximation used. This approximation linearises the Einstein field equations of General Relativity, and yields the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) h^{\alpha\beta} = \frac{16\pi G}{c_0^4} T^{\alpha\beta} \tag{3}$$

which for empty spacetime has the homogeneous form

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) h^{\alpha\beta} = 0 \tag{4}$$

In both electromagnetism and gravitation the field equations are taken to be fundamental and in both cases, wave motion is a consequence of an analysis of these field equations within the context of a Gauge Transformation. Thus, an accelerated charged particle is a source of electromagnetic radiation (Maxwell's equations) and an accelerated mass is a source of gravitational waves (Einstein equations). In both cases, the waves propagate at light speed (in free space) and the energy scales according to an inverse square law over all frequencies where frequency shifts may occur. However, electromagnetic waves are vector fields which interact over intermediate scales whereas gravitational waves are metric tensor fields which interact over very large scales. Further, electromagnetic waves originate from the small-scale incoherent acceleration of charged particles with low electric field strengths whereas as gravitational waves are taken to be the 'product' of bodies with high mass densities undergoing coherent motion. The lowest 'order' of the radiation field pattern for an electromagnetic wave is a dipole and that of a gravitational wave is a quadrupole. In this paper we consider another similarity between the two fields, namely, that like electromagnetic waves, gravitational waves can be scattered by a mass, the object being taken to change the wave-velocity by an amount that is strictly less than light speed.

2.3 On the Cosmological Constant

In 1917, Einstein considered Newton's field equation - Equation (1). From this equation it is clear that $\rho(\mathbf{r})$ must tend to zero as the domain over which the equation applies becomes infinite unless the gravitation field is infinite as well. But this is incompatible with there being no net gravitational force (as compounded in the equation $\nabla\phi(\mathbf{r}) = 0$) on matter in an extended uniformly dense universe. If the distribution of mass over a region of space with radius R were uniform with constant mass density ρ_0 say, then

$$\phi = \rho_0 G \int\limits_{\mathbf{r}\in\mathbb{R}^3} \frac{1}{r} d^3\mathbf{r} = 4\pi\rho_0 G \int\limits_0^R r dr = 2\pi\rho_0 G R^2$$

showing that $\phi \to \infty$ as $R \to \infty$. A resolution to this problem is possible if we consider the following modification to Newton's field equation:

$$\nabla^2\phi - \Lambda\phi = 4\pi G\rho$$

whose Green's function solution for uniform density ρ_0 is

$$\phi = \rho_0 G \int\limits_{\mathbf{r}\in\mathbb{R}^3} \frac{\exp(-\sqrt{\Lambda}r)}{r} d^3\mathbf{r} = 4\pi\rho_0 G \int\limits_0^R \exp(-\sqrt{\Lambda}r) r dr$$

$$= -4\pi\rho_0 G \left[\frac{r\exp(-\sqrt{\Lambda}r)}{\sqrt{\Lambda}} + \frac{\exp(-\sqrt{\Lambda}r)}{\Lambda} \right]_0^R = \frac{4\pi\rho_0 G}{\Lambda}, \; R \to \infty$$

giving a uniform constant potential so that $\nabla\phi = 0$.

The factor Λ is the Cosmological Constant and was originally considered by Einstein to be erroneous when the universe was understood to be expanding and not static but has more recently been reconsidered as a possible explanation for the force associated with dark energy that appears to be counteracting gravity causing the universe to expand at an increasing pace.

In terms of Einstein's field equations the Cosmological Constant is introduce by modifying Equation (2) to the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c_0^4} T_{\mu\nu}$$

where $g_{\mu\nu}$ is the metric tensor.

3 Gravity as a Low Frequency Scattering Effect

Neither Equation (1) nor Equation (2) explain why Gravity exists only how it behaves. Equation (1) requires that the gravitational force be transmitted instantaneously (instantaneous action at a distance) which is incompatible with the concept of any field 'radiating' with a maximum velocity of light-speed. Equation (2) describes the local curvature of space-time due to energy and momentum, other particles being taken to move along trajectories determined by the geometry of space-time, or, paraphrasing a quotation from John Wheeler, 'Space-time tells matter how to move, and matter tells space-time how to curve' [5]. However, in both cases, it is important to respect a statement from Isaac Newton himself - 'I have told how it works not why it works'.

Embodied in Equation (2) is the concept of a gravitational force 'radiating' at a finite speed which eliminates the concept of instantaneous action at a distance. In this context, and, on a strictly phenomenological basis, we consider Equation (1) in terms of a limiting case in which $\phi(\mathbf{r})$ is taken to be an ultra-low frequency wave function scattering from an inhomogeneity with wave velocity $c(\mathbf{r})$ which is of compact support and is strictly less than or equal to light speed.

3.1 Derivation of a Newtonian Gravitation Potential from the Wave Equation

Consider Φ be a solution to the equation

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\right)\Phi(\mathbf{r},t) = 0 \tag{5}$$

Let

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2}[1+\gamma(\mathbf{r})],\ c(\mathbf{r}) \leq c_0\forall\mathbf{r}$$

where $\gamma(\mathbf{r}) \geq 0$ is a dimensionless function of compact support $\gamma(\mathbf{r})\exists\forall\mathbf{r} \in \mathbb{R}^3$, so that we can write

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\mathbf{r},t) = -\frac{\gamma(\mathbf{r})}{c_0^2}\frac{\partial^2}{\partial t^2}\Phi(\mathbf{r},t)$$

Let $\Phi(\mathbf{r},t) = \phi(\mathbf{r},\omega)\exp(i\omega t)$ where ω is the angular temporal frequency. Then with $k = \omega/c_0 = 2\pi/\lambda$ (where λ is the wavelength),

$$\left(\nabla^2 + k^2\right)\phi(\mathbf{r},k) = -k^2\gamma(\mathbf{r})\phi(\mathbf{r},k) \tag{6}$$

whose Green's function solution is

$$\phi(\mathbf{r},k) = \phi_i(\mathbf{r},k) + \phi_s(\mathbf{r},k) \tag{7}$$

where ϕ_s is the scattered wave function given by (Born series solution)

$$\phi_s(\mathbf{r},k) = k^2 g(r,k)\otimes_{\mathbf{r}}\gamma(\mathbf{r})\phi(\mathbf{r},k) = k^2 g(r,k)\otimes_{\mathbf{r}}\gamma(\mathbf{r})\phi_i(\mathbf{r},k)$$
$$+k^4 g(r,k)\otimes_{\mathbf{r}}\gamma(\mathbf{r})\left[g(r,k)\otimes_{\mathbf{r}}\gamma(\mathbf{r})\phi_i(\mathbf{r},k)\right] + ... \tag{8}$$

$\phi_i(\mathbf{r},k)$ is the incident wave function which satisfies

$$\left(\nabla^2 + k^2\right)\phi_i(\mathbf{r},k) = 0,$$

$g(r,k)$ is the out-going free-space Green's

$$g(r,k) = \frac{\exp(ikr)}{4\pi r}$$

and $\otimes_{\mathbf{r}}$ denotes the (three-dimensional) convolution integral for $\mathbf{r} \in \mathbb{R}^3$.

For empty space, and, according to the Newtonian model of gravity, the gravitational potential is a scalar and is taken to satisfy the Laplace equation $\nabla^2\phi(\mathbf{r}) = 0$ which might be regarded - given Equation (6) - as the limit of a wave equation where the characteristic speed of transmission tends to infinity, i.e. $k = \omega/c_0 \to 0$. This leads to the principle of instantaneous action at a distance when it becomes impossible to associate a wave-length with a given frequency. Einstein's model for gravity yields a wave equation for the propagation of gravitational waves which are tensorial fields, compress and stretch space-time, are time-varying and whose amplitude spectrum depends on the source that is emitting the gravitational waves. In the following theorem, we consider the scattering of an existing scalar wave-field in the limit as $k \to 0$.

Theorem 3.1 Equation (7) yields an exact solution for the scattered field in the limit as $k \to 0$, i.e.

$$\phi_s(\mathbf{r}) = \lim_{k\to 0} \frac{\phi(\mathbf{r},k) - \phi_i(\mathbf{r},k)}{k^2} = g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k)$$

Proof of Theorem 3.1 From Equations (7) and (8) it is clear that we can write

$$\phi(\mathbf{r},k) - \phi_i(\mathbf{r},k) = k^2 g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k) + k^4 g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r}) \left[g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k)\right] + ...$$

so that

$$\frac{\phi(\mathbf{r},k) - \phi_i(\mathbf{r},k)}{k^2} = g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k) + k^2 g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r}) \left[g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k)\right] + ...$$

and thus, since

$$\lim_{k\to 0} \left[k^2 g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r}) \left[g(r,k) \otimes_{\mathbf{r}} \gamma(\mathbf{r})\phi_i(\mathbf{r},k)\right] + ...\right] = 0$$

the result is obtained.

Corollary 3.1 If the support of $\gamma(\mathbf{r})$ is finite, we can consider an exact solution for the scattered field given by (in the asymptotic sense as $k \to 0$ but where, for notational convenience, we replace k with k_0 to denote the ultra-low frequency case where $k_0 << 1$)

$$\phi_s(\mathbf{r},k_0) = \frac{k_0^2}{4\pi r} \otimes_{\mathbf{r}} \gamma(\mathbf{r}),\ k_0 << 1 \tag{9}$$

which is the solution of

$$\nabla^2\phi_s(\mathbf{r},k_0) = -k_0^2\gamma(\mathbf{r}) \tag{10}$$

Equation (10) is Poisson's equation for low value coefficient k_0^2 given that the Green's function solution to this equation is

$$\phi_s(\mathbf{r}, k_0) = \frac{k_0^2}{4\pi r} \otimes_{\mathbf{r}} \gamma(\mathbf{r})$$

Thus, in a phenomenological context, we consider a Newtonian gravitation potential to be the scattered wave function when the wavelength approaches infinity, the origin of a (Newtonian) gravitational field then being taken to be due to the scattering of ultra-long wavelength scalar waves incident upon an inhomogeneous object that changes the wave velocity to a velocity strictly less than light speed. Note that the asymptotic limit $k_0 \to 0$ used, implies that $\phi_s(\mathbf{r}, k_0)$ will be a relatively weak field.

Remark 3.1 The Newtonian theory of gravity is conventionally taken to be a static model for gravity whereas Einstein's theory of gravity implies wave properties as defined by Equation (3) under the weak field approximation. In the context of a Newtonian gravitational field being the 'product' of a low frequency scattering effect of a scalar wave-field, Equation (4) can be cast in the form

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) h^{\alpha\beta}(\mathbf{r}, t) = -\frac{\gamma(\mathbf{r})}{c_0^2}\frac{\partial^2}{\partial t^2} h^{\alpha\beta}(\mathbf{r}, t)$$

to which a tensorial Green's function solution may be applied to evaluate the tensorial scattered field $h_s^{\alpha\beta}(\mathbf{r}, t)$ under the Born approximation, giving, for an out-going wave

$$h_s^{\alpha\beta}(\mathbf{r}, t) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} \int\limits_{\mathbf{r}' \in \mathbb{R}^3} \frac{\Gamma^{\alpha\beta}(\mathbf{r}', t- \mid \mathbf{r} - \mathbf{r}' \mid /c_0)}{4\pi \mid \mathbf{r} - \mathbf{r}' \mid} d^3\mathbf{r}'$$

$$\sim \frac{1}{4\pi c_0^2 r}\frac{\partial^2}{\partial t^2} \int\limits_{\mathbf{r}' \in \mathbb{R}^3} \Gamma^{\alpha\beta}(\mathbf{r}', t- \mid \mathbf{r} - \mathbf{r}' \mid /c_0) d^3\mathbf{r}' \tag{11}$$

as $r \to \infty$ where, for an incident tensorial wave-field $h_i^{\alpha\beta}(\mathbf{r}, t)$, $\Gamma^{\alpha\beta}(\mathbf{r}, t) = \gamma(\mathbf{r}) h_i^{\alpha\beta}(\mathbf{r}, t)$.

Remark 3.2 Poisson's equation for the gravitational potential can be recovered from Einstein's equation under the weak field approximation by considering the 00-component of the stress-energy tensor. In this sense, the low-frequency scattering theory presented here, which allows the scattered field $\phi_s(\mathbf{r}, k_0)$ to be cast in terms of Poisson's equation with coefficient k_0^2 - Equation (10) - can, on a strictly phenomenological basis, be interpreted in terms of the scattering of a gravitational wave propagating in a near-Cartesian coordinate (flat-space) system.

3.2 Dimensional Analysis

In Equation (10), $\phi_s(\mathbf{r}, k_0)$ is dimensionless whereas in Equation (1) the dimensions of $\phi(\mathbf{r})$ are $[\mathrm{L}]^2[\mathrm{T}]^{-2}$. We can evaluate the wavelength associated with a Newtonian gravitational scattering model from the equation $k^2 = 4\pi G/c_0^2$ given that, on the grounds of a dimensional analysis, we can write

$$G\left[\frac{1}{r}\otimes_{\mathbf{r}}\rho(\mathbf{r})\right] := \frac{k_0^2 c_0^2}{4\pi}\left[\frac{1}{r}\otimes_{\mathbf{r}}\gamma(\mathbf{r})\right] \quad [\mathrm{L}]^2[\mathrm{T}]^{-2}$$

This is equivalent to considering the case where the two potentials are related by $\phi(\mathbf{r}) := c_0^2\phi_s(\mathbf{r}, k_0)$ and yields a wavelength (with $c_0 = 2.9979 \times 10^8$) of $\lambda = 6.5 \times 10^{13}$ metres and a corresponding frequency of 4.6×10^{-6}Hz. Given that one light year is 9.4605×10^{15} metres this equates to a wavelength of approximately 0.007 of a light year. In this respect, we are attempting to explain the generation of a (Newtonian) gravitational field in terms of the scattering of an ultra-long wavelength scalar wave, an approach that represents a 'paradigm shift' in regard to the origins of a field with respect to the *a priori* existence of a wave-field. Here the scattering of scalar gravitational waves (in the Newtonian sense) is taken to generate a gravitational field.

4 Diffraction of an Ultra-low Frequency Scattered Wave-field

For notational convenience, let the potential $\phi_s^{(0)}(\mathbf{r}, k_0 := \phi_s(\mathbf{r}, k_0)$ be taken to be the solution to

$$\nabla^2\phi_s^{(0)}(\mathbf{r}, k_0) = -k_0^2\gamma(\mathbf{r}),\ k_0 << 1 \tag{12}$$

and consider the wave function $\phi_s(\mathbf{r}, k)$ for $k >> 1$ given by

$$\phi_s(\mathbf{r}, k) = k^2 g(r, k)\otimes_{\mathbf{r}}\gamma(\mathbf{r})\phi_i(\mathbf{r}, k) \tag{13}$$

which is the solution to Equation (6) under the Born approximation when $k^2\|\gamma(\mathbf{r})\| << 1$.

This solution for $\phi_s(\mathbf{r}, k)$ provides a solution for the (near-field) diffraction pattern generated by a scattering function $\gamma(\mathbf{r})\exists\forall\mathbf{r}\in\mathbb{R}^3$. However, it is important to note that the potential $\phi_s^{(0)}(\mathbf{r}, k_0)$ exists both within the scatterer and beyond the (compact) support of the scatterer. In this context, the question then arises as to what the effect will be of $\phi_s(\mathbf{r}, k)$ scattering from $\phi_s^{(0)}(\mathbf{r}, k_0)$, $\forall\mathbf{r}\notin\mathbb{R}^3$ rather than from the scatterer itself given that the field $\phi_s^{(0)}(\mathbf{r}, k_0)$ is taken to exist within and beyond the finite spatial extent of the scatterer, noting that $\phi_s^{(0)}(\mathbf{r}, k_0)$ is not of compact support as it is given by the convolution of a function of compact support with the inverse function $r^{-1},\ r\to\infty$.

Thus, since, from Equation (15), we can write

$$\gamma(\mathbf{r}) = -\frac{1}{k_0^2}\nabla^2\phi_s^{(0)}(\mathbf{r}, k_0),\ k_0 << 1$$

we study an equation for $\phi_s(\mathbf{r}, k)$ - Equation (13) - given by

$$\phi_s(\mathbf{r}, k) = -\frac{k^2}{k_0^2}g(r, k)\otimes_{\mathbf{r}}[\phi_i(\mathbf{r}, k)\nabla^2\phi_s^{(0)}(\mathbf{r}, k_0)] \tag{14}$$

where

$$\phi_s^{(0)}(\mathbf{r}, k_0) = \frac{k_0^2}{4\pi r} \otimes_{\mathbf{r}} \gamma(\mathbf{r}) \tag{15}$$

and the convolution over $\mathbf{r}$ in Equation (14) is taken to be over all space whereas the convolution over $\mathbf{r}$ in Equation (15) is over $\mathbb{R}^3$.

4.1 Far-field Analysis for an Infinitely Thin Scattering Function

Consider the case where the function $\gamma(\mathbf{r})$ is taken to be infinitely thin (in dimension z) so that we can write $\gamma(\mathbf{r}) = \gamma(x, y)\delta(z)$ and where a simple plane wave-field travelling in the z-direction is incident on this 'infinitely thin' scattering function so that we can write $\phi_i(\mathbf{r}, k) = \exp(ikz)$.

In this case, given that, from Equation (15)

$$\nabla^2 \phi_s^{(0)}(\mathbf{r}, k_0) = \nabla^2 \left[\frac{k_0^2}{4\pi r} \otimes_{\mathbf{r}} \gamma(x, y)\delta(z)\right] = \nabla^2 \left[\frac{k_0^2}{4\pi r} \otimes_x \otimes_y \gamma(x, y)\right]$$

where $\otimes_x$ and $\otimes_y$ denote the covolution integrals over x and y, respectively, the scattered field $\phi_s(\mathbf{r}, k)$ can be written in the form

$$\phi_s(\mathbf{r}, k) = -k^2 g(r, k) \otimes_{\mathbf{r}} \exp(ikz)\nabla^2 \left[\frac{1}{4\pi r} \otimes_x \otimes_y \gamma(x, y)\right] \tag{16}$$

Theorem 4.1 In the far-field Equation (16) becomes

$$\phi_s(r, k) = \frac{\exp(ikr)}{16\pi^2 r} A_s(u, v, z, k)$$

where

$$A_s(u, v, z, k) = -k^2 \mathcal{F}_2 \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int \frac{dz}{\sqrt{x^2 + y^2 + z^2}} \otimes_x \otimes_y \gamma(x, y)\right]$$
$$+k^2 \mathcal{F}_2 \left[\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \otimes_x \otimes_y \gamma(x, y)\right] \tag{17}$$

is the (foward) scattering amplitude and $\mathcal{F}_2$ is the two-dimensional Fourier operator:

$$\mathcal{F}_2[f(x, y)] = \int\int f(x', y') \exp(-iux') \exp(-ivy') dx'dy', \quad u = \frac{kx}{z}, \ v = \frac{ky}{z}$$

Proof of Theorem 4.1 From Equation (16), let

$$f(\mathbf{r}, k) = \exp(ikz) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(\frac{1}{4\pi r} \otimes_x \otimes_y \gamma(x, y)\right) + \exp(ikz) \frac{\partial^2}{\partial z^2} \left(\frac{1}{4\pi r} \otimes_x \otimes_y \gamma(x, y)\right)$$

so that

$$\phi_s(\mathbf{r},k) = -k^2 g(r,k)\otimes_{\mathbf{r}} f(\mathbf{r},k) = -k^2\int \frac{\exp(ik\mid \mathbf{r}-\mathbf{r}'\mid)}{4\pi\mid \mathbf{r}-\mathbf{r}'\mid} f(\mathbf{r}',k)d^3\mathbf{r}'$$

$$= -k^2\frac{\exp(ikr)}{4\pi r}\int \exp(-ik\hat{\mathbf{n}}\cdot\mathbf{r}')f(\mathbf{r}',k)d^3\mathbf{r}'$$

in the far field, given that, for $\mid \mathbf{r}' \mid / \mid \mathbf{r} \mid << 1$,

$$\mid \mathbf{r}-\mathbf{r}'\mid \sim r-\hat{\mathbf{n}}\cdot\mathbf{r}',\ \hat{\mathbf{n}}=\frac{\mathbf{r}}{r},\ r\equiv\mid \mathbf{r}\mid$$

Further, if both $x^2/z^2 << 1$ and $y^2/z^2 << 1$ then $\hat{\mathbf{n}} \simeq \mathbf{r}/z$ so that with $u = kx/z$ and $v = ky/z$ we can write

$$\phi_s(\mathbf{r},k) = -k^2\frac{\exp(ikr)}{4\pi r}\int\int\int \exp(-iux')\exp(-ivx')\exp(-ikz')f(\mathbf{r}',k)dx'dy'dz'$$

Interchanging the independent vectors $\mathbf{r}$ and $\mathbf{r}'$ (for notational convenience alone) we can then write

$$\phi_s(\mathbf{r}',k) = -k^2\frac{\exp(ikr')}{4\pi r'}\int\int\int \exp(-iux)\exp(-ivx)\exp(-ikz)f(x,y,z,k)dxdydz$$

$$= k^2\frac{\exp(ikr')}{16\pi^2 r'}A_s(u,v,z,k)$$

where

$$A_s(u,v,z,k) =$$

$$-k^2\int\int dxdy\exp(-iux)\exp(-ivy)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\int\frac{dz}{\sqrt{x^2+y^2+z^2}}\otimes_x\otimes_y\gamma(x,y)\right)$$

$$-k^2\int\int dxdy\exp(-iux)\exp(-ivy)\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\otimes_x\otimes_y\gamma(x,y)\right)$$

$$= -k^2\int\int dxdy\exp(-iux)\exp(-ivy)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\int\frac{dz}{\sqrt{x^2+y^2+z^2}}\otimes_x\otimes_y\gamma(x,y)\right)$$

$$+k^2\int\int dxdy\exp(-iux)\exp(-ivy)\left(\frac{z}{\sqrt{(x^2+y^2+z^2)^3}}\otimes_x\otimes_y\gamma(x,y)\right)$$

and the result is obtained.

Theorem 4.2 In the limit $z \to 0$, Equation (17) can be written as

$$A_s(u,v,k) \sim -2\pi k^2\sqrt{u^2+v^2}\widetilde{\gamma}(u,v) \tag{18}$$

where (with $\leftrightarrow$ denoting the two-dimensional Fourier transformation)

$$\widetilde{\gamma}(u,v)\leftrightarrow\gamma(x,y)$$

Proof of Thoerem 4.2 Noting that [6]

$$\int\frac{dz}{\sqrt{x^2+y^2+z^2}}=\ln\mid z+\sqrt{z^2+x^2+y^2}\mid$$

and using the approximation[1]

$$\ln\mid z+\sqrt{z^2+x^2+y^2}\mid\sim 1-\frac{1}{\mid z+\sqrt{z^2+x^2+y^2}\mid}$$

we obtain

$$A_s(u,v,k)\sim k^2\mathcal{F}_2\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{1}{\sqrt{x^2+y^2}}\otimes_x\otimes_y\gamma(x,y)\right]$$

$$=k^2\mathcal{F}_2\left[\frac{1}{\sqrt{x^2+y^2}}\otimes_x\otimes_y\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\gamma(x,y)\right],\ z\to 0$$

Finally, using the convolution theorem, and, noting that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\gamma(x,y)\leftrightarrow-(u^2+v^2)\widetilde{\gamma}(u,v)$$

where

$$\widetilde{\gamma}(u,v)=\mathcal{F}_2[\gamma(x,y)]$$

and that [9]

$$\frac{1}{\sqrt{x^2+y^2}}\leftrightarrow\frac{2\pi}{\sqrt{u^2+v^2}}$$

the result is obtained.

Remark 4.1 The scattering amplitude generated by $\gamma(x,y)\delta(z)$ itself is given by

$$A_s(u,v,k)\sim k^2\widetilde{\gamma}(u,v)\tag{19}$$

which is the result of applying a far-field analysis to Equation (13). Thus, Equation (18) differs from the usual far field diffraction pattern - Equation (19) - for an infinitely thin scatterer, the difference being compounded in the extra spatial frequency factor $\sqrt{u^2+v^2}$.

Remark 4.2 For a point scatterer when $\gamma(x,y)=\delta(x)\delta(y)$ and $\widetilde{\gamma}(u,v)=1$ Equation (19), shows that the intensity $\mid A_s(u,v,k)\mid^2$ is proportional to λ^{-4} (Rayleigh scattering) whereas in regard to Equation (18), the intensity is proportional to λ^{-6}.

---

[1]Using the series representation for an independent (real) variable s: $\ln s=\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\frac{s-1}{s}\right)^n,\ s\geq\frac{1}{2}$ [7]

4.2 Case Study: Diffraction from a Gaussian Scatterer

Consider a Gaussian scatterer (a unit amplitude Gaussian function with a real constant a) given by

$$\gamma(r) = \exp(-ar^2), \ r = \sqrt{x^2 + y^2}$$

the idea being to model a spiral galaxy in plane as a 'Gaussian mass object'.

In terms of the wavelength λ, the analytical solutions for the intensity $| A_s(u, v, k) |^2$ generated by the diffraction from an infinitely thin scatterer $\gamma(x, y)$ - Equation (19) - and the diffraction from the field $\nabla^2 \phi_s^0(\mathbf{r}, k_0)$ - Equation (18) - are given by (evaluating the two-dimensional Fourier transforms)

$$| A_s(r, \lambda) |^2 = \frac{16\pi^6}{a^2\lambda^4} \exp\left(-\frac{2\pi^2 r^2}{a\lambda^2 z^2}\right) \text{ and } | A_s(r, \lambda) |^2 = \frac{4^4\pi^{12} r^2}{a^2 z^2 \lambda^6} \exp\left(-\frac{2\pi^2 r^2}{a\lambda^2 z^2}\right)$$

respectively.

Note that the diffraction for a scattering function produces a pattern whose intensity peaks at the centre for $r = 0$ but that diffraction from a low frequency scattered field $\nabla^2 \phi_s^0(\mathbf{r}, k_0)$ has an intensity that is zero at $r = 0$ and produces a pattern characterised by a ring which has maximum intensity when $r = \sqrt{a} z\lambda/\sqrt{2}\pi$, i.e. when $\partial | A_s(r, \lambda) |^2 / \partial r = 0$. Further, with regard to the principal remit of this section, we note that the intensity generated by the scatterer γ scales as λ^{-4} whereas that generated by the field $\nabla^2 \phi_s^0(\mathbf{r}, k_0)$ scales as λ^{-6}.

5 Conclusion

In the context of the λ^{-6} scaling law derived in the previous section, it is noted that, in the **optical spectrum**, Einstein rings appear to be blue [3], [8]. In this respect, the scaling law derived may be a validation to the (low-frequency) scattering model for gravity considered in this paper. However, it should be noted that this 'evidence' must be considered in terms of the scaler wave-field model used where a light wave is taken to be a scalar wave function (no polarisation effects) and a gravitational wave is based on the weak-field approximation for the 00-component of the stress-energy tensor. This allows a Newtonian scalar gravitational field model to be recovered which, in turn, is taken to be the limit of a scattering effect for the case when $k \to 0$, i.e. Equation (9). By way of an analogy, the approach taken is similar to that used in optics to model diffraction phenomena.

In geometric optics, rays of light are taken to be 'bent' or refracted by a dielectric and no wavelength dispersive effects can be modelled. By modelling light as a wave-field and computing the scattered field (diffraction under the Born approximation), dispersive effects are obtained. Einstein's field equations represent a geometric model for gravity in which light is 'bent' by the curvature of space-time and consequently wavelength dispersion is not an embodiment of the approach used. In this context, the model developed here considers the diffraction of light from a gravitational field, a field which is, itself, a 'by-product' of a ultra-low frequency scattering effect.

Given the recent detection of gravitational waves [10], an investigation into the scattering characteristics of such waves and a theoretical develop thereof is timely. This requires a full tensorial method to be developed starting with the weak (far-field) scattering condition under the Born approximation compounded in Equation (11). While gravitational wave source theory has and continuous to be developed, there appears to have been a rather limited amount of work undertaken on gravitational wave scattering theory when compared to the scattering theories developed for non-relativistic and relativistic quantum mechanics and (non-relativistic) electromagnetism, for example, a full investigation of such a theory lying beyond the scope of this paper and being left for future consideration.

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