Mathematica Aeterna, Vol. 4, 2014, no. 6, 597 - 611

Global well-posedness for Gross-Pitaevskii hierarchies

Chuangye Liu

School of Mathematics and Statistics,
Central China Normal University,
Luo-Yu Road 152,
Wuhan 430079, China

Minmin Liu

School of science,
Wuhan Institute of Technology,
Wuhan 430073, China

Abstract

The purpose of this paper is to establish the global well-posedness of solutions to Gross-Pitaevskii (GP) infinite linear hierarchy of equations on $\mathbb{R}^n$, $n \geq 1$. More precisely, by introducing a suitable solution space $\mathscr{H}_\xi^\alpha$ with $\xi > 1$ we prove that there exists a unique global solution to the GP hierarchy. In particular, the solution can belong to the space that of the initial data. In this respect, it is new.

Mathematics Subject Classification: 35Q55, 81V70.

Keywords: Gross-Pitaevskii hierarchy, global solutions.

1 Introduction

In the present paper, we investigate the global well-posedness of solutions to Gross-Pitaevskii (GP) infinite linear hierarchy of equations on $\mathbb{R}^n$, $n \geq 1$, with focusing and defocusing interaction. Motivated by recent experimental realizations of Bose-Einstein condensation the theory of dilute, inhomogeneous Bose systems is currently a subject of intensive studies in physics (see [6, 16, 17, 18]). It is well known that the dynamics of Bose-Einstein condensates are well described by the Gross-Pitaevskii equation (see [13, 14, 19]). On a rigorous derivation of this equation from the basic many-body Schrödinger equation we refer to [4, 7, 8, 9, 10, 11, 12] and the reference therein. In their program an important step is to prove uniqueness to the GP hierarchy(see [9, 15]). Recently, T.Chen and N.Pavlović started to investigate the Cauchy problem for the GP hierarchy, using a Picard-type fixed point argument(see [2]). Later, They presented a new proof in which the approximate solution sequence are produced by truncating the initial data, for detail we refer to [3]. By careful verifying, we find that the contraction mapping in [2] seems wrong. In order to use the Banach fixed-point theorem, we must modify the solution space which was introduced in [2]. In the modified solution space, $\mathscr{H}_\xi^\alpha$, we will prove global existence and uniqueness of solutions in spaces $\mathscr{H}_\xi^\alpha$ by the Banach fixed point theorem. We note that the initial data and solutions given by the local theory don't belong to the same space(see [2, 3, 5]). But, the GP hierarchy can be solved in the modified space such that the solution and initial data belong to the same space. Incidentally, for a very recent work on the global analysis of GP hierarchy we refer to [20].

As follows, we denote by x a general variable in $\mathbb{R}^n$ and by $\mathbf{x} = (x_1, \cdots, x_N)$ a point in $\mathbb{R}^{Nn}$. We will also use the notation $\mathbf{x}_k = (x_1, \ldots, x_k) \in \mathbb{R}^{kn}$. For a function f on $\mathbb{R}^{kn}$ we let

$$(\Theta_\sigma f)(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

for any permutation $\sigma \in \Pi_k$ (Π_k denotes the set of permutations on k elements). Then, each Θ_σ is a unitary operator on $L^2(\mathbb{R}^{kn})$. A bounded operator A on $L^2(\mathbb{R}^{kn})$ is called *k-partite symmetric* or simply *symmetric* if

$$\Theta_\sigma A \Theta_{\sigma^{-1}} = A \tag{1}$$

for every $\sigma \in \Pi_k$. Evidently, a density operator $\gamma^{(k)}$ on $L^2(\mathbb{R}^{kn})$ (i.e., $\gamma^{(k)} \geq 0$ and $\mathrm{tr}\gamma^{(k)} = 1$) with the kernel function $\gamma^{(k)}(\mathbf{x}_k; \mathbf{x}_k')$ is k-partite symmetric if and only if

$$\gamma^{(k)}(x_1, \ldots, x_k; x_1', \ldots, x_k') = \gamma^{(k)}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x_{\sigma(1)}', \ldots, x_{\sigma(k)}')$$

for any $\sigma \in \Pi_k$.

Also, we set

$$L_s^2(\mathbb{R}^{kn}) = \left\{ f \in L^2(\mathbb{R}^{kn}) : \ \Theta_\sigma f = f, \ \forall \sigma \in \Pi_k \right\},$$

equipped with the inner product of $L^2(\mathbb{R}^{kn})$. Clearly, $L_s^2(\mathbb{R}^{kn})$ is a Hilbert subspace of $L^2(\mathbb{R}^{kn})$. It is easy to check that any k-partite symmetric operator on $L^2(\mathbb{R}^{kn})$ preserves $L_s^2(\mathbb{R}^{kn})$.

Definition 1.1. *Given $n \geq 1$, the n-dimensional Gross-Pitaevskii (GP) hierarchy refers to a sequence $\{\gamma^{(k)}(t)\}_{k \geq 1}$ of k-partite symmetric density operators on $L^2(\mathbb{R}^{kn})$, where $t \geq 0$, which satisfy the Gross-Pitaevskii infinite* *linear hierarchy of equations,*

$$i\partial_t \gamma^{(k)}(t) = \left[-\Delta^{(k)}, \gamma^{(k)}(t)\right] + \mu B_{k+1}\gamma^{(k+1)}(t), \quad \Delta^{(k)} = \sum_{j=1}^{k} \Delta_{x_j},\ \mu = \pm 1, \quad (2)$$

with initial conditions

$$\gamma^{(k)}(0) = \gamma_0^{(k)}, \quad k = 1, 2, \dots .$$

Here, Δ_{x_j} refers to the usual Laplace operator with respect to the variables $x_j \in \mathbb{R}^n$ and the operator B_{k+1} is defined by

$$B_{k+1}\gamma^{(k+1)} = \sum_{j=1}^{k} \mathrm{tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}\right]$$

where the notation tr_{k+1} indicates that the trace is taken over the $(k+1)$-th variable.

As in [2], we refer to (2) as the cubic GP hierarchy. For $\mu = 1$ or $\mu = -1$ we refer to the corresponding GP hierarchies as being defocusing or focusing, respectively. We note that the cubic GP hierarchy accounts for two-body interactions between the Bose particles (e.g., see [6, 11] and references therein for details).

Remark 1.1. *In terms of the kernel functions $\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k)$, we can rewrite (2) as follows:*

$$\left(i\partial_t + \triangle_{\pm}^{(k)}\right) \gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \mu \left[B_{k+1}\gamma^{(k+1)}(t)\right](\mathbf{x}_k; \mathbf{x}'_k), \quad (3)$$

where $\triangle_{\pm}^{(k)} = \sum_{j=1}^{k}(\Delta_{x_j} - \Delta_{x'_j})$, with initial conditions

$$\gamma^{(k)}(0, \mathbf{x}_k; \mathbf{x}'_k) = \gamma_0^{(k)}(\mathbf{x}_k; \mathbf{x}'_k), \quad k = 1, 2, \dots .$$

*In particular, the action of B_{k+1} on density operators with smooth kernel functions, $\gamma^{(k+1)}(\mathbf{x}_{k+1}; \mathbf{x}'_{k+1})$
$\in \mathcal{S}(\mathbb{R}^{(k+1)n} \times \mathbb{R}^{(k+1)n})$, is given by*

$$B_{k+1} := \sum_{j=1}^{k} B_{j,k+1} \quad (4)$$

and

$$\begin{aligned} B_{j,k+1}\gamma^{(k+1)}(\mathbf{x}_k; \mathbf{x}'_k) = \int & dx_{k+1}dx'_{k+1}\gamma^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) \\ & \times \delta(x'_{k+1} - x_{k+1})\left[\delta(x_j - x_{k+1}) - \delta(x'_j - x_{k+1})\right]. \end{aligned} \quad (5)$$

The action of B_{k+1} can be extended to generic density operators.

Remark 1.2. *Let $\varphi_0 \in \mathrm{H}^1(\mathbb{R}^n)$, then one can easily verify that a particular solution to* (3) *with initial conditions*

$$\gamma_0^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} \varphi_0(x_j)\overline{\varphi_0(x'_j)}, \quad k = 1, 2, \ldots,$$

is given by

$$\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} \varphi(t, x_j)\overline{\varphi(t, x'_j)}, \quad k = 1, 2, \ldots,$$

where $\varphi(t, x)$ satisfies the cubic non-linear Schrödinger equation

$$i\partial_t \varphi = -\Delta\varphi + \mu|\varphi|^2\varphi, \quad \varphi(0, \cdot) = \varphi_0, \tag{6}$$

which is defocusing if $\mu = 1$, and focusing if $\mu = -1$.

The GP hierarchy (2) can be written in the integral form

$$\gamma^{(k)}(t) = e^{it\Delta_{\pm}^{(k)}}\gamma_0^{(k)} - i\mu \int_0^t ds\ e^{i(t-s)\Delta_{\pm}^{(k)}} B_{k+1}\gamma^{(k+1)}(s),\ k = 1, 2, \ldots. \tag{7}$$

is given by

$$\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} \varphi(t, x_j)\overline{\varphi(t, x'_j)}, \quad k = 1, 2, \ldots,$$

where $\varphi(t, x)$ satisfies the cubic non-linear Schrödinger equation

$$i\partial_t \varphi = -\Delta\varphi + \mu|\varphi|^2\varphi, \quad \varphi(0, \cdot) = \varphi_0, \tag{8}$$

which is *defocusing* if $\mu = 1$, and *focusing* if $\mu = -1$.

The GP hierarchy (2) can be written in the integral form

$$\gamma^{(k)}(t) = e^{it\Delta_{\pm}^{(k)}}\gamma_0^{(k)} - i\mu \int_0^t ds\ e^{i(t-s)\Delta_{\pm}^{(k)}} B_{k+1}\gamma^{(k+1)}(s),\ k = 1, 2, \ldots. \tag{9}$$

2 Main Results

These are the main results of the paper.

In order to state our main results, we require some more notation. We will use $\gamma^{(k)}, \rho^{(k)}$ for denoting either (density) operators or kernel functions. For $k \geq 1$ and $\alpha > 0$, we denote by $\mathrm{H}_k^\alpha = \mathrm{H}^\alpha(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$ the space of measurable functions $\gamma^{(k)} = \gamma^{(k)}(\mathbf{x}_k, \mathbf{x}'_k)$ in $L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})$ such that

$$\|\gamma^{(k)}\|_{\mathrm{H}_k^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{kn} \times \mathbb{R}^{kn})} < \infty,$$

where

$$S^{(k,\alpha)} := \prod_{j=1}^{k} \left[(1-\Delta_{x_j})^{\frac{\alpha}{2}} (1-\Delta_{x_j'})^{\frac{\alpha}{2}} \right].$$

Evidently, H_k^α is a Hilbert space with the inner product

$$\langle \gamma^{(k)}, \rho^{(k)} \rangle := \left\langle S^{(k,\alpha)}\gamma^{(k)},\ S^{(k,\alpha)}\rho^{(k)} \right\rangle_{L^2(\mathbb{R}^{kn}\times\mathbb{R}^{kn})}.$$

Moreover, the norm $\|\cdot\|_{\mathrm{H}_k^\alpha}$ is invariance under the action of $e^{it\Delta_\pm^{(k)}}$, that is,

$$\|e^{it\Delta_\pm^{(k)}}\gamma^{(k)}\|_{\mathrm{H}_k^\alpha} = \|\gamma^{(k)}\|_{\mathrm{H}_k^\alpha}$$

because $e^{it\Delta_\pm^{(k)}}$ commutates with Δ_{x_j} for any j.

Given $\xi > 0$ and $\alpha > 0$, we define

$$\mathscr{H}_\xi^\alpha = \left\{ \Gamma = \{\gamma^{(k)}\}_{k\geq 1} \in \bigoplus_{k=1}^{\infty} \mathrm{H}_k^\alpha :\ \|\Gamma\|_{\mathscr{H}_\xi^\alpha} := \sum_{k=1}^{\infty} \xi^{k^2} \|\gamma^{(k)}\|_{\mathrm{H}_k^\alpha} < \infty \right\}. \quad (10)$$

Evidently, $\mathscr{H}_\xi^\alpha$ is a Banach space equipped with the norm $\|\cdot\|_{\mathscr{H}_\xi^\alpha}$. We remark that the following space

$$\mathcal{H}_\xi^\alpha = \left\{ \Gamma = \{\gamma^{(k)}\}_{k\geq 1} \in \bigoplus_{k=1}^{\infty} \mathrm{H}_k^\alpha :\ \|\Gamma\|_{\mathcal{H}_\xi^\alpha} := \sum_{k=1}^{\infty} \xi^{k} \|\gamma^{(k)}\|_{\mathrm{H}_k^\alpha} < \infty \right\}. \quad (11)$$

is introduced in [2]. And spaces which are similar to (11) are used in the isospectral renormalization group analysis of spectral problems in quantum field theory (see [1]).

Definition 2.1. *For $T > 0$, $\Gamma(t) = \{\gamma^{(k)}(t)\}_{k\geq 1} \in C([0,T], \mathscr{H}_\xi^\alpha)$ is said to be a local (mild) solution to the GP hierarchy (2) if for every $k = 1, 2, \ldots,$*

$$\gamma^{(k)}(t) = e^{it\Delta_\pm^{(k)}}\gamma_0^{(k)} - i\mu \int_0^t ds\ e^{i(t-s)\Delta_\pm^{(k)}} B_{k+1}\gamma^{(k+1)}(s), \quad \forall t \in [0,T],$$

holds in H_k^α.

In order to write the equations above in a more compact form, we introduce the notation below [2]. Set

$$\hat{\Delta}_\pm \Gamma := \left\{ \Delta_\pm^{(k)} \gamma^k \right\}_{k\geq 1} \quad (12)$$

and

$$\hat{B}\Gamma := \left\{ B_{k+1}\gamma^{k+1} \right\}_{k\geq 1}. \quad (13)$$

Then, the GP hierarchy can be rewritten as

$$\left(i\partial_t + \hat{\Delta}_\pm\right)\Gamma = \mu\hat{B}\Gamma. \tag{14}$$

Also, it can be rewritten as in the integral form

$$\Gamma(t) = e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s). \tag{15}$$

Following [2], in order to solve the equation (15) we also deal with the auxiliary equation

$$\Xi(t) = \hat{B}e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\hat{B}e^{i(t-s)\hat{\Delta}_\pm}\Xi(s). \tag{16}$$

Let $\mathscr{R}(n)$ denote the set below

$$\mathscr{R}(n) = \begin{cases} (\frac{1}{2},\infty), & n=1, \\ (\frac{n-1}{2},\infty), & n=2, n>4, \\ [1,\infty), & n=3, \end{cases} \tag{17}$$

where the set $\mathscr{R}(n)$ was first introduced in [2]. And let

$$C_\xi := \sup_{k\geq 1}\{k\xi^{1-2k}\} < \infty, \tag{18}$$

for any $\xi > 1$. It is time to state our main results. They are the following two theorems.

Theorem 2.2 (local solution)**.** *Assume that* $\alpha \in \mathscr{R}(n)$ *and* $\xi > 1$. *Let* $T = \frac{1}{4(AC_\xi)^2}$ *and* $I = [0,T]$. *Suppose* $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1} \in \mathscr{H}_\xi^\alpha$. *Then, the following hold.*

(i) *There exists a unique solution* $\Xi(t) \in L^1_{t\in I}\mathscr{H}_\xi^\alpha)$ *to the system*(16). *Moreover, the following estimate*

$$\|\Xi(t)\|_{L^1_{t\in I}\mathscr{H}_\xi^\alpha} \leq \|\Gamma_0\|_{\mathscr{H}_\xi^\alpha} \tag{19}$$

holds.

(ii) *There exists a solution* $\Gamma(t) \in C(I,\mathscr{H}_\xi^\alpha)$ *to the system*(15) *with the initial data* Γ_0. *In particular, this solution has the property that*

$$\|\Gamma(t)\|_{C(I,\mathscr{H}_\xi^\alpha)} \leq 2\|\Gamma_0\|_{\mathscr{H}_\xi^\alpha} \tag{20}$$

and

$$\|\hat{B}\Gamma(t)\|_{L^1_{t\in I}\mathscr{H}_\xi^\alpha} \leq \|\Gamma_0\|_{\mathscr{H}_\xi^\alpha}. \tag{21}$$

(iii) *For any interval J with $0 \in J$, in the space*

$$\mathscr{W}_{\xi}^{\alpha}(J) := \left\{\Gamma(t) \in C(J, \mathscr{H}_{\xi}^{\alpha}) : \hat{B}\Gamma(t) \in L^1_{t\in J}\mathscr{H}_{\xi}^{\alpha}\right\}, \tag{22}$$

there exists a unique solution to the system(15) *with the initial data* Γ_0.

Remark 2.1. *Here and there, the constant $A = A(n,\alpha)$ is fixed and from the lemma 3.1.*

Remark 2.2. *Let $\lambda \in (0, \frac{1}{\sqrt{\xi}})$ be a constant, and set*

$$\gamma_0^{(k)}(\mathbf{x}_k;\mathbf{x}'_k) := \prod_{j=1}^{k} \varphi_k(x_j)\overline{\varphi_k(x'_j)}, \quad k = 1,2,\ldots,$$

with $\|\varphi_k\|_{\mathrm{H}_k^\alpha} = \lambda^k$. Then, it is easy to verify that $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1} \in \mathscr{H}_{\xi}^{\alpha}$.

Theorem 2.3 (global solution)**.** *Aussem that $\alpha \in \mathscr{R}(n)$ and $\xi > 1$. And suppose $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1} \in \mathscr{H}_{\xi}^{\alpha}$. Then, there existence a global solution $\Gamma \in C(\mathbb{R}, \mathscr{H}_{\xi}^{\alpha})$ to the system*(15) *with the initial data Γ_0. Moreover, the following estimates*

$$\|\Gamma\|_{C(I_j,\mathscr{H}_{\xi}^{\alpha})} \leq 2^j\Gamma_0\|_{\mathscr{H}_{\xi}^{\alpha}} \tag{23}$$

and

$$\|\hat{B}\Gamma(t)\|_{L^1_{t\in[0,jT]}\mathscr{H}_{\xi}^{\alpha}} \leq (2^j - 1)\|\Gamma_0\|_{\mathscr{H}_{\xi}^{\alpha}} \tag{24}$$

hold, where T was defined in theorem 2.2 and $I_j = [(j-1)T, jT]$, $j = 1,2,\ldots$.

In addition, if two solutions to (15) *with the same initial data belong to $\mathscr{W}_{\xi}^{\alpha}(J)$ for all finite interval $J \ni 0$, then the two solutions are equal.*

3 Proof of Theorem 2.2

In order to prove the Theorem 2.2, we need the following estimate.

3.1 Preliminary estimate

Lemma 3.1. *Suppose that $\alpha \in \mathscr{R}(n)$. Let $\gamma^{(k)}(t)$ be the unique solution of*

$$\left(i\partial_t + \Delta_{\pm}^{(k)}\right)\gamma^{(k)}(t) = 0 \tag{25}$$

with initial condition

$$\gamma^{(k)}(0,\cdot) = \gamma_0^{(k)} \in \mathrm{H}_k^\alpha. \tag{26}$$

Then, there exists a constant $A = A(n,\alpha)$ such that

$$\|B_{l,k+1}\gamma^{(k+1)}\|_{L^2_{t\in\mathbb{R}}\mathrm{H}_k^\alpha} \leq A\|\gamma_0^{(k+1)}\|_{\mathrm{H}_{k+1}^\alpha} \tag{27}$$

for all $l = 1,2,\ldots,k$.

Proof. The lemma was proved in [2]. □

3.2 The proof of Theorem 2.2

Proof. (i) Let $I = [0, T]$ and $T > 0$ to be choosed later. We defined

$$\Phi(\Xi)(t) := \hat{B} e^{it\hat{\Delta}_\pm} \Gamma_0 - i\mu \int_0^t ds \hat{B} e^{i(t-s)\hat{\Delta}_\pm} \Xi(s) \tag{28}$$

for any $\Xi \in L^1_{t\in I}\mathscr{H}^\alpha_\xi$. In terms of components, we can rewrite it as

$$\Phi(\Xi)^{(k)}(t) = B_{k+1} e^{it\Delta_\pm^{(k+1)}} \gamma_0^{(k+1)} - i\mu \int_0^t ds B_{k+1} e^{i(t-s)\Delta_\pm^{(k+1)}} \Xi^{(k+1)}(s), \;\; k \geq 1 \tag{29}$$

if $\Xi(t) = \{\Xi^{(k)}(t)\}_{k\geq 1}$ and $\Gamma_0 = \{\gamma_0^{(k)}\}_{k\geq 1}$. In order to apply Banach fixed point theorem, we prove firstly that $\Phi(\Xi) \in L^1_{t\in I}\mathscr{H}^\alpha_\xi$ when $\Xi \in L^1_{t\in I}\mathscr{H}^\alpha_\xi$. In fact, for any $k \geq 1$, we have

$$\begin{aligned}
&\|B_{k+1} e^{it\Delta_\pm^{(k+1)}} \gamma_0^{(k+1)}\|_{L^1_{t\in I}\mathrm{H}^\alpha_k} \\
&\leq \sum_{l=1}^{k} \|B_{l,k+1} e^{it\Delta_\pm^{(k+1)}} \gamma_0^{(k+1)}\|_{L^1_{t\in I}\mathrm{H}^\alpha_k} \\
&\leq \sum_{l=1}^{k} \sqrt{T} \|B_{l,k+1} e^{it\Delta_\pm^{(k+1)}} \gamma_0^{(k+1)}\|_{L^2_{t\in I}\mathrm{H}^\alpha_k} \\
&\leq Ak\sqrt{T} \|\gamma_0^{(k+1)}\|_{\mathrm{H}^\alpha_{k+1}}
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the lemma 3.1. Then, we obtain the following estimate

$$\begin{aligned}
&\|\hat{B} e^{it\hat{\Delta}_\pm} \Gamma_0\|_{L^1_{t\in I}\mathscr{H}^\alpha_\xi} \\
&= \sum_{k=1}^{\infty} \xi^{k^2} \|B_{k+1} e^{it\Delta_\pm^{(k+1)}} \gamma_0^{(k+1)}\|_{L^1_{t\in I}\mathrm{H}^\alpha_k} \\
&\leq \sum_{k=1}^{\infty} \xi^{k^2} Ak\sqrt{T} \|\gamma_0^{(k+1)}\|_{\mathrm{H}^\alpha_{k+1}} \\
&\leq A\sqrt{T} \sup_{k\geq 1}\{k\xi^{1-2k}\} \sum_{k=1}^{\infty} \xi^{k^2} \|\gamma_0^{(k)}\|_{\mathrm{H}^\alpha_k} \\
&= AC_\xi \sqrt{T} \|\Gamma_0\|_{\mathscr{H}^\alpha_\xi}
\end{aligned} \tag{30}$$

where the notation C_ξ was defined in (18).

Similarly, for the second term in the right hand of equation (29), we have

$$\begin{aligned}
&\| \int_0^t ds B_{k+1} e^{i(t-s)\Delta_{\pm}^{(k+1)}} \Xi^{(k+1)}(s) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_k} \\
&\leq \sum_{l=1}^{k} \int_0^T ds \| B_{l,k+1} e^{i(t-s)\Delta_{\pm}^{(k+1)}} \Xi^{(k+1)}(s) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_k} \\
&\leq \sum_{l=1}^{k} \int_0^T ds \sqrt{T} \| B_{l,k+1} e^{i(t-s)\Delta_{\pm}^{(k+1)}} \Xi^{(k+1)}(s) \|_{L^2_{t\in I}\mathrm{H}^{\alpha}_k} \\
&\leq \sum_{l=1}^{k} \int_0^T ds \sqrt{T} A \| \Xi^{(k+1)}(s) \|_{\mathrm{H}^{\alpha}_{k+1}} \\
&= Ak\sqrt{T} \| \Xi^{(k+1)}(t) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_{k+1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| \int_0^t ds \hat{B} e^{i(t-s)\hat{\Delta}_{\pm}} \Xi(s) \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \\
&= \sum_{k=1}^{\infty} \xi^{k^2} \| \int_0^t ds B_{k+1} e^{i(t-s)\Delta_{\pm}^{(k+1)}} \Xi^{(k+1)}(s) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_k} \\
&\leq \sum_{k=1}^{\infty} \xi^{k^2} Ak\sqrt{T} \| \Xi^{(k+1)}(t) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_{k+1}} \\
&\leq A\sqrt{T} \sup_{k\geq 1}\{k\xi^{1-2k}\} \sum_{k=1}^{\infty} \xi^{k^2} \| \Xi^{(k+1)}(t) \|_{L^1_{t\in I}\mathrm{H}^{\alpha}_{k+1}} \\
&= AC_{\xi}\sqrt{T} \| \Xi \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}}.
\end{aligned} \tag{31}$$

Combing (30) with (31), we deduce that

$$\| \Phi(\Xi) \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \leq AC_{\xi}\sqrt{T} \left[\| \Gamma_0 \|_{\mathscr{H}^{\alpha}_{\xi}} + \| \Xi \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \right], \tag{32}$$

where $\mu = \pm 1$ was used. It implies that the mapping

$$\Phi : L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi} \mapsto L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi} \tag{33}$$

is well defined. And by the inequality (31), we obtain

$$\| \Phi(\Xi_1) - \Phi(\Xi_2) \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \leq AC_{\xi}\sqrt{T} \| \Xi_1 - \Xi_2 \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \tag{34}$$

for any $T > 0$. Now We choose $T = \frac{1}{4(AC_{\xi})^2}$, then by (34) the following inequality

$$\| \Phi(\Xi_1) - \Phi(\Xi_2) \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \leq \frac{1}{2} \| \Xi_1 - \Xi_2 \|_{L^1_{t\in I}\mathscr{H}^{\alpha}_{\xi}} \tag{35}$$

holds for all $\Xi_1, \Xi_2 \in L^1_{t\in I}\mathscr{H}^\alpha_\xi$. It concludes that Φ is a contraction mapping on $L^1_{t\in I}\mathscr{H}^\alpha_\xi$. By Banach fixed point theorem there exists a unique $\Xi \in L^1_{t\in I}\mathscr{H}^\alpha_\xi$ such that

$$\Xi = \Phi(\Xi). \tag{36}$$

Consequently, there exists a unique solution Ξ of (16) in $L^1_{t\in I}\mathscr{H}^\alpha_\xi$. In addition, the following estimate

$$\|\Xi\|_{L^1_{t\in I}\mathscr{H}^\alpha_\xi} \le AC_\xi\sqrt{T}\left[\|\Gamma_0\|_{\mathscr{H}^\alpha_\xi} + \|\Xi\|_{L^1_{t\in I}\mathscr{H}^\alpha_\xi}\right] \tag{37}$$

holds because of (32) and (36). By the choice of T, (37) implies that the inequality (19) holds.

(ii) Following [2], we define

$$\Gamma(t) := e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\Xi(s) \tag{38}$$

where Ξ is the unique solution of (16) in $L^1_{t\in I}\mathscr{H}^\alpha_\xi$. Then,

$$\begin{aligned} \|\Gamma(t)\|_{\mathscr{H}^\alpha_\xi} &\le \|\Gamma_0\|_{\mathscr{H}^\alpha_\xi} + \int_0^t ds\|\Xi(s)\|_{\mathscr{H}^\alpha_\xi} \\ &\le \|\Gamma_0\|_{\mathscr{H}^\alpha_\xi} + \|\Xi(t)\|_{L^1_{t\in I}\mathscr{H}^\alpha_\xi} \\ &\le 2\|\Gamma_0\|_{\mathscr{H}^\alpha_\xi} \end{aligned} \tag{39}$$

where we used the unitary of operator $e^{it\hat{\Delta}_\pm}$ with respect to $\mathscr{H}^\alpha_\xi$ in the first inequality and the estimate (19) in last inequality. Hence, $\Gamma \in C(I, \mathscr{H}^\alpha_\xi)$ and satisfies the estimate (20). The continuity with respect to $t \in I$ follows from the fact $\{e^{it\hat{\Delta}_\pm}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. In fact, the first term in the right hand of equation (38) is continuous. And for the second term, we have

$$\begin{aligned} &\int_0^{t+\tau} ds e^{i(t+\tau-s)\hat{\Delta}_\pm}\Xi(s) - \int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\Xi(s) \\ &\quad = \int_0^{t+\tau} ds\left[e^{i(t+\tau-s)\hat{\Delta}_\pm} - e^{i(t-s)\hat{\Delta}_\pm}\right]\Xi(s) \\ &\quad + \int_t^{t+\tau} ds e^{i(t-s)\hat{\Delta}_\pm}\Xi(s) \end{aligned} \tag{40}$$

for fixed $t \in I$ and any τ such that $t + \tau \in I$. Then, the Lebesgue Dominated Convergence theorem implies that the second term in the right hand of equation (38) is continuous at t.

We note that

$$\hat{B}\Gamma = \hat{B}e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\hat{B}e^{i(t-s)\hat{\Delta}_\pm}\Xi(s) \tag{41}$$

by the definition (38). Since Ξ is the solution of the equation (16), we obtain

$$\hat{B}\Gamma = \hat{B}e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\hat{B}e^{i(t-s)\hat{\Delta}_\pm}\Xi(s) = \Xi. \tag{42}$$

That is

$$\hat{B}\Gamma = \Xi. \tag{43}$$

Then, inserting the equation (43) into the equation (38), we deduce

$$\Gamma(t) = e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s). \tag{44}$$

It implies that $\Gamma(t)$ is a solution of the equation (15). Then the estimate (21) follows from (43) and (19).

(iii) Suppose $\Gamma_1(t), \Gamma_2(t) \in \mathscr{W}_\xi^\alpha(J)$ are two solutions to (15) with the same initial datum $\Gamma_0 \in \mathscr{H}_\xi^\alpha$. Let $\widetilde{\Gamma}(t) = \Gamma_1(t) - \Gamma_2(t)$, then $\widetilde{\Gamma}(t)$ satisfies that

$$\widetilde{\Gamma}(t) = -i\mu\int_0^t e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\widetilde{\Gamma}(s)ds \tag{45}$$

and

$$\hat{B}\widetilde{\Gamma}(t) \in L^1_{t\in J}\mathscr{H}_\xi^\alpha \tag{46}$$

Next, we will prove that $\widetilde{\Gamma}(t) = 0$ in any compact interval $[T_1, T_2] \subset J$ with $T_1 < 0 < T_2$. Set

$$T^* = \sup\left\{0 \le t \le T_2 | \widetilde{\Gamma}(s) = 0,\ s \in [0,t]\right\}. \tag{47}$$

If $T^* < T_2$, then by translation $s \to T^* + s$ we obtain the equation

$$\widetilde{\Gamma}(T^* + \tau) = -i\mu\int_0^\tau e^{i(\tau-s)\hat{\Delta}_\pm}\hat{B}\widetilde{\Gamma}(T^* + s)ds \tag{48}$$

for any $\tau \in (0, T_2 - T^*)$. We note that

$$\hat{B}\widetilde{\Gamma}(T^* + \tau) = -i\mu\int_0^\tau \hat{B}e^{i(\tau-s)\hat{\Delta}_\pm}\hat{B}\widetilde{\Gamma}(T^* + s)ds \tag{49}$$

and

$$\hat{B}\widetilde{\Gamma}(T^* + \tau) \in L^1_{\tau\in[0,T_2-T^*]}\mathscr{H}_\xi^\alpha. \tag{50}$$

Then, the uniqueness result in the part (i) of theorem 2.2 implies that

$$\hat{B}\widetilde{\Gamma}(T^* + \tau) = 0, \quad a.e.\ \tau \in (0, \tilde{T}) \tag{51}$$

where $\tilde{T} = \min\{\frac{1}{4(AC_\xi)^2}, T_2 - T^*\}$. Inserting the equation (51) into (48) results $\widetilde{\Gamma}(t) = 0$ for all $t \in [0, T^* + \tilde{T}]$. It is contradictive to the definition of T^*. Thus, $T^* = T_2$. It concludes that $\Gamma_1(t) = \Gamma_2(t)$ for all $t \in [0, T_2]$. Similarly, we can prove that $\Gamma_1(t) = \Gamma_2(t)$ for all $t \in [T_1, 0]$. Consequently, we prove that $\Gamma_1(t) = \Gamma_2(t)$ for any compact interval $[T_1, T_2] \subset J$ with $T_1 < 0 < T_2$. It implies that $\Gamma_1(t) = \Gamma_2(t)$ in the interval J. It completes the proof. □

3.3 The proof of theorem 2.3

Proof. We will construct a solution by using the theorem 2.2 again and again. Let $\Xi_1(t) := \Xi(t)$ and $\Gamma_1(t) := \Gamma(t)$, $t \in [0, T]$, where $\Xi(t), \Gamma(t)$ are from the theorem 2.2. And set $I_j = [(j-1)T, jT]$, $j = 1, 2, \cdots$. Now we consider the following equations:

$$\Xi_j(t) = \hat{B}e^{i(t-(j-1)T)\hat{\Delta}_\pm}\Gamma_{j-1}((j-1)T) - i\mu\int_{(j-1)T}^{t} ds\hat{B}e^{i(t-s)\hat{\Delta}_\pm}\Xi_j(s) \tag{52}$$

and

$$\Gamma_j(t) = e^{i(t-(j-1)T)\hat{\Delta}_\pm}\Gamma_{j-1}((j-1)T) - i\mu\int_{(j-1)T}^{t} dse^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma_j(s) \tag{53}$$

for $t \in I_j$. We note that the equations above for $j = 1$ are exact the equations (16) and (15). Applying the part (ii) of the theorem 2.2, $\Gamma_1(t) = \Gamma(t) \in C(I_1, \mathscr{H}_\xi^\alpha)$. Then, $\Gamma_1(T) \in \mathscr{H}_\xi^\alpha$ makes sense. Next, we solve the equation (52) for $j = 2$ by using the theorem 2.2. Then, we can construct a solution $\Gamma_2(t) \in C(I_2, \mathscr{H}_\xi^\alpha)$ to the equation (53) for $j = 2$ just as do the part (ii) of the theorem 2.2 . In addition, the following estimates

$$\|\Gamma_2\|_{C(I_2,\mathscr{H}_\xi^\alpha)} \leq 2\Gamma(T)\|_{\mathscr{H}_\xi^\alpha} \leq 2^2\|\Gamma_0\|_{\mathscr{H}_\xi^\alpha} \tag{54}$$

and

$$\|\hat{B}\Gamma_2(t)\|_{L^1_{t\in I_2}\mathscr{H}_\xi^\alpha} \leq \|\Gamma(T)\|_{\mathscr{H}_\xi^\alpha} \leq 2\|\Gamma_0\|_{\mathscr{H}_\xi^\alpha} \tag{55}$$

hold. By the mathematical induction, we deduce that

$$\Gamma_j(t) \in C(I_j, \mathscr{H}_\xi^\alpha),\ \Gamma_j((j-1)T) = \Gamma_{j-1}((j-1)T) \tag{56}$$

and

$$\|\Gamma_j\|_{C(I_j,\mathscr{H}_\xi^\alpha)} \leq 2^j\|\Gamma_0\|_{\mathscr{H}_\xi^\alpha} \tag{57}$$

and

$$\|\hat{B}\Gamma_j(t)\|_{L^1_{t\in I_j}\mathscr{H}^\alpha_\xi} \leq 2^{j-1}\|\Gamma_0\|_{\mathscr{H}^\alpha_\xi}. \tag{58}$$

Now, we extend the domain of $\Gamma(t)$ to (T,∞) such that

$$\Gamma(t) = \Gamma_j(t),\ t \in I_j,\ j = 2,\dots. \tag{59}$$

This definition makes sense because of (56). Then, $\Gamma \in C([0,\infty),\mathscr{H}^\alpha_\xi)$ is a solution to the GP hierarchy (15). In fact, suppose that

$$\Gamma(t) = e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s),\ t \in [0,(j-1)T] \tag{60}$$

then (53) and (60) imply that for $t \in I_j$

$$\begin{aligned}\Gamma(t) &= e^{i(t-(j-1)T)\hat{\Delta}_\pm}\Gamma((j-1)T) - i\mu\int_{(j-1)T}^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s)\\ &= e^{i(t-(j-1)T)\hat{\Delta}_\pm}\left[e^{i(j-1)T\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^{(j-1)T} ds e^{i((j-1)T-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s)\right]\\ &\quad - i\mu\int_{(j-1)T}^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s)\\ &= e^{it\hat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds e^{i(t-s)\hat{\Delta}_\pm}\hat{B}\Gamma(s).\end{aligned} \tag{61}$$

Consequently, mathematical induction implies that Γ is a solution to the GP hierarchy (15) on $[0,\infty)$. Further more, the estimates (23) and (24) are from (57) and (58). Similarly, we can deal with GP hierarchy (15) for $t \in (-\infty,0)$. Finally, the uniqueness property of the solution follows from the part (iii) of the theorem 2.2 under the assumptions that the solutions belong to the spaces $\mathscr{W}^\alpha_\xi(J)$. It completes the proof. □

ACKNOWLEDGEMENTS. C.Liu is partially supported by NSFC grants (11101171,11071095) and Hubei Key Laboratory of Mathematical Sciences.

References

[1] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, Smooth Feshbach map and operator-theoretic renormalization group methods, *J. Funct. Anal.* **203** (1), 44-92 (2003).

[2] T.Chen and N.Pavlović, On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies, *Discrete Contin. Dyn. Syst.* **27** (2), 715-739 (2010).

[3] T.Chen and N.Pavlović, A new proof of existence of solutions for focusing and defocusing Groos-Pitaevskii hierarchies, *Pro.Amer.Math.Soc.*,141,279-293,(2013).

[4] T. Chen and N. Pavlovic, The quintic NLS as the mean field limit of a Boson gas with three-body interactions, *J. Funct. Anal.* **260** (4), 959-997 (2011).

[5] Z. Chen and C. Liu, On the Cauchy problem for Gross-Pitaevskii hierarchies, *J. Math. Phys.* **52**, 032103 (2011).

[6] F.Dalfovo, S.Giorgini, L.P.Pitaevskii, and S.Stringari, Theory of Bose-Einstein condensation in trapped gases, *Rev. Mod. Phys.* **71**, 463-512 (1999).

[7] A.Elgart, L.Erdös, B.Schlein, and H.T.Yau, Gross-Pitaevskii equation as the mean field limit of weakly coupled bosons, *Arch. Rat. Mech. Anal.* **179**(2), 265-283 (2006).

[8] L.Erdös, B.Schlein, and H.T.Yau, Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate, *Commun. Pure Appl. Math.* **59**(12), 1659-1741 (2006).

[9] L.Erdös, B.Schlein, and H.T.Yau, Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems, *Invent. Math.* **167**, 515-614 (2007).

[10] L.Erdös, B.Schlein, and H.T.Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate, *Ann. Math.*,170,291-370(2010).

[11] L.Erdös, B.Schlein, and H.T.Yau, Rigorous derivation of the Gross-Pitaevskii equation, *Phys.Rev.Lett.* **98**, 040404 (2007).

[12] L.Erdös and H.T.Yau, Derivation of the nonlinear Schrödinger equation from a many body Coulomb system, *Adv. Theor. Math. Phys.* **5**(6), 1169-1205 (2001).

[13] E.P.Gross, Structure of a quantized vortex in boson systems, *Nuovo Cimento* **20**, 454-466 (1961).

[14] E.P.Gross, Hydrodynamics of a superfluid condensate, *J.Math. Phys.* **4**, 195-207 (1963).

[15] S.Klainerman and M.Machedon, On the uniqueness of solutions to the Gross-Pitaevskii hierarchy, *Commun.Math.Phys.* **279**, 169-185 (2008).

[16] E.H.Lieb and R.Seiringer, Proof of Bose-Einstein condensation for dilute trapped gases, *Phys. Rev. Lett.* **88**, 170409 (2002).

[17] E.H.Lieb, R.Seiringer, and J.Yngvason, Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional, *Phys. Rev. A* **61**, 043602 (2000).

[18] E.H.Lieb, R.Seiringer, and J.Yngvason, A rigorous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas, *Commun. Math. Phys.* **224**, 17-31 (2001).

[19] L.P.Pitaevskii, Vortex lines in an imperfect Bose gas, *Sov. Phys. JETP* **13**, 451-454 (1961).

[20] K.Taliferro and T.Chen, Positive Semidefiniteness and Global Well-Posedness of Solutions to the Gross-Pitaevskii Hierarchy, http://arxiv.org/abs/1305.1404

Received: June, 2014