Mathematica Aeterna, Vol. 6, 2016, no. 4, 521 - 526

Generalized Derivations of Hom-Lie color algebras

Yi Zhao

School of Mathematics and Statistics
Northeast Normal University

Abstract

In this paper, we give some basic properties of the generalized derivation algebra GDer(L) of a Hom-Lie color algebra L. In particular, we prove that GDer(L) = QDer(L) + QC(L).

Mathematics Subject Classification: 17A40, 17B10, 17B56

Keywords: Hom-Lie color algebras; Generalized derivations; Quasiderivations; Centroids.

1 Introduction

Hom-Lie color algebras are a generalization of Lie algebras, where the classical Jacobi identity is twisted by a linear map.The purpose of this paper is to generalize some beautiful results to the generalized derivation algebra of a Hom-Lie color algebra. In this paper, we mainly study the derivation algebra Der(L), the center derivation algebra ZDer(L), the quasiderivation algebra QDer(L), and the generalized derivation algebra GDer(L) of a Hom-Lie color algebra L.

2 Preliminary Notes

Throughout this paper $\mathbf{K}$ is a field of characteristic zero . A vector space V is Γ-graded, Let V and W be two Γ-graded vectors spaces. A linear map $f : V \to W$ is said to be homogeneous of degree $\xi \in \Gamma$, if $f(x)$ is homogeneous of degree $\gamma + \xi$ for all the element $x \in V_\gamma$. The set of all such maps is denoted by $Hom(V, W)_\xi$. It is a subspace of $Hom(V, W)$, the vector space of all linear maps from V into W.

Definition 2.1 *[1] Let* $\mathbf{K}$ *and* Γ *be an abelian group,A map* $\Gamma \times \Gamma \to \mathbf{K}^*$ *is called a skew-symmetric bi-character on* Γ *if the following identities hold,for all x,y,z,in* Γ

(1) $\varepsilon(x, y)\varepsilon(y, x) = 1$,

(2) $\varepsilon(x, y + z) = \varepsilon(x, y)\varepsilon(x, z)$,

(3) $\varepsilon(x + y, z) = \varepsilon(x, z)\varepsilon(y, z)$,

If x *and* y *are two homogeneous elements of degree* θ *and degree* μ*,respectively and* ε *is a skew-symmetric bi-character,then we shorten the notation by writing* $\varepsilon(x, y)$ *instead of* $\varepsilon(\theta, \mu)$.

Definition 2.2 *[1] A Hom-Lie color algebra is a quadruple* $(L, [\cdot, \cdot], \varepsilon, \alpha)$ *consisting of a* Γ*-graded vector space* L*, a bi-character* ε*, an even bilinear mapping* $[\cdot, \cdot] : L \times L \to L$ *(i.e.* $[L_\theta, L_\mu] \subseteq L_{\theta+\mu}$ *for all* $\theta, \mu \in \Gamma$*) and an even homomorphism* $\alpha : L \to L$ *such that for homogeneous elements* $x, y, z \in L$ *we have*

(1) $[x, y] = -\varepsilon(x, y)[y, x]$,

(2) $\varepsilon(z, x)[\alpha(x), [y, z]] + \varepsilon(x, y)[\alpha(y), [z, x]] + \varepsilon(y, z)[\alpha(z), [x, y]] = 0$.

Let$(L, [\cdot, \cdot], \varepsilon, \alpha)$ be a Hom Lie color algebra.It is called multiplicative Hom Lie color algebra if $\alpha[x, y] = [\alpha(x), \alpha(y)]$.

Definition 2.3 *[3] Let* $(L, [\cdot, \cdot], \alpha)$ *be a Hom-Lie color algebra and define the following subvector space* $\mho$ *of* $\mathrm{End}(L)$ *consisting of even linear maps* u *on* L *as follows:*

$$\mho = \{u \in \mathrm{End}(L)|\ u\alpha = \alpha u\}$$

and $\sigma : \mho \to \mho$; $\sigma(u) = \alpha u$*. Then* $\mho$ *is a Hom-Lie color algebra over* $\mathbf{K}$ *with the bracket*

$$[D_\theta, D_\mu] = D_\theta D_\mu - \varepsilon(\theta, \mu)D_\mu D_\theta$$

for all $D_\theta, D_\mu \in \mathrm{hg}(\mho)$.

Definition 2.4 *[3] Let* $(L, [\cdot, \cdot], \alpha)$ *be a multiplicative Hom-Lie color algebra. A homogeneous bilinear map* $D : L \to L$ *is said to be an* α^k*-derivation of* L*, where* $k \in \mathbf{N}$*, if it satisfies*

$$D\alpha = \alpha D,$$

$$[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D(y)] = D([x, y]),$$

$\forall x \in \mathrm{hg}(L),\ y \in L$.

We denote the set of all α^k*-derivations by* $\mathrm{Der}_{\alpha^k}(L)$*, then* $\mathrm{Der}(L) := \bigoplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(L)$ *provided with the super-commutator and the following even map*

$$\tilde{\alpha} : \mathrm{Der(L)} \to \mathrm{Der(L)}; \quad \tilde{\alpha}(\mathrm{D}) = \mathrm{D}\alpha$$

is a Hom-subalgebra of $\mho$ *and is called the derivation algebra of* L.

Definition 2.5 *[1] An endomorphism $D \in \mathrm{hg}(Der(L))$ is said to be a homogeneous generalized α^k-derivation of L, if there exist two endomorphisms $D', D'' \in hg(\mathrm{End}(L))$ such that such that*

$$D\alpha = \alpha D, D\alpha' = \alpha' D, D\alpha'' = \alpha'' D$$

$$[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D'(y)] = D''([x, y]), \tag{1.1}$$

for all $x \in \mathrm{hg}(L)$,$y \in L$If f is a quasiderivation of L, for convenience ,we write all triple(f, f',f") satisfied (1.1) as $\Gamma(L)$.

Definition 2.6 *[1] An endomorphism $D \in \mathrm{hg}(Der(L))$ is said to be a homogeneous α^k-quasiderivation , if there exists an endomorphism $D' \in hg(\mathrm{End}(L))$ such that*

$$D\alpha = \alpha D, D\alpha' = \alpha' D$$

$$[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D(y)] = D'([x, y]), \tag{1.2}$$

for all $x \in \mathrm{hg}(L), y \in L$.

Let $\mathrm{GDer}_{\alpha^k}(L)$ and $\mathrm{QDer}_{\alpha^k}(L)$ be the sets of homogeneous generalized α^k-derivations and of homogeneous α^k-quasiderivations, respectively. That is

$$\mathrm{GDer}(L) := \bigoplus_{k \geq 0} \mathrm{GDer}_{\alpha^k}(L), \quad \mathrm{QDer}(L) := \bigoplus_{k \geq 0} \mathrm{QDer}_{\alpha^k}(L).$$

It is easy to verify that both $\mathrm{GDer}(L)$ and $\mathrm{QDer}(L)$ are Hom-subalgebras of $\mho$ (see Proposition 2.1)

Definition 2.7 *[1] If $\mathrm{C}(L) := \bigoplus_{k \geq 0} \mathrm{C}_{\alpha^k}(L)$, with $\mathrm{C}_{\alpha^k}(L)$ consisting of $D \in hg(\mathrm{End}(L))$ satisfying*

$$D\alpha = \alpha D,$$

$$[D(x), \alpha^k(y)] = \varepsilon(D, x)[\alpha^k(x), D(y)] = D([x, y]),$$

for all $x \in \mathrm{hg}(L), y \in L$, then $\mathrm{C}(L)$ is called an α^k-centroid of L.

Definition 2.8 *[1] If $\mathrm{QC}(L) := \bigoplus_{k \geq 0} \mathrm{QC}_{\alpha^k}(L)$ with $\mathrm{QC}_{\alpha^k}(L)$ consisting of $D \in hg(\mathrm{End}(L))$ such that*

$$[D(x), \alpha^k(y)] = \varepsilon(D, x)[\alpha^k(x), D(y)],$$

for all $x \in \mathrm{hg}(L), y \in L$, then $\mathrm{QC}(L)$ is called an α^k-quasicentroid of L.

Define $\mathrm{ZDer}(L) := \bigoplus_{k \geq 0} \mathrm{Der}_{\alpha^k}(L)$, where $\mathrm{Der}_{\alpha^k}(L)$ consists of $D \in hg(\mathrm{End}(L))$ such that

$$[D(x), \alpha^k(y)] = D([x, y]) = 0,$$

for all $x \in \mathrm{hg}(L),\ y \in L$.

Definition 2.9 *[3] Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie color algebra. If $\mathrm{Z}(L) := \bigoplus_{\theta \in \Gamma} \mathrm{Z}_\theta(L)$, with $\mathrm{Z}_\theta(L) = \{x \in L_\theta | [x, y] = 0, \forall x \in \mathrm{hg}(L), y \in L\}$, then $\mathrm{Z}(L)$ is called the center of L.*

3 Main Results

Lemma 3.1 *Let $(L, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie color algebra. Then the following statements hold:*

(1) $\mathrm{GDer}(L), \mathrm{QGer}(L)$ *and* $\mathrm{C}(L)$ *are Hom-subalgebras of* $\mho$.

(2) $\mathrm{ZDer}(L)$ *is a Hom-ideal of* $\mathrm{Der}(L)$.

Proof. Assume that $D_1 \in \mathrm{GDer}_{\alpha^k}(L)$, $D_2 \in \mathrm{GDer}_{\alpha^s}(L)$, $\forall x \in \mathrm{hg}(L)$ and $y \in L$. We have

$$\begin{aligned}[(\tilde{\alpha}(D_1))(x), \alpha^{k+1}(y)] &= [(D_1\alpha)(x), \alpha^{k+1}(y)] = \alpha[D_1(x), \alpha^k(y)] \\ &= \tilde{\alpha}(D_1^{''})([x,y]) - \varepsilon(D_1, x)[\alpha^{k+1}(x), \tilde{\alpha}(D_1^{'})(y)]).\end{aligned}$$

Since both $\tilde{\alpha}(D_1^{''})$ and $\tilde{\alpha}(D_1^{'})$ are in $hg(\mathrm{End}(L))$, $\tilde{\alpha}(D_1) \in \mathrm{GDer}_{\alpha^{k+1}}(L)$.

We also have

$$\begin{aligned}[D_1D_2(x), \alpha^{k+s}(y)] &= D_1^{''}D_2^{''}([x,y]) + \varepsilon(D_1,D_2)\varepsilon(D_1,x)\varepsilon(D_2,x)[\alpha^{s+k}(x), D_2^{'}D_1^{'}(y)] \\ &-\varepsilon(D_1,D_2)\varepsilon(D_1,x)[\alpha^k(x), D_1^{'}(y)]) - \varepsilon(D_2,x)D_1^{''}([\alpha^s(x), D_2^{'}(y)])\end{aligned}$$

and

$$\begin{aligned}[D_2D_1(x), \alpha^{k+s}(y)] &= D_2^{''}D_1^{''}([x,y]) + \varepsilon(D_2,D_1)\varepsilon(D_2,x)\varepsilon(D_1,x)[\alpha^{s+k}(x), D_1^{'}D_2^{'}(y)] \\ &-\varepsilon(D_2,D_1)\varepsilon(D_2,x)D_1^{''}([\alpha^s(x), D_2^{'}(y)]) - \varepsilon(D_1,x)D_2^{''}([\alpha^k(x), D_1^{'}(y)]).\end{aligned}$$

Thus for all $x \in \mathrm{hg}(L)$ and $y \in L$, we have

$$[[D_1,D_2](x), \alpha^{k+s}(y)] = [D_1'', D_2''](]x,y]) - \varepsilon(D_1,x)\varepsilon(D_2,x)[\alpha^{k+s}(x), [D_1', D_2'](y)].$$

Since both $[D_1', D_2']$ and $[D_1'', D_2'']$ are in $hg(\mathrm{End}(L))$, $[D_1,D_2] \in \mathrm{GDer}_{\alpha^{k+s}}(L)$, $\mathrm{GDer}(L)$ is a Hom-subalgebra of $\mho$.

Similarly, $\mathrm{QGer}(L)$ is a Hom-subalgebra of $\mho$.

(2) Assume that $D_1 \in \mathrm{ZDer}_{\alpha^k}(L), D_2 \in \mathrm{Der}_{\alpha^s}(L), \forall x \in \mathrm{hg}(L)$, $y \in L$. Then

$$[\tilde{\alpha}(D_1)(x), \alpha^{k+1}(y)] = \alpha([D_1(x), \alpha^k(y)]) = \alpha D_1([x,y]) = \tilde{\alpha}(D_1)([x,y]) = 0.$$

So $\tilde{\alpha}(D_1) \in \mathrm{ZDer}_{\alpha^{k+1}}(L)$. Note that

$$[[D_1,D_2]([x,y])] = D_1D_2([x,y]) - \varepsilon(D_1,D_2)D_2D_1([x,y]) = 0$$

and

$$\begin{aligned}[[D_1,D_2](x), \alpha^{s+k}(y)] &= [(D_1D_2 - \varepsilon(D_1,D_2)D_2D_1)(x), \alpha^{s+k}(y)] \\ &= -\varepsilon(D_2,D_1)\varepsilon(D_2,x)[D_\theta(\alpha^s(x)), \alpha^k(D_\mu(y))] = 0.\end{aligned}$$

Then $[D_1,D_2] \in \mathrm{ZDer}_{\alpha^{k+s}}(L)$, Thus $\mathrm{ZDer}(L)$ is a Hom-ideal of $\mathrm{Der}(L)$. □

Lemma 3.2 *Let* $(L, [\cdot, \cdot], \alpha)$ *be a multiplicative Hom-Lie color algebra. Then*

(1) $[\mathrm{Der}(L), \mathrm{C}(L)] \subseteq \mathrm{C}(L)$.

(2) $[\mathrm{QDer}(L), \mathrm{QC}(L)] \subseteq \mathrm{QC}(L)$.

Proof. Assume that $D_1 \in \mathrm{Der}_{\alpha^k}(L), D_2 \in \mathrm{C}_{\alpha^s}(L), \forall x \in \mathrm{hg}(L)$ and $y \in L$. We have

$$\begin{aligned}[D_1D_2(x), \alpha^{k+s}(y)] &= D_1([D_2(x), \alpha^s(y)]) - \varepsilon(D_1, D_2)\varepsilon(D_1, x)[\ \alpha^k(D_2(x)), D_1(\alpha^s(y))] \\ &= D_1D_2([x, y]) - \varepsilon(D_1, D_2)\varepsilon(D_1, x)\varepsilon(D_2, x)[\alpha^{k+s}(x), D_2D_1(y)],\end{aligned}$$

and

$$\begin{aligned}[D_2D_1(x), \alpha^{k+s}(y)] &= D_2([D_1(x), \alpha^k(y)]) \\ &= D_2D_1([x, y]) - \varepsilon(D_1, x)\varepsilon(D_2, x)[\alpha^{k+s}(x), D_2D_1(y)].\end{aligned}$$

On the other hand,

$$\begin{aligned}[D_1D_2(x), \alpha^{k+s}(y)] &= D_1([D_2(x), \alpha^s(y)]) - \varepsilon(D_1, D_2)\varepsilon(D_1, x)[\alpha^k(D_2(x)), D_1(\alpha^s(y))] \\ &= \varepsilon(D_2, x)[D_1(\alpha^s(x)), \alpha^k(D_2(y))] + \varepsilon(D_1, x)\varepsilon(D_2, x)[\alpha^{k+s}(x), D_1D_2(y)] \\ &\quad -\varepsilon(D_1, D_2)\varepsilon(D_1, x)\varepsilon(D_2, x)[\alpha^{k+s}(x), D_2D_1(y)]\end{aligned}$$

$$[D_2D_1(x), \alpha^{k+s}(y)] = \varepsilon(D_2, D_1)\varepsilon(D_2, x)[D_1(\alpha^s(x)), \alpha^k(D_2(y))].$$

Then

$$\begin{aligned}[[D_1, D_2](x), \alpha^{k+s}(y)] &= [D_1D_2(x), \alpha^{k+s}(y)] - \varepsilon(D_1, D_2)[D_2D_1(x), \alpha^{k+s}(y)] \\ &= \varepsilon(D_1, x)\varepsilon(D_2, x)[\alpha^{k+s}(x), [D_1, D_2](y)]\end{aligned}$$

Thus $[D_1, D_2] \in \mathrm{C}_{\alpha^{s+k}}(L)$, and we get $[\mathrm{Der}(L), \mathrm{C}(L)] \subseteq \mathrm{C}(L)$.

(2) Similar to the proof of (1). □

Theorem 3.3 *Let* $(L, [\cdot, \cdot], \alpha)$ *be a multiplicative Hom-Lie color algebra.Then*

$$\mathrm{GDer}(L) = \mathrm{QDer}(L) + \mathrm{QC}(L).$$

Proof. Let $D_1 \in \mathrm{GDer}_{\alpha^k}(L)$. Then for all $x, y \in \mathrm{hg}(L)$, there exist $D_1', D_1'' \in \mathrm{End}(L)$ such that

$$[D_1(x), \alpha^k(y)] + \varepsilon(D_1, x)[\alpha^k(x), D_1'(y)] = D_1''([x, y]).$$

Since $\varepsilon(D_1, y)\varepsilon(x, y)[\alpha^k(y), D_1(x)] + \varepsilon(x, y)[D_1'(y), \alpha^k(x)] = \varepsilon(x, y)D_1''([y, x])$,

$$[D_1'(y), \alpha^k(x)] + \varepsilon(D_1, y)[\alpha^k(y), D_1(x)] = D_1''([y, x]).$$

Hence $D_1' \in \mathrm{GDer}_{\alpha^k}(L)$. For all $x, y \in \mathrm{hg}(L)$, we have

$$[\frac{D_1 + D_1'}{2}(x), \alpha^k(y)] + \varepsilon(D_1, x)[\alpha^k(x), \frac{D_1 + D_1'}{2}(y)] = D_1''([x, y]),$$

and

$$[\frac{D_1 - D_1'}{2}(x), \alpha^k(y)] - \varepsilon(D_1, x)[\alpha^k(x), \frac{D_1 - D_1'}{2}(y)] = 0,$$

Hence

$$D_1 \in \mathrm{QDer}(L) + \mathrm{QC}(L),$$

and

$$\mathrm{GDer}(L) \subseteq \mathrm{QDer}(L) + \mathrm{QC}(L).$$

It is easy to vertify that $\mathrm{QDer}(L) + \mathrm{QC}(L) \subseteq \mathrm{GDer}(L)$. Therefore $\mathrm{QDer}(L) + \mathrm{QC}(L) = \mathrm{GDer}(L)$. □

ACKNOWLEDGEMENTS. The authors thank Professor Liangyun Chen for their helpful com- ments and suggestions. We also give our special thanks to referees for many helpful suggestions.

References

[1] Abdaoui K., Ammar F. and Makhlouf A., *Constructions and cohomology of color Hom-Lie algebras.* arXiv:1307.2612 (2013).

[2] Ammar F. and Makhlouf A., *Hom-Lie superalgebras and Hom-Lie admissible superalgebras.* J. Algebra 324 (2010), no. 7, 1513-1528.

[3] Ammar F., Ayadi I., Mabrouk S. and Makhlouf A., *Quadratic color Hom-Lie algebras.* arXiv:1204.5155 (2012).

[4] Ammar F., Makhlouf A. and Saadoui N, *Cohomology of Hom-Lie superalgebras and q-deforemed Witt superalgebra.* arXiv:1204.6244 (2012).

[5] Benoist Y., *La partie semi-simple de l'algèbre des dérivations d'une algèbre de Lie nilpotente*, C. R. Acad. Sci. Paris 307 (1988), 901-904.

[6] Chen L., Ma Y. and Ni L., *Gerneralized derivations of Lie color algebras.* Results Math. 63 (2013), no. 3-4, 923-936.

[7] Gohr A., *On Hom-algebras with surjective twisting*, J. Algebra 324 (2010), no. 7, 1483-1491.

[8] Hartwig J., Larsson D. and Silvestrov S., *Deformations of Lie algebras using σ-derivations.* J. Algebra 295 (2006), no. 2, 314-361.

Received: July, 2016