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Fuzzy preorder sets and extensional subobjects

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Abstract

In this paper, we investigate two categories of fuzzy preorder sets and extesional subobjects in complete residuated lattices.

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1 Introduction

Hájek [5] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure [1-6, 10]. Močkoř[7-9] introduced two categories of fuzzy similar relations and extensional subobjects in an MV-algebras. In particular, quasi-reflective subcategories were investigated. Zhang [6,10] introduced the category of preorder sets.

In this paper, we investigate two categories of fuzzy preorder sets and extensional subobjects in complete residuated lattices. In particular, we show that the category of extensional subobjects is a quasi-reflective subcategory of the category of preorder sets.

Definition 1.1 [1,5] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to,^* 0, 1)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$.

Lemma 1.2 [1,5] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \le z$, $(x \odot y) \le (x \odot z)$, $x \to y \le x \to z$ and $z \to x \le y \to x$.
(2) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.
(3) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.
(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(6) $x \odot y = (x \to y^*)^*$ and $x \to y = y^* \to x^*$.
(7) $x \odot (x \to y) \le y$ and $y \to z \le x \odot y \to x \odot z$.
(8) $(x \to y) \odot (y \to z) \le x \to z$.
(9) $x \le y \to z$ iff $y \le x \to z$.
(10) $y \to z \le (x \to y) \to (x \to z)$ and $y \to z \le (z \to x) \to (y \to x)$.
(11) $(\bigwedge_{i\in\Gamma} x_i) \to (\bigwedge_{i\in\Gamma} y_i) \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.
(12) $(\bigvee_{i\in\Gamma} x_i) \to (\bigvee_{i\in\Gamma} y_i) \ge \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

Definition 1.3 [1,5] Let X be a set. A function $e_X : X \times X \to L$ is called fuzzy preorder if it satisfies

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,
(E2) transitive if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$,

The pair (X, e_X) is a fuzzy preorder set.

The category **pROrd** is the object (X, e_X) and a morphism from (X, e_X) to (Y, e_Y) with $f \in L^{X\times Y}$ and a morphism from (Y, e_Y) to (Z, e_Z) with $g \in L^{Y\times Z}$ satisfying the following conditions:

(1) $e_X(z, x) \odot f(x, y) \le f(z, y)$, $\forall x, z \in X, y \in Y$,
(2) $e_Y(y, w) \odot f(x, y) \le f(x, w)$, $\forall x \in X, y, w \in Y$,
(3) $g \circ f(x, z) = \bigvee_{y\in Y}(f(x, y) \circ g(y, z))$.

The category **pOrd** is the object (X, e_X) and a morphism $f : (X, e_X) \to (Y, e_Y)$ with $e_X(x, y) \le e_Y(f(x), f(y))$.

Remark 1.4 We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy preorder set from Lemma 1.2 (8).

2 Fuzzy preorder sets and extensional subobjects

Theorem 2.1 *Define* F : **pOrd** $\to$ **pRord** *as* $F(X, e_X) = (X, e_X)$ *and* $F(f)(x, y) = e_Y(f(x), y)$ *for a morphism* $f : (X, e_X) \to (Y, e_Y)$ *in* **pOrd**. *Then* F *is a functor.*

Proof First, we show that **pRord** is a category. If $f : X \times Y \to L$ and $g : Y \times Z \to L$ is a morphism, then $g \circ f : X \times Z \to L$ is a morphism from the following statement (1) and (2).

(1) $e_X(w,x) \odot (g \circ f)(x,z) = e_X(w,x) \odot \bigvee_{y \in Y}(f(x,y) \odot g(y,z)) = \bigvee_{y \in Y}(e_X(w,x) \odot f(x,y) \odot g(y,z) \leq \bigvee_{y \in Y}(f(w,y) \odot g(y,z)) = (g \circ f)(w,z)$.

(2) $(g \circ f)(x,z) \odot e_Z(z,w) = \bigvee_{y \in Y}(f(x,y) \odot g(y,z)) \odot e_Z(z,w) = \bigvee_{y \in Y}(f(x,y) \odot g(y,z) \odot e_Z(z,w)) \leq \bigvee_{y \in Y}(f(x,y) \odot g(y,w)) = (g \circ f)(x,w)$.

We will show that F is a functor. Then $F(f)$ is a morphism in **pRord** from:

$$\begin{aligned} e_X(z,x) \odot F(f)(x,y) &= e_X(z,x) \odot e_Y(f(x),y) \leq e_Y(f(z),f(x)) \odot e_Y(f(x),y) \\ &\leq e_Y(f(z),y) = F(f)(z,y), \end{aligned}$$

$$F(f)(x,y) \odot e_Y(y,w) = e_Y(f(x),y) \odot e_Y(y,w) \leq e_Y(f(x),w) = F(f)(x,w).$$

Moreover, $F(g \circ f) = F(g) \circ F(f)$ from:

$$\begin{aligned} F(g \circ f)(x,z) &= e_Z(g \circ f(x),z), \\ F(g) \circ F(f)(x,z) &= \bigvee_{y \in Y}(F(f)(x,y) \odot F(g)(y,z) \\ &= \bigvee_{y \in Y}(e_Y(f(x),y) \odot e_Z(g(y),z) \\ &\leq \bigvee_{y \in Y}(e_Z(g(f(x)),g(y)) \odot e_Z(g(y),z) \\ &\leq e_Z(g(f(x)),z), \\ F(g) \circ F(f)(x,z) &= \bigvee_{y \in Y}(e_Y(f(x),y) \odot e_Z(g(y),z) \\ &\geq e_Y(f(x),f(x)) \odot e_Z(g(f(x)),z) = e_Z(g(f(x)),z). \end{aligned}$$

Definition 2.2 [7-9] A map $s : X \to L$ is called an extensional subobject of (X, e_X) in **pOrd** if $s(x) \odot e_X(x,y) \leq s(y)$.

A map $r : X \times L \to L$ is called an extensional subobject of (X, e_X) in **pRord** if it satisfies

(1) $e_X(z,x) \odot r(x,a) \leq r(z,a)$, $\forall x, z \in X, a \in L$,

(2) $r(x,a) \odot (a \to b) \leq r(x,b)$, $\forall x \in X, a, b \in L$.

Remark 2.3 (1) If $s : (X, e_X) \to (L, \to)$ is a morphism in **pOrd**, then $e_X(x,y) \leq s(x) \to s(y)$. Hence $s : X \to L$ is an extensional subobject of (X, e_X) in **pOrd**

(2) If $r : (X, e_X) \to (L, \to)$ is a morphism in **pROrd**, then $r \in L^{X \times L}$ is an extensional subobject of (X, e_X) in **pRord**.

$$\begin{aligned} E(X, e_X) &= \{s : \text{ is an extensional subobject of } (X, e_X) \text{ in } \mathbf{pOrd}\} \\ R(X, e_X) &= \{s : \text{ is an extensional subobject of } (X, e_X) \text{ in } \mathbf{pRord}\} \end{aligned}$$

Theorem 2.4 *There exists a map $F : E(X, e_X) \to R(X, e_X)$ with $F(s) \in$* **pRord**. *Moreover, $F : (E(X, e_X), e_{L^X}) \to (F(E(X, e_X)), e_{F(X)})$ is an order-isomorphism where $e_{F(X)} : F(E(X, e_X)) \times F(E(X, e_X)) \to L$ as*

$$e_{F(X)}(F(s), F(t)) = \bigwedge_{(x,a)\in F(E(X,e_X))} (F(s)^{-1}(x, a) \to F(t)^{-1}(x, a)).$$

Proof For $s \in E(X, e_X)$, put $F(s)(x, y) = s(x) \to y$. Since $s(x) \odot e_X(x, z) \le s(z)$, $F(s)$ is an extensional subobject in **pRord** from

$$\begin{aligned} e_X(z, x) \odot F(s)(x, y) &= e_X(z, x) \odot (s(x) \to y) \\ &\le (s(z) \to y) = F(s)(z, y), \end{aligned}$$

$$\begin{aligned} F(s)(x, y) \odot e_Y(y, w) &= (s(x) \to y) \odot (y \to w) \\ &\le s(x) \to y = F(s)(x, w). \end{aligned}$$

$$\begin{aligned} e_{F(X)}(F(s), F(t)) &= \bigwedge_{(x,a)\in X\times L}(F(s)^{-1}(x, a) \to F(t)^{-1}(x, a)) \\ &= \bigwedge_{(x,a)\in X\times L}((a \to s(x)) \to (a \to t(x))) \\ &\le \bigwedge_{x\in X}((s(x) \to s(x)) \to (s(x) \to t(x))) \\ &= \bigwedge_{x\in X}(s(x) \to t(x)) = e_{L^X}(s, t). \end{aligned}$$

Since $(a \to s(x)) \to (a \to t(x)) \ge s(x) \to t(x)$ from Lemma 1.2 (10), we have

$$\begin{aligned} e_{F(X)}(F(s), F(t)) &= \bigwedge_{(x,a)\in X\times L}((a \to s(x)) \to (a \to t(x))) \\ &\ge \bigwedge_{x\in X}(s(x) \to t(x)) = e_{L^X}(s, t). \end{aligned}$$

Hence $F : (E(X, e_X), e_{L^X}) \to (F(E(X, e_X)), e_{F(X)})$ is an order-isomorphism, that is, F is bijective and $e_{F(X)}(F(s), F(t)) = e_{L^X}(s, t)$.

Definition 2.5 [8] A category **L** is called a quasi-reflective subcategory in **K** if there exists a functor $G : \mathbf{K} \to \mathbf{L}$ such that for any object $a \in \mathbf{K}$, there is a morphism $u_a : a \to G(a)$ such that for any $b \in \mathbf{L}$ and $f : a \to b$ in **K**, there exists a morphism $\bar{f} : G(a) \to b$ such that $f = \bar{f} \circ u_a$.

The category **Ext** is the object $(E(X, e_X), e_{L^X})$ and a morphism $\alpha : (E(X, e_X), e_{L^X}) \to (E(Y, e_X), e_{L^Y})$ with $e_{L^X}(s, t) \le e_{L^Y}(\alpha(s), \alpha(t))$.

Theorem 2.6 *There exists a functor $D :$* **pOrd** $\to$ **Ext**. *Moreover,* **Ext** *is a quasi-reflective subcategory in* **pOrd**.

Proof Let (X, e_X) be an object of **pOrd** and $f : (X, e_X) \to (Y, e_Y)$ be a morphism in **pOrd**. Define a functor $D :$ **pOrd** $\to$ **Ext** as follows

$$D(X, e_X) = (E(X, e_X), e_{L^X})$$

$$D(f)(s)(y) = \bigvee_{x\in X} (s(x) \odot e_Y(f(x), y)),\ s \in E(X, e_X)$$

Since $D(f)(s)(y) \odot e_Y(y, w) = \bigvee_{x\in X}(s(x) \odot e_Y(f(x), y)) \odot e_Y(y, w) \le \bigvee_{x\in X}(s(x) \odot e_Y(f(x), w)) = D(f)(s)(w)$, then $D(f)(s) \in E(Y, e_Y)$. Moreover, if $g : (Y, e_Y) \to (Z, e_Z)$ is a morphism in **pOrd**, then

$$\begin{array}{ll} D(g\circ f)(s)(z) & = \bigvee_{x\in X}(s(x) \odot e_Z(f(x), z)),\ s \in E(X, e_X) \\ (D(g)\circ D(f))(s)(z) & = \bigvee_{y\in Y}(D(f)(s)(y) \odot e_Z(g(y), z)) \\ & = \bigvee_{y\in Y}(\bigvee_{x\in X}(s(x) \odot e_Y(f(x), y)) \odot e_Z(g(y), z)) \\ & \le \bigvee_{y\in Y}(\bigvee_{x\in X}(s(x) \odot e_Z(g(f(x)), g(y))) \odot e_Z(g(y), z)) \\ & \le \bigvee_{x\in X}(s(x) \odot e_Z(g(f(x)), z)) \\ (D(g)\circ D(f))(s)(z) & = \bigvee_{y\in Y}(\bigvee_{x\in X}(s(x) \odot e_Y(f(x), y)) \odot e_Z(g(y), z)) \\ & \ge \bigvee_{x\in X}(s(x) \odot e_Y(f(x), f(x))) \odot e_Z(g(f(x)), z)) \\ & = \bigvee_{x\in X}(s(x) \odot e_Z(g(f(x)), z)) \end{array}$$

Hence $D(g \circ f) = D(g) \circ D(f)$. We have $e_{L^Y}(D(f)(s), D(f)(t)) \ge e_{L^X}(s, t)$ from:

$$\begin{array}{l} \bigvee_{x\in X}(s(x) \odot e_Z(f(x), z)) \to \bigvee_{x\in X}(t(x) \odot e_Z(f(x), z)) \\ \ge \bigwedge_{x\in X}((s(x) \odot e_Z(f(x), z)) \to (t(x) \odot e_Z(f(x), z)) \text{ (by Lemma 1.2 (12))} \\ \ge \bigwedge_{x\in X}(s(x) \to t(x)) \text{ (by Lemma 1.2 (7))}. \end{array}$$

Thus, D is a functor. Define $\delta : (X, e_X^{-1}) \to (E(X, e_X), e_{L^X})$ as $\delta(x) = (e_X)_x$. Then $\delta(x) = (e_X)_x \in E(X, e_X)$ is well defined because $(e_X)_x(y) \odot e_X(y, w) \le (e_X)_x(w)$ and δ is an order-isomorphism from:

$$\begin{array}{ll} e_{L^X}(\delta(x), \delta(y)) & = e_{L^X}((e_X)_x, (e_X)_y) \\ & = \bigwedge_{z\in X}((e_X)_x(z) \to (e_X)_y(z)) \\ & = \bigwedge_{z\in X}(e_X(x, z) \to e_X(y, z)) \\ & = e_X(y, x) = e_X^{-1}(x, y). \end{array}$$

Let $f : (X, e_X^{-1}) \to (E(Y, e_Y), e_{L^X})$ be a morphism in **pOrd**;i.e. $e_{L^X}(f(x), f(y)) \ge e_X^{-1}(x, y) = e_X(y, x)$.

We define $\bar{f}(s)(b) = \bigvee_{x\in X}(f(x)(b) \odot e_{L^X}(\delta(x), s)$. Since $f(x) \in E(Y, e_Y)$, we have $\bar{f}(s) \in E(Y, e_Y)$ from:

$$\begin{array}{ll} \bar{f}(s)(b) \odot e_Y(b, c) & = \bigvee_{x\in X}(f(x)(b) \odot e_{L^X}((e_X)_x, s) \odot e_Y(b, c) \\ & \le \bigvee_{x\in X}(f(x)(c) \odot e_{L^X}((e_X)_x, s) = \bar{f}(s)(c) \end{array}$$

Since $e_{L^X}((e_X)_x, s) \to e_{L^X}((e_X)_x, t) \ge \bigwedge_{y\in X}(((e_X)_x(y) \to s(y)) \to ((e_X)_x(y) \to t(y))) \ge \bigwedge_{y\in X}(s(y) \to t(y))$, we have

$$\begin{array}{l} \bar{f}(s)(b) \to \bar{f}(t)(b) \\ = \Big(\bigvee_{x\in X}(f(x)(b) \odot e_{L^X}((e_X)_x, s)\Big) \to \Big(\bigvee_{x\in X}(f(x)(b) \odot e_{L^X}((e_X)_x, t)\Big) \\ \ge \bigwedge_{x\in X}\Big(f(x)(b) \odot e_{L^X}((e_X)_x, s) \to (f(x)(b) \odot e_{L^X}((e_X)_x, t)\Big) \\ \ge \bigwedge_{x\in X}(s(x) \to t(x)) = e_{L^X}(s, t), \end{array}$$

Then $e_{L^Y}(\bar{f}(s),\bar{f}(t)) = \bigwedge_{b\in Y}(\bar{f}(s)(b) \to \bar{f}(t)(b)) \geq e_{L^X}(s,t)$. We have $\bar{f}(\delta(a))(b) = \bar{f}((e_X)_a)(b) = f(a)(b)$ because

$$\begin{array}{ll}\bar{f}((e_X)_a)(b) & = \bigvee_{x\in X}(f(x)(b)\odot e_{L^X}((e_X)_x,(e_X)_a) \\ & = \bigvee_{x\in X}(f(x)(b)\odot e_X(a,x)) \leq \bigvee_{x\in X}(f(x)(b)\odot e_{L^Y}(f(x),f(a))) \\ & = \bigvee_{x\in X}(f(x)(b)\odot \bigwedge_{y\in Y}(f(x)(y)\to f(a)(y))) \\ & \leq f(a)(b). \\ \bar{f}((e_X)_a)(b) & = \bigvee_{x\in X}(f(x)(b)\odot e_{L^X}((e_X)_x,(e_X)_a) \\ & = \bigvee_{x\in X}(f(x)(b)\odot e_X(a,x)) \geq f(a)(b).\end{array}$$

Hence $\bar{f} : (E(X,e_X),e_{L^X}) \to (E(Y,e_Y),e_{L^Y})$ such that $f = \bar{f}\circ\delta$.

Then $\bar{f}$ is the smallest morphism such that $f(a)(b) = \bar{f}((e_X)_a)(b)$;i.e, if $f(a)(b) = g((e_X)_a)(b)$, then $\bar{f} \leq g$.

$$\begin{array}{ll}\bar{f}(s)(b) & = \bigvee_{x\in X}(f(x)(b)\odot e_{L^X}((e_X)_x,s) \\ & \leq \bigvee_{x\in X}(f(x)(b)\odot e_{L^Y}(g((e_X)_x),g(s)) \\ & = \bigvee_{x\in X}(f(x)(b)\odot e_{L^Y}(f(x),g(s))) \\ & = \bigvee_{x\in X}(f(x)(b)\odot \bigwedge_{y\in Y}(f(x)(y)\to g(s)(y))) \\ & = \bigvee_{x\in X}(f(x)(b)\odot (f(x)(b)\to g(s)(b))) \\ & \leq g(s)(b).\end{array}$$

We define a fuzzy preorder $d_{E(X)} : E(X,e_X)\times E(X,e_X) \to L$ as

$$d_{E(X)}(s,t) = \begin{cases} \bigvee_{x\in X}(s(x)\odot t(x)) & \text{if } s\neq t \\ 1 & \text{if } s = t. \end{cases}$$

The category **dExt** is the object $(E(X,e_X),d_{E(X)})$ and a morphism
$\alpha : (E(X,e_X),d_{E(X)}) \to (E(Y,e_Y),d_{E(Y)})$ with $d_{E(X)}(s,t) \leq d_{E(Y)}(\alpha(s),\alpha(t))$.

Theorem 2.7 *There exists a functor G :* **pOrd** $\to$ **dExt**. *Moreover,* **dExt** *is a quasi-reflective subcategory in* **pOrd**.

Proof Let (X,e_X) be an object of **pOrd** and $f : (X,e_X) \to (Y,e_Y)$ be a morphism in **pOrd**. Define a functor G : **pOrd** $\to$ **dExt** as follows

$$G(X,e_X) = (E(X,e_X),d_{E(X)})$$

$$G(f)(s)(y) = \bigvee_{x\in X}(s(x)\odot e_Y(f(x),y)),\ s\in E(X,e_X).$$

$G(f)(s)(y)\odot e_Y(y,w) = \bigvee_{x\in X}(s(x)\odot e_Y(f(x),y))\odot e_Y(y,w) \leq \bigvee_{x\in X}(s(x)\odot e_Y(f(x),w)) = G(f)(s)(w)$. Hence $G(f)(s)\in E(Y,e_Y)$.

Define $\delta : (X, e_X^{-1}) \to (E(X, e_X), d_{E(X)})$ as $\delta(x) = (e_X)_x$. Then $\delta(x) = (e_X)_x \in E(X, e_X)$ because $(e_X)_x(y) \odot e_X(y, w) \leq (e_X)_x(w)$ and

$$\begin{aligned} d_{E(X)}(\delta(x), \delta(y)) &= d_{E(X)}((e_X)_x, (e_X)_y) = \bigvee_{z \in X}((e_X)_x(z) \odot (e_X)_y(z)) \\ &= \bigvee_{z \in X}(e_X(x, z) \odot e_X(y, z)) \geq e_X(y, x) = e_X^{-1}(x, y). \end{aligned}$$

Hence $\delta(x)$ is a morphism in **pOrd**. Moreover, $G(f)$ is a morphism in **dExt** from

$$\begin{aligned} &d_{E(X)}(G(f)(s), G(f)(t)) \\ &= \bigvee_{y \in Y}(G(f)(s)(y) \odot G(f)(t)(y)) \\ &= \bigvee_{y \in Y}\Big(\bigvee_{x \in X}(s(x) \odot e_Y(f(x), y)) \odot \bigvee_{x \in X}(t(x) \odot e_Y(f(x), y))\Big) \\ &\geq \bigvee_{x \in X}(s(x) \odot e_Y(f(x), f(x))) \odot \bigvee_{x \in X}(t(x) \odot e_Y(f(x), f(x))) \\ &\geq \bigvee_{x \in X}(s(x) \odot t(x)) = d_{E(X)}(s, t) \end{aligned}$$

Let $f : (X, e_X^{-1}) \to (E(Y, e_Y), d_{E(Y)})$ be a morphism in **pOrd**, that is, $d_{E(Y)}(f(x), f(y)) \geq e_X^{-1}(x, y)$. We define $\bar{f}(s)(b) = \bigvee_{x \in X}(f(x)(b) \odot d_{E(X)}((e_X)_x, s)$. Then $\bar{f}(s)(b) \geq \bigvee_{x \in X}(f(x)(b) \odot s(x)$. Since $f(x) \in E(Y)$, we have

$$\begin{aligned} \bar{f}(s)(b) \odot e_Y(b, c) &= \bigvee_{x \in X}(f(x)(b) \odot d_{E(X)}((e_X)_x, s) \odot e_Y(b, c) \\ &= \bigvee_{x \in X}(f(x)(c) \odot d_{E(X)}((e_X)_x, s) = \bar{f}(s)(c). \end{aligned}$$

Moreover, $\bar{f} : (E(X, e_X), d_{E(X)}) \to (E(Y, e_Y), d_{E(Y)})$ is a morphism in **dExt** from

$$\begin{aligned} &d_{E(Y)}(\bar{f}(s), \bar{f}(t)) \\ &= \bigvee_{b \in Y}(\bar{f}(s)(b) \odot \bar{f}(t)(b)) \\ &\geq \bigvee_{b \in Y}\bigvee_{x, z \in X}\Big(\bigvee_{x \in X}(f(x)(b) \odot f(z)(b)) \odot (s(x) \odot t(z))\Big) \\ &\geq \bigvee_{x, z \in X}\Big(d_{E(Y)}(f(x), f(z)) \odot (s(x) \odot t(z))\Big) \\ &\geq \bigvee_{x, z \in X}\Big(e_X(z, x) \odot (s(x) \odot t(z))\Big) \\ &\geq \bigvee_{x \in X}(s(x) \odot t(x)) = d_{E(X)}(s, t). \end{aligned}$$

$$\begin{aligned} \bar{f}((e_X)_a)(b) &= \bigvee_{x \in X}(f(x)(b) \odot (e_X)_a(x) \geq f(a)(b). \\ \bar{f}((e_X)_a)(b) &= \bigvee_{x \in X}(f(x)(b) \odot (e_X)_a(x) \\ &\leq \bigvee_{x \in X}(f(x)(b) \odot d_{E(Y)}(f(x), f(a)) \\ &\leq \bigvee_{x \in X}(f(x)(b) \odot e_{L^Y}(f(x), f(a)) \ \ (d_{E(Y)} \leq e_{L^Y}) \\ &\leq \bigvee_{x \in X}(f(x)(b) \odot (f(x)(b) \to f(a)(b)) \leq f(a)(b). \end{aligned}$$

Hence there exists $\bar{f} : (E(X, e_X), d_{E(X)}) \to (E(Y, e_Y), d_{E(Y)})$ such that $f = \bar{f} \circ \delta$.

The category **dRExt** is the object $E(X, e_X)$ and a morphism $f : (E(X, e_X), d_{E(X)}) \to (E(Y, e_Y), d_{E(Y)})$ with $f \in L^{E(X, e_X) \times E(Y, e_Y)}$ satisfying the following conditions:

(1) $d_{E(X)}(s_1, s) \odot f(s, t) \leq f(s_1, t)$, $\forall s, s_1 \in E(X, e_X), t \in E(Y, e_Y)$,

(2) $d_{E(Y)}(t, t_1) \odot f(s, t) \leq f(s, t_1)$, $\forall s \in E(X, e_X), t, t_1 \in E(Y, e_Y)$.

Theorem 2.8 *There exists a functor* H : **pRord** $\to$ **dRExt**. *Moreover,* **dRExt** *is a quasi-reflective subcategory in* **pRord**.

Proof Let (X, e_X) be an object of **pRord** and $f : (X, e_X) \to (Y, e_Y)$ be a morphism in **pRord**. Let $e_X : (X, e_X^{-1}) \to (E(X, e_X), d_{E(X)})$ be a morphism in **pOrd**.

$$\begin{array}{ll} d_{E(X)}((e_X)_x, (e_X)_y) & = \bigvee_{z\in X}((e_X)_x(z) \odot (e_X)_y(z)) \\ & = \bigvee_{z\in X}(e_X(x, z) \odot e_X(y, z)) \\ & \geq e_X(y, x) = e_X^{-1}(x, y). \end{array}$$

Then $F : (X, e_X) \times E(X, e_X) \to L$ defined as $F(x, s) = d_{E(X)}((e_X)_x, s)$ is a morphism in **pRord** because

$$\begin{array}{ll} e_X(z, x) \odot F(x, s) & = e_X(z, x) \odot d_{E(X)}((e_X)_x, s) \\ & = e_X(z, x) \odot \bigvee_{w\in X}((e_X)_x(w) \odot s(w)) \\ & \leq \bigvee_{w\in X}(e_X(z, w) \odot s(w)) = F(z, s) \\ F(x, s) \odot d_{E(X)}(s, t) & = d_{E(X)}((e_X)_x, s) \odot d_{E(X)}(s, t) \\ & \leq d_{E(X)}((e_X)_x, t) = F(x, t). \end{array}$$

Define a functor H : **pRord** $\to$ **dRExt** as follows

$$H(X, e_X) = (E(X, e_X), d_{E(X)})$$

$$H(f)(s, t) = \bigvee_{x\in X} \bigvee_{y\in Y} (f(x, y) \odot F(y, t) \odot F(x, s)),\ s \in E(X, e_X), t \in E(Y, e_Y)$$

Then $H(f)$ is a morphism in **dRExt** from:

$$\begin{array}{l} d_{E(X)}(s_1, s) \odot H(f)(s, t) \\ = d_{E(X)}(s_1, s) \odot \bigvee_{x\in X} \bigvee_{y\in Y}(f(x, y) \odot F(y, t) \odot F(x, s)) \\ \leq \bigvee_{x\in X} \bigvee_{y\in Y}(f(x, y) \odot F(y, t) \odot F(x, s_1)) \\ = H(f)(s_1, t), \\ H(f)(s, t) \odot d_{E(Y)}(t, t_1) \\ = \bigvee_{x\in X} \bigvee_{y\in Y}(f(x, y) \odot F(y, t) \odot F(x, s)) \odot d_{E(Y)}(t, t_1) \\ \leq \bigvee_{x\in X} \bigvee_{y\in Y}(f(x, y) \odot F(y, t_1) \odot F(x, s)) \\ = H(f)(s, t_1). \end{array}$$

Let $f : X \times E(Y, e_Y) \to L$ be a morphism in **pRord**. We define $\bar{f}(s, t) = \bigvee_{x\in X}(f(x, t) \odot F(x, s))$. Since $f(x, b) \odot e_Y(b, c) \leq f(x, c)$, we have

$$\begin{array}{ll} \bar{f}(s, b) \odot e_Y(b, c) & = \bigvee_{x\in X}(f(x, b) \odot F(x, s) \odot e_Y(b, c) \\ & \leq \bigvee_{x\in X}(f(x, c) \odot d_{E(X)}(x, s) = \bar{f}(s, c). \end{array}$$

Since $d_{E(Y)}(t, t_1) \odot f(x, t) \leq f(x, t_1)$, we have

$$\begin{array}{ll} d_{E(Y)}(t, t_1) \odot \bar{f}(s, t) & = d_{E(Y)}(t, t_1) \odot \bigvee_{x\in X}(f(x, t) \odot F(x, s)) \\ & = \bigvee_{x\in X}((f(x, t_1) \odot F(x, s)) = \bar{f}(s, t_1) \end{array}$$

$$\begin{aligned}(\bar f \circ F)(a,t) &= \bigvee_{s\in E(X,e_X)}(F(a,s)\odot \bar f(s,t)\\ &= \bigvee_{s\in L^{(X,e_X)}}(F(a,s)\odot \bigvee_{x\in X}(f(x,t)\odot F(x,s))\\ &\geq F(a,(e_X)_a)\odot \bigvee_{x\in X}(f(x,t)\odot F(x,(e_X)_a))\\ &= \bigvee_{x\in X}(f(x,t)\odot F(x,(e_X)_a))\geq f(a,t),\end{aligned}$$

$$\begin{aligned}&(\bar f \circ F)(a,t)\\ &= \bigvee_{s\in L^{(X,e_X)}}(F(a,s)\odot \bar f(s,t)\\ &= \bigvee_{s\in L^{(X,e_X)}}\bigvee_{x\in X}(d_{E(X)}((e_X)_a,s)\odot d_{E(X)}((e_X)_x,s)\odot f(x,t)\\ &\leq \bigvee_{s\in E(X,e_X)}\bigvee_{x\in X}(d_{E(X)}((e_X)_x,(e_X)_a)\odot f(x,t)\\ &\leq \bigvee_{s\in E(X,e_X)}\bigvee_{x\in X}(e_{L^X}((e_X)_x,(e_X)_a)\odot f(x,t)\\ &\leq \bigvee_{x\in X}(e_X(a,x)\odot f(x,t))\leq f(a,t).\end{aligned}$$

Hence there exist a morphism $\bar f : E(X,e_X)\times E(Y,e_Y)\to L$ such that $f=\bar f\circ F$.

Moreover, for each $(F\circ f): X\times Y\to L$, there exists $\overline{F\circ f}: E(X,e_X)\times E(Y,e_Y)\to L$ such that $H(f)=\overline{F\circ f}$ from:

$$\begin{aligned}H(f)(s,t) &= \bigvee_{x\in X}\bigvee_{y\in Y}(f(x,y)\odot F(y,t)\odot F(x,s))\\ &= \bigvee_{x\in X}(F\circ f)(x,t)\odot F(x,s))=\overline{F\circ f}(s,t).\end{aligned}$$

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