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# Generalized Dunkl-Williams inequality

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**Abstract**

Dunkl and Williams showed that

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{4\|u-v\|}{\|u\| + \|v\|}.$$

for any nonzero elements $u, v$ in a normed linear space $X$. Pečarić and Rajić gave a refinement and, moreover, a generalization to operators $A, B$ belong to the algebra $B(\mathcal{H})$ of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$, such that $|A|, |B|$ are invertible as follows:

$$|A|A|^{-1} - B|B|^{-1}|^2 \leq |A|^{-1}(r|A-B|^2 + s(|A| - |B|)^2)|A|^{-1},$$

where $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$.
In this paper, we generalized this inequality in the framework of Hilbert C*-modules. As a consequence we investigate this inequality without assumption of the invertibility of the absolute value of operator $B$.

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# 1 Introduction

Suppose that $B(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$.

In [2], Dunkl and Williams showed that for any nonzero elements $u, v$ in a normed linear space $X$

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{4\|u-v\|}{\|u\| + \|v\|}. \tag{1}$$

Pečarić and Rajić [6] gave the following refinement of (1): For any nonzero elements $u, v$ in a normed linear space $X$

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{\sqrt{2\|u-v\|^2 + 2(\|u\| - \|v\|)^2}}{\max\{\|u\|, \|v\|\}}. \tag{2}$$

Also in [6] they generalized the inequality (2) to the operators $A, B$ belong to the algebra $B(\mathcal{H})$ such that $|A|, |B|$ are invertible as follows:

**Theorem 1.1.** *Let $A, B \in B(\mathcal{H})$ of all bounded linear operators acting on a complex Hilbert space $H$ such that $|A|$ and $|B|$ are invertible, and let $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|A|A|^{-1} - B|B|^{-1}|^2 \leq |A|^{-1}(r|A-B|^2 + s(|A| - |B|)^2)|A|^{-1}.$$

*The equality holds if and only if $(r-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1})$.*

Where $|C| = (C^*C)^{\frac{1}{2}}$ denotes the absolute value of $C \in B(\mathcal{H})$. Note that $A \geq 0$ means that $\langle Ax, x\rangle \geq 0$ for all $x \in H$, and $A \leq 0$ represents that $-A \geq 0$, and $A \geq B$ if $A$ and $B$ are self-adjoint operators and $A - B \geq 0$, for any $A, B \in B(\mathcal{H})$.

# 2 Preliminaries

Let us recall some definitions and basic properties of C*-algebras and Hilbert C*-modules that we need in the rest of the parer. A Banach $*$-algebra $\mathcal{A}$ is called a C*-algebra if it satises $\|a^*a\|^2 = \|a\|$ for any $a \in \mathcal{A}$. An element $a$ of a C*-algebra $\mathcal{A}$ is positive if there exists $b \in \mathcal{A}$ such that $a = b^*b$. We write $a \geq 0$ to mean that $a$ is positive. The relation " $\leq$ " given by

$$a \le b \text{ if and only if } b - a \text{ is positive}$$

defines a partial ordering on $\mathcal{A}$. Let $\mathcal{A}$ be a C*-algebra then the absolute value of $a$ is defined by $|a| = (a^*a)^{\frac{1}{2}}$. For undefined notions and more details on C*-algebra theory, we refer to [5].

Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{H}$ be a right $\mathcal{A}$-module. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ that possesses the following properties,
$(i)$ $\langle u, u\rangle \ge 0$, for all $u \in \mathcal{H}$ and $\langle u, u\rangle = 0$ if and only if $u = 0$;
$(ii)$ $\langle u, \alpha v + \beta w\rangle = \alpha\langle u, v\rangle + \beta\langle u, w\rangle$, for all $\alpha, \beta \in \mathbb{C}$ and $u, v, w \in \mathcal{H}$;
$(iii)$ $\langle u, v\rangle = \langle v, u\rangle^*$, for all $u, v \in \mathcal{H}$;
$(iv)$ $\langle u, va\rangle = \langle u, v\rangle a$, for all $a \in \mathcal{A}$ and $u, v \in \mathcal{H}$;
The action of $\mathcal{A}$ on $\mathcal{H}$ is $\mathbb{C}$- and $\mathcal{A}$-linear, i.e., $\mu(ua) = u(\mu a) = (\mu u)a$ for every $\mu \in \mathbb{C}$, $a \in \mathcal{A}$ and $u \in \mathcal{H}$. For $u \in \mathcal{H}$, we define $\|u\| = \|\langle u, u\rangle\|^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over $\mathcal{A}$.
The C*-algebra $\mathcal{A}$ itself can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = a^*b$ for any $a, b \in \mathcal{A}$. For a C*-algebra $\mathcal{A}$ the standard Hilbert $\mathcal{A}$-module $\ell^2(\mathcal{A})$ is defined by

$$\ell^2(\mathcal{A}) = \{\{a_j\}_{j\in\mathbb{N}} : \sum_{j\in\mathbb{N}} a_j^* a_j \ \ \textit{converges in } \mathcal{A}\}$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j\in\mathbb{N}}, \{b_j\}_{j\in\mathbb{N}}\rangle = \sum_{j\in\mathbb{N}} a_j^* b_j$. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert modules over C*-algebra $\mathcal{A}$. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a mapping $T^* : \mathcal{K} \to \mathcal{H}$ satisfying $\langle Tu, v\rangle = \langle u, T^*v\rangle$ where $u \in \mathcal{H}$ and $v \in \mathcal{K}$. The mapping $T^*$ is called the adjoint of $T$.

For every $u$ in Hilbert C*-module $\mathcal{H}$ we define the absolute value of $u$ as the unique positive square root of $\langle u, u\rangle$, that is, $|u| = \langle u, u\rangle^{\frac{1}{2}}$. We refer the reader to [3, 7, 8] for more information on Hilbert C*-modules.

The following Lemma is the Bohr inequality in Hilbert C*-modules (see [4]).

**Lemma 2.1.** *Suppose that* $u, v \in \mathcal{H}$.
$(i)$ *If* $\lambda > 0$, *then*

$$|u \mp v|^2 \le (1 + \lambda)|u|^2 + (1 + \frac{1}{\lambda})|v|^2,$$

$(ii)$ *If* $\lambda < 0$, *then*

$$|u \mp v|^2 \ge (1 + \lambda)|u|^2 + (1 + \frac{1}{\lambda})|v|^2.$$

*Furthermore, in* $(i)$ *and* $(ii)$ *the equality holds if and only if* $\lambda u = v$.

In 2011, Dadipour and Moslehian [1] introduced operator version of the Dunkl-Williams inequality with respect to the $p$-angular distance as a generalization of the Theorem 1.1 as follows:

**Theorem 2.2.** *Let $a, b$ in C*-algebra $\mathcal{A}$ such that $|a|$ and $|b|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$. Then*

$$\begin{aligned}&|a|a|^{p-1} - b|b|^{p-1}|^2 \\ &\qquad \leq |a|^{p-1}[r|a-b|^2 + s(|a|^{1-p}|b|^p - |b|)(|b|^p|a|^{1-p} - |b|)]|a|^{p-1}.\end{aligned}$$

*The equality holds if and only if $(r-1)(a-b)|a|^{p-1} = b(|a|^{p-1} - |b|^{p-1})$.*

In this paper they extend Theorem 2.2 to the Hilbert C*-modules case.

**Theorem 2.3.** *Let $u, v$ be elements of a Hilbert C*-module $\mathcal{H}$ such that $|u|$ and $|v|$ are invertible, $\frac{1}{r} + \frac{1}{s} = 1$ $(r > 1)$ and $p \in \mathbb{R}$. Then*

$$\begin{aligned}&|u|u|^{p-1} - v|v|^{p-1}|^2 \\ &\qquad \leq |u|^{p-1}[r|u-v|^2 + s(|u|^{1-p}|v|^p - |v|)(|v|^p|u|^{1-p} - |v|)]|u|^{p-1}.\end{aligned}$$

*The equality holds if and only if $(r-1)(u-v)|u|^{p-1} = v(|u|^{p-1} - |v|^{p-1})$.*

We improve Theorem 2.3 without assumption of the invertibility of the absolute value of operator $v$.

In the sequel we denote $\mathcal{H}$ and $\mathcal{K}$ as Hilbert modules over a unital C*-algebra $\mathcal{A}$ with unit $e$.

# 3 Main Results

We have the following generalization of the Dunkl-Williams type inequality [2] in the framework of Hilbert C*-modules. As a result, Theorem 2.3 is extended without demanding the invertibility of $|v|$.

**Theorem 3.1.** *Let $u, v$ be two elements of $\mathcal{H}$. If $\mathcal{A}$ is unital and $a, b$ are elements in $\mathcal{A}$ such that $a$ is invertible, $\lambda > 0$, then*

$$|ua - vb|^2 \leq a^* \left( (1+\lambda)|u-v|^2 + (1+\frac{1}{\lambda})|vba^{-1} - v|^2 \right) a.$$

*The reverse inequality is valid for $\lambda < 0$. The equality holds if and only if $\lambda(u-v)a = v(a-b)$.*

*Proof.* First observe that

$$a^*|u-v|^2 a = a^*\langle u-v, u-v\rangle a = \langle (u-v)a, (u-v)a\rangle = |(u-v)a|^2. \quad (3)$$

Also,

$$\begin{aligned}|vba^{-1}-v|^2 &= \langle vba^{-1}-v, vba^{-1}-v\rangle \\ &= \langle vba^{-1}, vba^{-1}\rangle - \langle vba^{-1}, v\rangle - \langle v, vba^{-1}\rangle + \langle v, v\rangle \\ &= (a^*)^{-1}b^*|v|^2ba^{-1} - (a^*)^{-1}b^*|v|^2 - |v|^2ba^{-1} + |v|^2, \qquad (4)\end{aligned}$$

By multiplying $a^*$ and $a$ from the left and right (4), respectively, we have

$$\begin{aligned}a^*|vba^{-1}-v|^2a &= b^*|v|^2b - b^*|v|^2a - a^*|v|^2b + a^*|v|^2a \\ &= \langle vb, vb\rangle - \langle vb, va\rangle - \langle va, vb\rangle + \langle va, va\rangle \\ &= -\langle vb, va-vb\rangle + \langle va, va-vb\rangle \\ &= \langle va-vb, va-vb\rangle \\ &= |v(a-b)|^2. \qquad (5)\end{aligned}$$

Using (3), (5) and the part $(i)$ of Lemma 2.1, we obtain

$$\begin{aligned}|ua-vb|^2 &= |(u-v)a+v(a-b)|^2 \\ &\le (1+\lambda)|(u-v)a|^2 + (1+\frac{1}{\lambda})|v(a-b)|^2 \\ &= (1+\lambda)a^*|u-v|^2a + (1+\frac{1}{\lambda})a^*|vba^{-1}-v|^2a \\ &= a^*\left((1+\lambda)|u-v|^2 + (1+\frac{1}{\lambda})|vba^{-1}-v|^2\right)a.\end{aligned}$$

The reverse inequality followed from (3), (5) and the part $(ii)$ of Lemma 2.1. The equality case follows from Lemma 2.1. □

**Lemma 3.2.** *Let $u \in \mathcal{H}$ and $a, b$ be two elements of $C^*$-algebra $\mathcal{A}$ such that $a$ is invertible then*

$$||u|ba^{-1} - |u|\,|^2 = |uba^{-1} - u|^2.$$

*Proof.* By definition of $|u|^2 = \langle u, u\rangle$, we have

$$\begin{aligned}|uba^{-1}-u|^2 &= \langle uba^{-1}-u, uba^{-1}-u\rangle \\ &= \langle uba^{-1}, uba^{-1}\rangle - \langle uba^{-1}, u\rangle - \langle u, uba^{-1}\rangle + \langle u, u\rangle \\ &= (a^*)^{-1}b^*|u|^2ba^{-1} - (a^*)^{-1}b^*|u|^2 - |u|^2ba^{-1} + |u|^2 \\ &= ((a^*)^{-1}b^*|u| - |u|)(|u|ba^{-1} - |u|) \\ &= (|u|ba^{-1} - |u|)^*(|u|ba^{-1} - |u|) \\ &= ||u|ba^{-1} - |u|\,|^2.\end{aligned}$$

Which complete the proof. □

The following Theorem follows by applying Theorem 3.1 and Lemma 3.2.

**Theorem 3.3.** *Let $u, v$ be two elements of $\mathcal{H}$. If $\mathcal{A}$ is unital and $a, b$ are elements in $\mathcal{A}$ such that $a$ is invertible, $\lambda > 0$, then*

$$|ua - vb|^2 \le a^* \left( (1+\lambda)|u-v|^2 + (1+\frac{1}{\lambda})||v|ba^{-1} - |v|\ |^2 \right) a.$$

*The reverse inequality is valid for $\lambda < 0$. The equality holds if and only if $\lambda(u-v)a = v(a-b)$.*

We have the following result.

**Corollary 3.4.** *Theorem 3.3 gives Theorem 2.3.*

*Proof.* Let us put $a = |u|^{p-1}, b = |v|^{p-1}, \lambda = r-1$ in Theorem 3.3. Then $a^* = |u|^{p-1}$ and $\frac{1}{\lambda} = \frac{1}{r-1} = s-1$, where $\frac{1}{r} + \frac{1}{s} = 1$, so

$$\begin{aligned} \left|u|u|^{p-1} - v|v|^{p-1}\right|^2 &\le |u|^{p-1} \left( r|u-v|^2 + s||v||v|^{p-1}|u|^{1-p} - |v|\ |^2 \right) |u|^{p-1} \\ &= |u|^{p-1} \left( r|u-v|^2 + s||v|^p|u|^{1-p} - |v|\ |^2 \right) |u|^{p-1} \\ &= |u|^{p-1} \left( r|u-v|^2 + s[(|v|^p|u|^{1-p} - |v|)^*(|v|^p|u|^{1-p} - |v|) \right) |u|^{p-1} \\ &= |u|^{p-1} \left( r|u-v|^2 + s[(|u|^{1-p}|v|^p - |v|)(|v|^p|u|^{1-p} - |v|) \right) |u|^{p-1}. \end{aligned}$$

The equality holds if and only if

$$\lambda(u-v)a = v(a-b) \Leftrightarrow (r-1)(u-v)|u|^{p-1} = v(|u|^{p-1} - |v|^{p-1}).$$

□

**Remark 3.5.** *Theorem 2.2 followed from Theorem 3.3 if we set $u = a, v = b, a = |a|^{p-1}, b = |b|^{p-1}$*

A special case of Theorem 3.3, where the Hilbert module is the $B(\mathcal{H})$ over itself gives rise to the main result of Pečarić and Rajić [6].

**Corollary 3.6.** *Theorem 3.3 gives Theorem 1.1 if we put $u = A, v = B, a = |A|^{-1}, b = |B|^{-1}$.*

We give the following Example for discussion on Theorem 3.3.

**Example 3.7.** *Let $\mathcal{A}$ be the $C^*$-algebra of the set of all diagonal matrices in $M_{2\times 2}(\mathbb{C})$ and suppose $\mathcal{A}$ is the Hilbert $\mathcal{A}$-module over itself. (Here, diagonal matrix means a $2\times 2$ matrix $(a_{ij})$ such that $a_{11} = a, a_{22} = b$ and $a_{12} = a_{21} = 0$, for $a, b \in \mathbb{C}$.) Consider the following matrices*

$$u = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

*and*

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

*Then we have*

$$\begin{aligned} D := & |ua - vb|^2 - a^* \left( (1+\lambda)|u-v|^2 + (1+\frac{1}{\lambda}) ||v|ba^{-1} - |v|\ |^2 \right) a \\ = & \begin{pmatrix} -4\lambda & 0 \\ 0 & -\frac{4\lambda^2+7\lambda+4}{\lambda} \end{pmatrix} \end{aligned}$$

*The matrix $D \leq 0$ if $\lambda > 0$ and $D \geq 0$ if $\lambda < 0$. Note that the matrix $b$ is not invertible.*

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