Global Integrability for Minimizers of Obstacle Problems of Anisotropic Functionals

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Abstract
This paper deals with minimizers $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ of $\mathcal{K}_{\psi,\theta}^{(p_i)}$-obstacle problems of anisotropic functionals whose prototype is

$$\int_\Omega (|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n})\, dx.$$

We show that higher integrability of $\theta_* = \max\{\theta, \psi\}$ forces minimizers u to have higher integrability as well.

Mathematics Subject Classification: 49N60, 35J60.

Keywords: Global integrability, minimizer, obstacle problem, anisotropic functional.

1 Introduction and Statement of Result

Let Ω be a bounded open subset of R^n, $n \geq 2$. For numbers $p_i > 1$, $i = 1, 2, \cdots, n$, we denote by p_m and $\overline{p}$ the maximum value and the harmonic mean of $p_i (i = 1, 2, \cdots, n)$, respectively, that is,

$$p_m = \max_{1 \leq i \leq n}\{p_i\}, \quad \overline{p} : \frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i}.$$

The anisotropic Sobolev spaces $W^{1,(p_i)}(\Omega)$ and $W_0^{1,(p_i)}(\Omega)$ are defined, respectively, by

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \cdots, n \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \cdots, n \right\}.$$

Let $f : \Omega \times \mathrm{R}^n \to [0, +\infty)$ be a Carathéodory function and satisfies

$$\sum_{i=1}^{n} |z_i|^{p_i} \le f(x, z) \le c_0 \sum_{i=1}^{n} (\varphi_i(x) + |z_i|)^{p_i} \tag{1.1}$$

for almost all $x \in \Omega$ and all $z \in \mathrm{R}^n$. The conditions on the functions $\varphi_i(x)(i = 1, 2, \cdots, n)$ in (1.1) will be given later. Consider the integral functional

$$\mathcal{I}(u; \Omega) = \int_{\Omega} f(x, Du(x)) dx \tag{1.2}$$

with the integrand f satisfies (1.1). The prototype of the integral (1.2) with f satisfies (1.1) is

$$\int_{\Omega} \left(|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n} \right) dx,$$

where the derivative $D_i u$ has the exponent p_i that might be different from the exponent p_j of the derivative $D_j v$ when $j \ne i$.

Let ψ be any function in Ω with values in $\mathrm{R} \cup \{\pm\infty\}$, and $\theta \in W^{1,(p_i)}(\Omega)$. We introduce

$$\mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega) = \left\{ v \in W^{1,(p_i)}(\Omega) : v \ge \psi, \text{ a.e., and } v - \theta \in W_0^{1,(p_i)}(\Omega) \right\}.$$

Definition 1.1 *By a solution to the anisotropic obstacle problem for the functional $\mathcal{I}$, we mean a function $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ such that*

$$\mathcal{I}(u; \Omega) \le \mathcal{I}(w; \Omega) \tag{1.3}$$

whenever $w \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$.

Minimizers of anisotropic integral functionals seem to be useful when dealing with some reinforced materials. Leonetti and Siepe obtained a global integrability result for minimizers of anisotropic functionals in [1]. Some other related results can be found in [2-4]. In the present paper, we consider obstacle problems of anisotropic functionals, the main result of this paper is the following theorem.

Theorem 1.2 *Let $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ be a solution to the anisotropic obstacle problem for the functional $\mathcal{I}$, $\theta_* = \max\{\theta, \psi\} \in \theta + W_0^{1,(q_i)}(\Omega)$, $0 \leq \varphi_i(x) \in L^{q_i}(\Omega)$ with $q_i > p_i$, $i = 1, 2, \cdots, n$. Moreover $\overline{p} < n$. Then*

$$u \in \theta_* + L_{weak}^t(\Omega),$$

where

$$t = \frac{\overline{p}\,\overline{p}^*}{\overline{p} - b\overline{p}^*} > \overline{p}^*$$

and b is any number such that

$$0 < b \leq \min_{i=1,2,\cdots,n} \left\{1 - \frac{p_i}{q_i}\right\} \ and \ b < \frac{\overline{p}}{\overline{p}^*}. \tag{1.4}$$

2 Proof of the Main Theorem

It is no loss of generality to assume $\theta \geq \psi$ almost everywhere on $\partial\Omega$ since otherwise $\mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ will be empty. Let $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ be a solution to the anisotropic obstacle problem for the functional $\mathcal{I}$. For $L \in (0, +\infty)$ and a function v, let $T_L(v)$ be the usual truncation of v at level L; that is,

$$T_L(v) = \begin{cases} v, & \text{for } |v| \leq L, \\ \text{sign}(v)L, & \text{for } |v| > L. \end{cases}$$

Let us consider

$$w = \theta_* + T_L(u - \theta_*) = \begin{cases} \theta_* + L, & \text{for } u - \theta_* > L, \\ u, & \text{for } |u - \theta_*| \leq L, \\ \theta_* - L, & \text{for } u - \theta_* < -L. \end{cases} \tag{2.1}$$

Our nearest goal is to show that $w \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$. Indeed, it is obvious that $w \in W^{1,(p_i)}(\Omega)$; in order to prove $w \geq \psi$ a.e., we notice that in the first case of (2.1), $w = \theta_* + L \geq \theta_* \geq \psi$, in the second case of (2.1), $w = u \geq \psi$ since $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$, and in the third case of (2.1), $w = \theta_* - L > u \geq \psi$; in order to prove $w - \theta \in W_0^{1,(p_i)}(\Omega)$, we notice that $\theta_* = \theta$ on $\partial\Omega$, and then $T_L(u - \theta_*) = 0$ on $\partial\Omega$, thus $w = \theta_* = \theta$ on $\partial\Omega$.

(2.1) implies

$$Dw = (Du)1_{\{|u-\theta_*|\leq L\}} + (D\theta_*)1_{\{|u-\theta_*|>L\}}, \tag{2.2}$$

where 1_E is the characteristic function of E, that is, $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ for $x \notin E$. By Definition 1.1 one has

$$\int_\Omega f(x, Du(x))dx \leq \int_\Omega f(x, Dw(x))dx,$$

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which together with (2.2) yields

$$\left(\int_{\{|u-\theta_*|\leq L\}}+\int_{\{|u-\theta_*|>L\}}\right)f(x,Du(x))dx$$
$$\leq \left(\int_{\{|u-\theta_*|\leq L\}}+\int_{\{|u-\theta_*|>L\}}\right)f(x,Dw(x))dx \quad (2.3)$$
$$= \int_{\{|u-\theta_*|\leq L\}}f(x,Du(x))dx+\int_{\{|u-\theta_*|>L\}}f(x,D\theta_*(x))dx.$$

By assumption, all integrals in (2.3) are finite and we drop the integrals over $\{|u-\theta_*|\leq L\}$ from both sides of (2.3) arriving at

$$\int_{\{|u-\theta_*|>L\}}f(x,Du(x))dx \leq \int_{\{|u-\theta_*|>L\}}f(x,D\theta_*(x))dx. \quad (2.4)$$

Using (1.1) and (2.4), we have

$$\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_iu-D_i\theta_*|^{p_i}dx$$
$$\leq 2^{p_m-1}\left(\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_iu|^{p_i}dx+\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_i\theta_*|^{p_i}dx\right)$$
$$\leq 2^{p_m-1}\left(\int_{\{|u-\theta_*|>L\}}f(x,Du(x))dx+\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_i\theta_*|^{p_i}dx\right)$$
$$\leq 2^{p_m-1}\left(\int_{\{|u-\theta_*|>L\}}f(x,D\theta_*(x))dx+\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_i\theta_*|^{p_i}dx\right)$$
$$\leq 2^{p_m-1}\left(c_0\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}(\varphi_i(x)+|D_i\theta_*|)^{p_i}dx+\int_{\{|u-\theta_*|>L\}}\sum_{i=1}^{n}|D_i\theta_*|^{p_i}dx\right)$$
$$\leq 2^{p_m-1}(c_0+1)\int_{\{|u-\theta_*|>L\}}(\varphi_i(x)+|D_i\theta_*|)^{p_i}dx. \quad (2.5)$$

Let

$$t_i=\frac{p_i}{1-b},$$

where b be any number satisfying (1.4). This ensures

$$\frac{t_i-p_i}{t_i}=b$$

does not depend on i. Let

$$M=\max_{1\leq i\leq n}\left(\int_{\{|u-\theta_*|>L\}}(\varphi_i(x)+|D_i\theta_*|)^{t_i}dx\right)^{\frac{p_i}{t_i}}.$$

(1.4) implies M is finite. Hölder inequality yields

$$
\begin{aligned}
&\int_{\{|u-\theta_*|>L\}} (\varphi_i(x) + |D_i\theta_*|)^{p_i}\, dx \\
\leq\ & \left(\int_{\{|u-\theta_*|>L\}} (\varphi_i(x) + |D_i\theta_*|)^{t_i}\, dx \right)^{\frac{p_i}{t_i}} |\{|u-\theta_*| > L\}|^b \\
\leq\ & M|\{|u-\theta_*| > L\}|^b.
\end{aligned}
\tag{2.6}
$$

Combining (2.5) with (2.6) yields

$$
\int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} |D_iu - D_i\theta_*|^{p_i} dx \leq M|\{|u-\theta_*| > L\}|^b.
$$

Following the idea of the proof of Theorem 2.1 in [4], we complete the proof of Theorem 1.1.

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