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Geometry of submersions on manifolds of nonnegative curvature

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Abstract

In the paper it is studied curvature properties of foliation generated by submersions on manifolds of nonnegative curvature. It is proved if length of gradient vector of every coordinate function of the submersion is constant on level surface then every leaf of the foliation generated be submersion is a manifold of nonnegative section curvature.

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1 Introduction

Let (M, g)-be a smooth riemannian manifold of dimension n, g-riemannian metric, T_pM-tangent space at a point $p \in M$, ∇ – Levi-Civita connection defined by the Riemannian metric g. We denote by $V(M)$ the set of all smooth vector fields defined on M. Throughout the paper, the smoothness means smoothness of class C^∞.

The curvature tensor R of the Levi-Civita connection ∇ is defined as a mapping $R : V(M) \times V(M) \times V(M) \to V(M)$ from the formula

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $[X, Y]$- Lie bracket of vector fields X, Y. The curvature tensor R is a tensor of type (3,1)[1].

Let X, Y-linearly independent vector fields, σ- two-dimensional plane generated by the pair X, Y. For the plane σ associate a real number K_σ :

$$K_\sigma = \frac{\langle R(X,Y)Y, X\rangle}{|X|^2|Y|^2 - (\langle X,Y\rangle)^2},$$

where $\langle X, Y\rangle$-the scalar product defined by the Riemannian metric g. Value K_σ is called the Riemann curvature relative to the plane σ or section curvature in the two-dimensional direction σ. The manifold M is called a manifold of non-negative curvature if $K_\sigma \geq 0$ for all σ.

Let F-be foliation of dimension k, where $0 < k < n$ [6]. Denote by L_p leaf of foliation F, passing through the point $p \in M$, by T_qF-the tangent space of leaf L_p at the point $q \in L_p$, by $H(q)$ - orthogonal complement of subspace T_qF. As result arises subbundle's $TF = \{T_qF\}$, $TH = \{H(q)\}$ of the tangent bundle TM and we have an orthogonal decomposition $TM = TF \oplus TH$. Thus every vector field X is decomposable as: $X = X^v + X^h$, where $X^v \in TF$, $X^h \in TH$. If X^h=0 (respectively $X^v = 0$), then the field X is called the vertical (respectively horizontal) vector field.

Riemannian metric g on a manifold M induces a Riemannian metric $\widetilde{g}$ on the leaf L_p. With respect to these metrics canonical injection $i : L_p \to M$ is an isometric immersion. Connection ∇ induces a connection $\widetilde{\nabla}$ on L_p, which coincides with the connection defined by the Riemannian metric $\widetilde{\nabla}$. This connection defines the sectional curvature of manifolds L_p [1]. In this paper we consider a foliation generated by submersion, and study the relationship between the sectional curvatures of manifolds M and L_p.

Recall that a differentiable mapping $f : M \to B$ of maximal rank, where M, B-are smooth manifold of dimension n, m respectively $n > m$, is called a submersion. By the theorem on the rank of a differentiable function for each point $p \in B$ of the full inverse image $f^{-1}(p)$ is a submanifold of dimension $k = n - m$. Thus submersion $f : M \to B$ generates a foliation F of dimension $k = n - m$ on a manifold M, whose leaves are submanifolds $L_p = f^{-1}(p) : p \in B$. Study of the geometry and topology of foliations generated submersions, is the subject of numerous studies [2],[3], in particular in [5]derived the fundamental equations of submersion.

If the foliation generated by submersion $f : M \to B$, the subspace T_qF coincides with the subspace $Kerdf_q$ of the tangent space T_qM, where df_q-the differential of the map f at the point q.

2 Main Results

Let's consider a submersion $f : M \to R^k$, $f(p) = \{f_1(p), f_2(p), \cdots f_k(p)\}$.

The following theorem shows the relationship between the sectional curvatures of the manifolds M and a leaf of the submersion f.

Theorem 2.1 *Let M-is a manifold of constant non-negative (positive) sectional curvatures, and $X(|gradf_i|^2) = 0$ for each vertical vector field X. Then, every leaf of foliation F is a submanifold of a non-negative (positive) sectional curvature.*

Proof. Let L-is a leaf of foliation F, $q \in L$, $V_1, V_2 \in T_qF$. It is known that, sectional curvature $K_\sigma, \widetilde{K}_\sigma$ of manifolds L, M in the direction of a two-dimensional subspace $\sigma \subset T_qL$ is connected by a relation (Gauss Equation)

$$K_\sigma = \widetilde{K}_\sigma + \sum_{i=1}^{k}\left[\frac{1}{|gradf_i|^2}\det\begin{pmatrix} h_i(V_1,V_1) & h_i(V_1,V_2) \\ h_i(V_1,V_2) & h_i(V_2,V_2)\end{pmatrix}\right],$$

where h_i-is hessian of function f_i, σ-is a plane, generated by vectors V_1, V_2.

Therefore it is sufficient to us to prove for i that,

$$\det\begin{pmatrix} h_i(V_1,V_1) & h_i(V_1,V_2) \\ h_i(V_1,V_2) & h_i(V_2,V_2)\end{pmatrix} > 0.$$

As it is known the hessian of the function f_i is defined by a relation $h_i(X,Y) = \langle \nabla_X Z, Y\rangle$, $Z = gradf_i$, where the Hessian tensor $\nabla_X Z$ is defined by a symmetric matrix A:

$$\nabla_X Z = AX$$

We know that if $X(|gradf_i|^2) = 0$ for each vertical vector field X, then each gradient line of the function f_i is a geodesic line of Riemannian manifold M [3]. By definition, the gradient line is a geodesic if and only if when $\nabla_N N = 0$, where $N = \frac{Z}{|Z|}$. We compute the covariant differential

$$\nabla_N N = \frac{1}{|Z|}\nabla_Z N = \frac{1}{|Z|}\left\{\frac{1}{|Z|}\nabla_Z Z + Z(\frac{Z}{|Z|})Z\right\} = 0$$

and find $\nabla_Z Z = \lambda Z$, where $\lambda = -|Z|Z(\frac{1}{|Z|})$. This means that the gradient vector Z is the eigenvector of the matrix A.

Let $X_1, X_2, \cdots, X_{n-1}, Z$-mutually orthogonal eigenvectors of the matrix A at the point q such, that $X_1, X_2, \cdots, X_{n-1}$ vectors of unit length. Z^0-the value of the gradient field at the point q. Locally, they can be continued to vector field $X_1, X_2, \cdots, X_{n-1}, Z$ in a neighborhood (say U) of a point q so that at each point of U they are generate orthogonal basis consisting of the eigenvectors.

Using vectors $X_1^0, X_2^0, \cdots, X_{n-1}^0, Z^0$ we introduce normal coordinate system of $(x_1, x_2, \cdots, x_n)$ in a neighborhood U of the point q. Components g_{ij} of the metric g and the connected components of Γ_{ij}^k in the normal coordinate system satisfy the conditions ([1],p.132)

$$g_{ij}(q) = \delta_{ij}, \Gamma_{ij}^k = 0.$$

We show that $X(\lambda) = 0$ for each vertical field X. We set $\varphi = \frac{1}{|Z|}$. Then $X(\lambda) = -X(|Z|)Z(\frac{1}{|Z|}) - |Z|X(Z(\frac{1}{|Z|}))$. From the condition of the theorem it follows that $X(|Z|) = 0$. We compute

$$X(Z(\frac{1}{|Z|})) = X(Z(\varphi)) = [X, Z](\varphi) - Z(X(\varphi)).$$

In [6], is shown that $X(|gradf_i|^2) = 0$ for each vertical vector field X if and only if, when $h_i(X, Z) = 0$. It follows that $[X, Z]$ is a vertical field. Therefore, under the condition of the theorem we have $X(Z(\varphi)) = 0$. Thus λ is a constant function on each leaf L.

Now denote by $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ the eigenvalues of the matrix A, corresponding to the eigenvectors $X_1, X_2, \cdots, X_{n-1}$. Then the matrix A has the form

$$A = \begin{pmatrix} \lambda_1 & 0 \cdots & 0 & 0 \\ 0 & \lambda_2 \cdots & 0 & 0 \\ \vdots & & & \\ 0 & 0 \ \cdots & \lambda_{n-1} & 0 \\ 0 & 0 \ \cdots & 0 & \lambda \end{pmatrix}.$$

By the definition of a normal coordinate system it takes place

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i, \nabla_{X_i} Z = \nabla_Z X_i$$

at the point q for each i, j.

Under the condition of the theorem the vector field $\nabla_X Z$ is vertical field. Therefore, the Codazzi equations are of the form [1]:

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

From Codazzi equation we get

$$\nabla_{X_i} AX_j = \nabla_{X_j} AX_i, \nabla_{X_i} AZ = \nabla_Z AX_i$$

at the point q for each vector field X_i. From this it follows that $X_i(\lambda_j)X_j = X_j(\lambda_i)X_i$ and $X_i(\lambda)Z = Z(\lambda_i)X_i$. By linear independence of the vector fields $X_1, X_2, \cdots, X_{n-1}, Z$ we have that $X_i(\lambda_j) = 0$ for $i \neq j$ and $Z(\lambda_i) = 0$ for all i.

On the other hand

$$\nabla_Z AX_i = X_i(\lambda) Z + \lambda \nabla_{X_i} Z$$

Given the equality $\nabla_{X_i} AZ = \nabla_Z AX_i$ $\nabla_{X_i} Z = \lambda_i X_i$ we find that

$$\lambda_i^2 X_i + Z(\lambda_i) X_i = X_i(\lambda) Z + \lambda\lambda_i X_i.$$

Since $Z(\lambda_i) = 0$, $X_i(\lambda) = 0$ at the point q, this implies that, if $\lambda_i \neq 0$, then $\lambda_i = \lambda$. In particular, this implies that if $\lambda_i \neq 0$, then $X(\lambda_i) = X(\lambda) = 0$ and $Z(\lambda) = Z(\lambda_i) = 0$ for all i.

Thus, in the neighborhood U at the point q of all nonzero eigenvalues of the matrix A are equal to constant λ. If $\lambda = 0$, then all principal curvatures of the level surface are equal to zero.

Suppose now that X, Y - the vertical fields on U , which coincide with the vectors V_1, V_2 at the point q. We can write the vector fields X, Y in the following form

$$X = \sum_i \varphi_i X_i, Y = \sum_i \psi_i X_i.$$

Then we have that

$$h_i(X,X) = \langle AX, X \rangle = \sum \varphi_i^2 \lambda_i,$$

$$h_i(Y,Y) = \langle AY, Y \rangle = \sum \psi_i^2 \lambda_i,$$

$$h_i(X,Y) = \langle AX, Y \rangle = \sum \varphi_i \psi_i \lambda_i.$$

Simple computation shows that

$$\det \begin{pmatrix} h_i(X,X) & h_i(X,Y) \\ h_i(X,Y) & h_i(Y,Y) \end{pmatrix} = \sum_{i<j} \lambda_i \lambda_j \lambda_{ij},$$

where $\lambda_{ij} = (\varphi_i \psi_j - \varphi_j \psi_i)^2 \geq 0$. Since all nonzero principal curvatures λ_i are equal to λ, we find that all the sectional curvatures of the surface L in the direction of the plane σ, defined by the vectors V_1, V_2, positive - in the case of $\widetilde{K}_\sigma > 0$ and non-negative - in the case of $\widetilde{K}_\sigma = 0$

The theorem is proved.

Note. Sectional curvatures of leaves are not necessarily constant.

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