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Global and blow-up solutions for a quasilinear parabolic equation with different kinds boundary condition

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Abstract

We investigate a quasilinear parabolic equation with a gradient term subject to homogeneous Dirichlet or Neumann boundary condition. By constructing auxiliary functions and using Hopf's maximum principle, we obtain the sufficient conditions for the existence of global and blow-up solutions, the upper bounds for the "blow-up time", the "upper estimates" of the "blow-up rate" and the "upper estimates" of the global solution. Finally, some application examples will be presented.

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1 Introduction

The global existence and blow-up solutions for the nonlinear evolution equations have been investigated extensively by many authors. In particular, for the parabolic equations with a gradient term, we refer to [1-6] etc. For example, P. Souplet and F. B. Weissler [1] studied semilinear parabolic equation

$$u_t = \Delta u + f(u, \nabla u), \quad \text{in } D \times (0, T),$$

subject to the homogeneous Dirichlet boundary condition. By using comparison principle and constructing self-similar lower solution, they obtained the sufficient conditions of global existence and blow-up solutions. Andreu [2] used similar method to study quasilinear parabolic equation

$$u_t = \Delta u^m + f(u, \nabla u^m), \quad \text{in } D \times (0, T).$$

Chen [3] considered the following semilinear parabolic equation

$$u_t = \Delta u + f(u) + g(u)|\nabla u|^2, \quad \text{in } D \times (0, T),$$

with homogeneous Dirichlet boundary condition. By estimating the integral of a ratio of one solution to the other, the author proved both global existence and blow up results. Then he used the same method to study more generalized equation with a gradient term, see [4].

For the nonlinear parabolic equations with Neumann boundary conditions, Lair and Oxley [5] considered quasilinear parabolic equation without a gradient term

$$u_t = \nabla \cdot (a(u)\nabla u) + f(u), \quad \text{in } D \times (0, T),$$

subject to homogeneous Neumann boundary conditions, they obtained the necessary and sufficient conditions for the global existence and blow-up solution by approximation method. Recently, Ding and Gao [6] have investigated initial boundary value problem of quasilinear parabolic equation with a gradient term

$$(g(u))_t = \Delta u + f(x, u, |\nabla u|^2, t), \quad \text{in } D \times (0, T),$$

subject to boundary flux $\frac{\partial u}{\partial n} = r(u)$, and obtained sufficient conditions for the global existence and blow-up solution, the upper estimate of global solution and blow-up time.

Motivated by the above works, in this article, we consider more generalized quasilinear parabolic equation with a gradient term

$$(g(u))_t = \nabla \cdot (a(u)b(x)c(t)\nabla u) + f(x, u, q, t), \quad \text{in } D \times (0, T), \tag{1.1}$$

subject to homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial D \times (0, T), \tag{1.2}$$

or Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial D \times (0, T), \tag*{(1.2)'}$$

and initial condition

$$u(x,0) = u_0(x) \geq 0, \quad \text{in } \overline{D}, \tag{1.3}$$

where $D \subseteq \mathbf{R}^N (N \geq 2)$ is a bounded domain with smooth boundary ∂D, $\overline{D}$ is the closure of D, $q = |\nabla u|^2$, n is the outer normal vector, $u_0(x) \in C^3(\overline{D})$ satisfies the compatibility conditions and T is the maximum existence time of $u(x,t)$. $a(u)b(x)c(t)$, $f(x,u,q,t)$ and $h(x,t)r(u)$ are nonlinear diffusion coefficient, reaction term and boundary flux, respectively. Let $\mathbf{R}^+ = (0,+\infty)$, $\overline{\mathbf{R}^+} = [0,+\infty)$, and suppose that the function $g(s) \in C^2(\mathbf{R}^+)$, $g'(s) > 0$ for any $s > 0$, $a(s) \in C^2(\mathbf{R}^+)$, $b(x) \in C^1(\overline{D})$, $c(t) \in C^1(\mathbf{R}^+)$, and $f(x,u,s,t) \in C^1(\overline{D} \times \mathbf{R}^+ \times \overline{\mathbf{R}^+} \times \overline{\mathbf{R}^+})$ is a nonnegative function. Under these assumptions, it is easy to know from the classical parabolic equation theory [7, Section 3] and Hopf's maximum principle [8] that problem (1.1)(1.2)(1.3) and problem (1.1)(1.2)$'$(1.3) both have a unique local positive solution $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ with some $T > 0$.

Our aim is to study existence of global or blow-up solutions, depending on the relations between the nonlinearites g, s, b, c, f and the nonlinear boundary conditions. For this purpose, we construct different auxiliary functions and use Hopf's maximum principle to establish the sufficient conditions for the global existence and blow-up of solutions, an upper bound for the "blow-up time", an upper estimate of the "blow-up rate" and an upper estimate of the global solution to problem (1.1)-(1.3), then some examples are given. The equation (1.1) describes nonlinear diffusion phenomenon in fields like physics, mechanics, biology and so on. For example, Newton flow in fluid dynamics, semiconductor, population dynamics model, see [9-12].

2 Blow-up solutions

In this section, we give sufficient conditions for the existence of blow-up solutions of problems (1.1)(1.2)(1.3) and (1.1)(1.2)$'$(1.3).

Theorem 2.1 *Let $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ be a solution of problem (1.1)(1.2)(1.3). Assume that the following conditions hold:*

(1) For any $(x,s,q,t) \in \overline{D} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$b(x) > 0,\ c(t) > 0; \tag{2.1}$$

(2) For any $(x,s,q,t) \in \overline{D} \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$a(s) \geq a'(s) \geq 0,\ f_q \geq 0,\ c'(t) \geq 0,\ g'(s) > 0,\ \ g''(s) \leq 0, \tag{2.2}$$

$$f_t(x,s,q,t) \geq \frac{c'(t)}{c(t)} f(x,s,q,t),\ f_s(x,s,q,t) \geq f(x,s,q,t) \geq 0; \tag{2.3}$$

(3) For any $x \in \{x \mid f(x, u_0, q_0, 0) = 0, x \in \overline{D}\}$,

$$\nabla(a(u_0)b(x)c(0)\nabla u_0) \geq 0; \tag{2.4}$$

(4) The constant

$$\beta = \min_{D_1}\{\frac{a(u_0)}{g'(u_0)e^{u_0}}[\nabla(a(u_0)b(x)c(0)\nabla u_0) + f(x, u_0, q_0, 0)]\} > 0, \tag{2.5}$$

where $D_1 = \{x \mid f(x, u_0, q_0, 0) \neq 0,\ x \in \overline{D}\} \neq \phi$, $q_0 = |\nabla u_0|^2$;

(5) The integration

$$\int_{M_0}^{+\infty} \frac{a(s)}{e^s - 1} ds < +\infty, \text{ where } M_0 = \max_{\overline{D}} u_0(x). \tag{2.6}$$

then the solution $u(x, t)$ *of system (1.1)(1.2)(1.3) must blow up in finite time* T *and*

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{e^s - 1} ds, \tag{2.7}$$

$$u(x, t) \leq \Phi^{-1}(\beta(T - t)), \tag{2.8}$$

where $\Phi(z) = \int_z^{+\infty} \frac{a(s)}{e^s-1} ds$, $z > 0$, *and* Φ^{-1} *is the inverse function of* Φ.

Proof of Theorem 2.1: Consider the auxiliary function

$$\Psi = -a(u)u_t + \beta e^u, \tag{2.9}$$

then we have

$$\nabla\Psi = -a' u_t \nabla u - a\nabla u_t + \beta e^u \nabla u, \tag{2.10}$$

$$\Delta\Psi = -a' u_t \Delta u - a'' q u_t - 2a' \nabla u \cdot \nabla u_t - a\Delta u_t + \beta e^u \Delta u + \beta e^u q, \tag{2.11}$$

and

$$\begin{aligned}
\Psi_t &= -a'(u_t)^2 + \beta e^u u_t - a(u_t)_t = -a'(u_t)^2 + \beta e^u u_t - a[\frac{1}{g'}(abc\Delta u + a'bcq + ac\nabla b \cdot \nabla u + f)]_t \\
&= -a'(u_t)^2 + \beta e^u u_t - \frac{a}{g'}(a'bcu_t\Delta u + abc'\Delta u + abc\Delta u_t + a''bcqu_t + a'bc'q + 2a'bc\nabla u \cdot \nabla u_t \\
&\quad + a'cu_t\nabla b \cdot \nabla u + ac'\nabla b \cdot \nabla u + ac\nabla b \cdot \nabla u_t + 2f_q\nabla u \cdot \nabla u_t + f_t + f_u u_t) \\
&\quad + \frac{ag''}{(g')^2}(abc\Delta u + a'bcq + ac\nabla b \cdot \nabla u + f)u_t
\end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\frac{abc}{g'}\Delta\Psi - \Psi_t$$

$$= -\frac{aa'bcg''}{(g')^2}qu_t + 2\frac{af_q}{g'}\nabla u\cdot\nabla u_t - \frac{a^2bcg''}{(g')^2}u_t\Delta u + (\frac{a^2bc'}{g'} + \beta\frac{abc}{g'}e^u)\Delta u$$
$$+(\frac{aa'bc'}{g'} + \beta\frac{abc}{g'}e^u)q + a'(u_t)^2 + (\frac{af_u}{g'} - \beta e^u - \frac{afg''}{(g')^2})u_t + \frac{a^2c'}{g'}\nabla b\cdot\nabla u$$
$$+(\frac{aa'c}{g'} - \frac{a^2cg''}{(g')^2})u_t\nabla b\cdot\nabla u + \frac{a^2c}{g'}\nabla b\cdot\nabla u_t + \frac{af_t}{g'}. \tag{2.13}$$

Using (2.10), we have

$$\nabla u_t = -\frac{1}{a}\nabla\Psi - \frac{a'}{a}u_t\nabla u + \beta\frac{1}{a}e^u\nabla u. \tag{2.14}$$

Now substituting (2.14) into (2.13) leads to

$$\frac{abc}{g'}\Delta\Psi + (\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u)\nabla\Psi - \Psi_t$$
$$= (-\frac{aa'bcg''}{(g')^2} - 2\frac{a'f_q}{g'})qu_t - \frac{a^2bcg''}{(g')^2}u_t\Delta u + (\frac{a^2bc'}{g'} + \beta\frac{abc}{g'}e^u)\Delta u$$
$$+(\frac{aa'bc'}{g'} + \beta\frac{abc}{g'}e^u + 2\beta\frac{f_q}{g'}e^u)q - \frac{a^2cg''}{(g')^2}u_t\nabla b\cdot\nabla u + (\frac{af_u}{g'} - \beta e^u - \frac{afg''}{(g')^2})u_t$$
$$+(\frac{a^2c'}{g'} + \beta\frac{ac}{g'}e^u)\nabla b\cdot\nabla u + a'(u_t)^2 + \frac{af_t}{g'}. \tag{2.15}$$

By (1.1), we have

$$\Delta u = \frac{1}{abc}(g'u_t - a'bcq - ac\nabla b\cdot\nabla u - f), \tag{2.16}$$

If we combine (2.15) and (2.16), we arrive at

$$\frac{abc}{g'}\Delta\Psi + (\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u)\nabla\Psi - \Psi_t$$
$$= -2\frac{a'f_q}{g'}qu_t + (\beta\frac{abc}{g'}e^u - \beta\frac{a'bc}{g'}e^u + 2\beta\frac{f_q}{g'}e^u - \frac{ag''}{g'})q + (\frac{af_u}{g'} + \frac{ac'}{c})u_t$$
$$-(\frac{ac'}{cg'} + \beta\frac{1}{g'}e^u)f + a'(u_t)^2 + \frac{af_t}{g'}. \tag{2.17}$$

In view of (2.9), it follows

$$u_t = -\frac{1}{a}\Psi + \beta\frac{1}{a}e^u, \tag{2.18}$$

then substituting (2.18) into (2.17) yields

$$\frac{abc}{g'}\Delta\Psi + (\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u)\nabla\Psi + (\frac{f_u}{g'} + \frac{c'}{c} + \frac{a'f_q}{ag'})\Psi - \Psi_t$$

$$= \beta\frac{1}{g'}(f_u e^u - f e^u) + \beta\frac{c'}{c}e^u + \frac{a}{g'}(f_t - \frac{c'}{c}f) + \beta(\frac{abc}{g'}e^u - \frac{a'bc}{g'}e^u)q$$

$$+2\beta(\frac{f_q}{g'}e^u - \frac{a'f_q}{ag'}e^u)q - \frac{ag''}{g'}q + a'(u_t)^2$$

$$= \beta\frac{1}{g'}e^{2u}(\frac{f}{e^u})' + \beta\frac{c'}{c}e^u + \frac{ac}{g'}(\frac{f}{c})' + \beta\frac{a^2bc}{g'}(\frac{e^u}{a})'q + 2\beta\frac{af_q}{g'}(\frac{e^u}{a})'q - \frac{ag''}{g'}q + a'(u_t)^2. \tag{2.19}$$

From assumptions (2.1)-(2.3), it follows that the right-hand side of (2.19) is nonnegative, i.e.

$$\frac{abc}{g'}\Delta\Psi + (\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u)\nabla\Psi + (\frac{f_u}{g'} + \frac{c'}{c} + \frac{a'f_q}{ag'})\Psi - \Psi_t \geq 0. \tag{2.20}$$

In fact, we can see from (2.4) and (2.5) that

$$\max_{\overline{D}} \Psi(x,0) = \max_{\overline{D}}\{-\frac{a(u_0)}{g'(u_0)}[\nabla(a(u_0)b(x)c(0)\nabla u_0) + f(x,u_0,q_0,0)] + \beta e^{u_0}\} \leq 0. \tag{2.21}$$

Also, on $\partial D \times (0,T)$, we have $u_t = 0$, therefore

$$\Psi = -a(0)u_t + \beta e^0 = \beta, \tag{2.22}$$

By combining (2.20)-(2.22) and using the Hopf's maximum principles, we find that the maximum of Ψ on $\overline{D} \times (0,T)$ is β, i.e.

$$\Psi \leq \beta \; in \; \overline{D} \times (0,T),$$

and by (2.9), we can see that

$$\frac{a(u)}{e^u - 1}u_t \geq \beta. \tag{2.23}$$

At the point $x_0 \in \overline{D}$, where $u_0(x_0) = M_0$, integrating (2.23) over $[0,t]$, we get

$$\frac{1}{\beta}\int_{M_0}^{u(x_0,t)} \frac{a(s)}{e^s - 1}ds \geq t. \tag{2.24}$$

This together with assumption (2.6) shows that $u(x,t)$ must blow up at the finite time T, moreover

$$T \leq \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{a(s)}{e^s - 1}ds. \tag{2.25}$$

For each fixed x, integrating the inequality (2.23) over $[t,s](0 < t < s < T)$ yields

$$\Phi(u(x,t)) \geq \Phi(u(x,t)) - \Phi(u(x,s)) = \int_{u(x,t)}^{u(x,s)} \frac{a(s)}{e^s - 1}ds = \int_t^s \frac{a(u)}{e^u - 1}u_t dt \geq \beta(s-t),$$

If we let $s \to T$,then formally

$$\Phi(u(x,t)) \geq \beta(T-t),$$

therefore

$$u(x,t) \leq \Phi^{-1}(\beta(T-t)).$$

The proof is completed.

Theorem 2.2 *Let $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ be a solution of problem (1.1)$(1.2)'$(1.3). Assume that the following conditions hold:*

(1) For any $(x,s,q,t) \in \overline{D} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$b(x) > 0, \ c(t) > 0; \tag{2.26}$$

(2) For any $(x,s,q,t) \in \overline{D} \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$a(s) \geq a'(s) \geq 0, \ f_q \geq 0, \ c'(t) \geq 0, \ g'(s) > 0, \ \ g''(s) \leq 0, \tag{2.27}$$

$$f_t(x,s,q,t) \geq \frac{c'(t)}{c(t)} f(x,s,q,t), \ f_s(x,s,q,t) \geq f(x,s,q,t) \geq 0; \tag{2.28}$$

(3) For any $x \in \{x \mid f(x,u_0,q_0,0) = 0, x \in \overline{D}\}$,

$$\nabla(a(u_0)b(x)c(0)\nabla u_0) \geq 0; \tag{2.29}$$

(4) The constant

$$\beta = \frac{a(m_0)}{g'(m_0)e^{m_0}} \min_{D_1} f(x,m_0,0,0) > 0, \tag{2.30}$$

where $D_1 = \{x \mid f(x,u_0,q_0,0) \neq 0, \ x \in \overline{D}\} \neq \phi$, $q_0 = |\nabla u_0|^2$;

(5) The integration

$$\int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds < +\infty, \ \text{where } m_0 = \min_{\overline{D}} u_0(x). \tag{2.31}$$

then the solution $u(x,t)$ of system (1.1)$(1.2)'$(1.3) must blow up in finite time T and

$$T \leq \frac{1}{\beta} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds. \tag{2.32}$$

Proof of Theorem 2.2: Let $\underline{u}$ be a solution of the following problem

$$(g(u))_t = \nabla \cdot (a(u)b(x)c(t)\nabla u) + f(x,u,q,t), \quad \text{in } D \times (0,T),$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial D \times (0,T),$$

$$u(x,0)=m_0, \quad \text{in } \overline{D},$$

where $q=|\nabla u|^2$, then $\underline{u}$ is a lower solution of (1.1)(1.2)$'$(1.3). In order to show that u blows up in finite time, we merely have to show that $\underline{u}$ blows up in some finite time.

Consider the auxiliary function

$$\Psi=-a(\underline{u})\underline{u}_t+\beta e^{\underline{u}}. \tag{2.33}$$

Going along the lines of Theorem 2.1 and under the assumptions of (2.26)-(2.28), it follows

$$\frac{abc}{g'}\Delta\Psi+(\frac{ac}{g'}\nabla b+2\frac{f_q}{g'}\nabla\underline{u})\nabla\Psi+(\frac{f_{\underline{u}}}{g'}+\frac{c'}{c}+\frac{a'f_q}{ag'})\Psi-\Psi_t\geq 0. \tag{2.34}$$

We can see from (2.29) and (2.30) that

$$\max_{\overline{D}}\Psi(x,0)=\max_{\overline{D}}\{-\frac{a(m_0)}{g'(m_0)e^{(m_0)}}[\nabla(a(m_0)b(x)c(0)\nabla m_0)+f(x,m_0,q_0,0)]+\beta e^{m_0}\leq 0. \tag{2.35}$$

Also, on $\partial D\times(0,T)$, it gives

$$\Psi=-a'\underline{u}_t\frac{\partial\underline{u}}{\partial n}-a\frac{\partial\underline{u}_t}{\partial n}+\beta e^{\underline{u}}\frac{\partial\underline{u}}{\partial n}=-a\frac{\partial\underline{u}_t}{\partial n}=0, \tag{2.36}$$

Now combining (2.34)-(2.36) and using the Hopf's maximum principles, we find that the maximum of Ψ on $\overline{D}\times(0,T)$ is β, i.e.

$$\Psi\leq\beta \; in \; \overline{D}\times(0,T),$$

and by (2.33), we can see that

$$\frac{a(\underline{u})}{e^{\underline{u}}}\underline{u}_t\geq\beta. \tag{2.37}$$

At the point $x_0\in\overline{D}$, where $\underline{u}_0(x_0)=M_0$, integrating (2.37) over $[0,t]$ yields

$$\frac{1}{\beta}\int_{M_0}^{\underline{u}(x_0,t)}\frac{a(s)}{e^s}ds\geq t. \tag{2.38}$$

This together with assumption (2.31) shows that $u(x,t)$ must blow up at the finite time T, moreover

$$T\leq\frac{1}{\beta}\int_{M_0}^{+\infty}\frac{a(s)}{e^s}ds. \tag{2.39}$$

The proof is completed.

3 Global solutions

In this section, we give nonexistence result of global solution of problem (1.1)(1.2)(1.3) and sufficient conditions for the existence of blow-up solution of problem (1.1)(1.2)$'$(1.3) under suitable conditions.

Theorem 3.1 *The system (1.1)(1.2)(1.3) does not have global solution for all* $t > 0$.

Proof of Theorem 3.1: We prove this theorem by contradiction. If (1.1)(1.2)(1.3) has a global solution u, then the following assumptions about $a(u)$ must hold

$$a(s) > 0,\ a'(s) \leq 0,\ a(s) \leq a'(s),\ \forall s \in \mathbf{R}^+.$$

It is clear that such $a(u)$ is non-existent. Therefore, the system (1.1)(1.2)(1.3) does not have the global solution.

Theorem 3.2 *Let* $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ *be a solution of (1.1)*(1.2)$'$*(1.3). Assume that the following conditions hold:*

(1) For any $(x, s, q, t) \in \overline{D} \times \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$b(x) > 0,\ c(t) > 0; \tag{3.1}$$

(2) For any $(x, s, q, t) \in \overline{D} \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$,

$$a'(s) \leq 0,\ a(s) + a'(s) \leq 0,\ f_q \leq 0,\ c'(t) \leq 0,\ g'(s) > 0,\ \ g''(s) \geq 0, \tag{3.2}$$

$$f_t(x, s, q, t) \leq \frac{c'(t)}{c(t)} f(x, s, q, t),\ f_s(x, s, q, t) + f(x, s, q, t) \leq 0; \tag{3.3}$$

(3) For any $x \in \{x \mid f(x, u_0, q_0, 0) = 0, x \in \overline{D}\}$,

$$\nabla(a(u_0)b(x)c(0)\nabla u_0) \geq 0; \tag{3.4}$$

(4) The constant

$$\alpha = \frac{a(M_0)}{g'(M_0)e^{M_0}} \max_{D_1} f(x, M_0, 0, 0) > 0, \tag{3.5}$$

where $D_1 = \{x \mid f(x, u_0, q_0, 0) \neq 0,\ x \in \overline{D}\} \neq \phi$, $q_0 = |\nabla u_0|^2$*;*

(5) The integration

$$\int_{M_0}^{+\infty} \frac{a(s)}{e^s} ds < +\infty,\ \text{where } M_0 = \max_{\overline{D}} u_0(x). \tag{3.6}$$

then the solution $u(x, t)$ *of system (1.1)*(1.2)$'$*(1.3) must be a global solution, and*

$$u(x, t) \leq \Psi^{-1}(\beta(T - t)), \tag{3.7}$$

where $\Psi(z) = \int_{M_0}^{z} \frac{a(s)}{e^{-s}} ds,\ z > 0,$ *and* Ψ^{-1} *is the inverse function of* Φ.

Proof of Theorem 3.2: Let $\overline{u}$ be a solution of the following problem

$$(g(u))_t = \nabla \cdot (a(u)b(x)c(t)\nabla u) + f(x,u,q,t), \quad \text{in } D\times(0,T),$$
$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial D\times(0,T),$$
$$u(x,0) = M_0, \quad \text{in } \overline{D},$$

where $q = |\nabla u|^2$, then $\overline{u}$ is an upper solution of (1.1)(1.2)$'$(1.3). We show that $\overline{u}$ must be a global solution, thus the solution u of (1.1)(1.2)$'$(1.3) must be a global solution.

Consider the auxiliary function

$$\Phi = -a(\overline{u})\overline{u}_t + \alpha e^{-\overline{u}}. \tag{3.8}$$

We first repeat the arguments of the proof of Theorem 2.1 by replacing Ψ and β by Φ and α, respectively, and under the assumptions of (3.1)-(3.3), it follows

$$\frac{abc}{g'}\Delta\Phi + (\frac{ac}{g'}\nabla b + 2\frac{f_{\overline{q}}}{g'}\nabla\overline{u})\nabla\Phi + (\frac{f_{\overline{u}}}{g'} + \frac{c'}{c} + \frac{a' f_{\overline{q}}}{ag'})\Phi - \Phi_t \le 0. \tag{3.9}$$

Then we can see from (3.4) and (3.5) that

$$\max_{\overline{D}}\Phi(x,0) = \max_{\overline{D}}\{-\frac{a(M_0)}{g'(M_0)e^{(M_0)}}[\nabla(a(M_0)b(x)c(0)\nabla M_0)+f(x,M_0,0,0)]+\alpha e^{-M_0}\} \le 0. \tag{3.10}$$

Also, on $\partial D\times(0,T)$, it gives

$$\frac{\partial\Phi}{\partial n} = -a'\overline{u}_t\frac{\partial\overline{u}}{\partial n} - a\frac{\partial\overline{u}_t}{\partial n} - \alpha e^{-\overline{u}}\frac{\partial\overline{u}}{\partial n} = -a\frac{\partial\overline{u}_t}{\partial n} = 0. \tag{3.11}$$

Now combining (3.9)-(3.11) and using the Hopf's maximum principles, we find that the maximum of Ψ on $\overline{D}\times(0,T)$ is β, i.e.

$$\Phi \ge 0 \ in \ \overline{D}\times(0,T),$$

and by (3.8), we can see that

$$\frac{a(\overline{u})}{e^{-\overline{u}}}\overline{u}_t \le \alpha. \tag{3.12}$$

At the point $x_0 \in \overline{D}$, where $\overline{u}_0(x_0) = M_0$, integrating (3.12) over $[0,t]$ leads to

$$\frac{1}{\alpha}\int_{M_0}^{\overline{u}(x_0,t)} \frac{a(s)}{e^{-s}}ds \le t. \tag{3.13}$$

This together with assumption (3.6) shows that $\overline{u}(x,t)$ must be a global solution, moreover

$$\Psi(\overline{u}(x,t)) - \Psi(\overline{u}_0(x,t)) = \int_{\overline{u}_0(x)}^{\overline{u}(x,s)} \frac{a(s)}{e^{-s}}ds \le \alpha t,$$

therefore

$$\overline{u}(x,t) \le \Phi^{-1}(\beta(T-t)).$$

The proof is completed.

4 Applications

In what follows, we present several examples to demonstrate the applications of Theorem 2.1, Theorem 2.2 and Theorem 3.2.

Example 1. Let u be a solution of

$$(u+\sqrt{u+1})_t = \nabla \cdot (e^{\frac{u}{2}}(2-\sum_{i=1}^{3} x_i^2)e^t \nabla u) + (9+q+\sum_{i=1}^{3} x_i^2)e^u e^t, \quad \text{in } D\times(0,T)$$

$$u=0, \quad \text{on } \partial D\times(0,T)$$

$$u(x,0)=u_0(x)=1-\sum_{i=1}^{3} x_i^2, \quad \text{in } \overline{D}$$

where $D=\{x=(x_1,x_2,x_3)\mid \sum_{i=1}^{3} x_i^2<1\}$, then we have

$$g(u)=u+\sqrt{u+1},\ a(u)=e^{\frac{u}{2}},\ b(x)=2-\sum_{i=1}^{3} x_i^2,\ c(t)=e^t,$$

$$f(x,u,q,t)=(9+q+\sum_{i=1}^{3} x_i^2)e^u e^t.$$

It is easy to verify that (2.1)-(2.4) hold. In order to determine β, we suppose $w=\sum_{i=1}^{3} x_i^2$. By (2.5), we find

$$\beta = \min_{D_1}\{\frac{a(u_0)}{g'(u_0)e^{u_0}}[\nabla(a(u_0)b(x)c(0)\nabla u_0)+f(x,u_0,q_0,0)]\}$$

$$=\min_{0\le w<1}\{\frac{2}{2+(2-w)^{-\frac{1}{2}}}[14w-2w^2-12+(9+5w)e^{\frac{1-w}{2}}]\} \doteq 2.0971.$$

It follows from Theorem 2.1 that $u(x,t)$ must blow up at a finite time T and

$$T\le \frac{1}{\beta}\int_{M_0}^{+\infty}\frac{a(s)}{e^s-1}ds = \frac{1}{\beta}\int_{1}^{+\infty}\frac{a(s)}{e^s-1}ds = \frac{1}{2.0971}\ln\frac{\sqrt{e}+1}{\sqrt{e}-1},$$

$$u(x,t)\le \Phi^{-1}(\beta(T-t))=\frac{1}{2}.$$

Example 2. Let u be a solution of

$$(u+\sqrt{u})_t = \nabla \cdot (e^{\frac{u}{2}}(1+\sum_{i=1}^{3} x_i^2)e^t \nabla u) + \exp(u+q+t+\sum_{i=1}^{3} x_i^2), \quad \text{in } D\times(0,T)$$

$$\frac{\partial u}{\partial n}=0, \quad \text{on } \partial D\times(0,T)$$

$$u(x,0) = u_0(x) = 1 + (1 - \sum_{i=1}^{3} x_i^2)^2, \quad \text{in } \overline{D}$$

where $D = \{x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1\}$, then we have

$$g(u) = u + \sqrt{u},\ a(u) = e^{\frac{u}{2}},\ b(x) = 1 + \sum_{i=1}^{3} x_i^2,\ c(t) = e^t,$$

$$f(x, u, q, t) = \exp(u + q + t + \sum_{i=1}^{3} x_i^2).$$

It is easy to verify that (2.26)-(2.29) hold. By (2.30), we find

$$\beta = \frac{a(m_0)}{g'(m_0)e^{m_0}} \min_{D_1} f(x, m_0, 0, 0)$$

$$= \frac{2}{3} e^{-\frac{1}{2}} \min_{D_1} \exp(1 + \sum_{i=1}^{3} x_i^2) = \frac{2}{3} e^{\frac{1}{2}}.$$

It follows from Theorem 2.2 that $u(x,t)$ must blow up at a finite time T and

$$T \le \frac{1}{\beta} \int_{m_0}^{+\infty} \frac{a(s)}{e^s} ds = \frac{3}{2} e^{-\frac{1}{2}} \int_{1}^{+\infty} \frac{e^{\frac{s}{2}}}{e^s} ds = \frac{3}{e},$$

$$u(x,t) \le \Phi^{-1}(\beta(T-t)) = 2\ln(3e^{-\frac{1}{2}}(T-t)^{-1}).$$

Example 3. Let u be a solution of

$$(e^u)_t = \nabla \cdot (e^{-u}(1 + \sum_{i=1}^{3} x_i^2) e^{-t} \nabla u) + \exp(-u - q - t - \sum_{i=1}^{3} x_i^2), \quad \text{in } D \times (0,T)$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial D \times (0,T)$$

$$u(x,0) = u_0(x) = 1 + (1 - \sum_{i=1}^{3} x_i^2)^2, \quad \text{in } \overline{D}$$

where $D = \{x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1\}$, then we have

$$g(u) = e^u,\ a(u) = e^{-u},\ b(x) = 1 + \sum_{i=1}^{3} x_i^2,\ c(t) = e^{-t},$$

$$f(x, u, q, t) = \exp(-u - q - t - \sum_{i=1}^{3} x_i^2).$$

It is easy to verify that (3.1)-(3.4) hold. By (3.5), we find

$$\alpha = \frac{a(M_0)}{g'(M_0)} e^{M_0} \max_{D_1} f(x, M_0, 0, 0)$$

$$= e^{-2} \max_{D_1} \exp(-2 - \sum_{i=1}^{3} x_i^2) = e^{-4}.$$

It follows from Theorem 3.2 that $u(x,t)$ must be a global solution and

$$u(x,t) \leq \Psi^{-1}(\alpha t) = e^{-4}t + 2.$$

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