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First order differential subordination for functions with one rotation which have positive real part

Jinjing Qiao

College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China
mathqiao@126.com

Qiannan Guo
College of Mathematics and Information Science,
Hebei University, Baoding, 071002, China
479818568@qq.com

Abstract

Sharp estimates on β are determined so that $p(z)$ is subordinate to some well known starlike functions $P(z)$ with Re $e^{i\theta_0}P(z) > 0$ for some $\theta_0 \in [0, 2\pi]$, whenever $1 + \beta zp'(z)$ is subordinate to $\frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0}$.

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1 Introduction

An analytic function $f(z)$ in $\mathbb{D}$ is subordinate to the analytic function $g(z)$ in $\mathbb{D}$ (or $g(z)$ is superordinate to $f(z)$), if there is an analytic function $\omega(z)$ in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = (g \circ \omega)(z)$. Moreover, if $g(z)$ is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (cf. [4, P_{52}]).

Let $p(z)$ be an analytic function in $\mathbb{D}$ and $p(0) = 1$. In 1935, Goluzin [5] investigated the first order differential subordination $zp'(z) \prec zq'(z)$ and obtained that if $zq'(z)$ is convex, then $p'(z) \prec q'(z)$ holds and the function $q'(z)$ is the best dominant. In the following time, many authors gave several generalizations of first order differential subordination. Nunokawa *et al.* [7] proved that if $1 + zp'(z) \prec 1 + z$, then $p(z) \prec 1 + z$. In 2007, Ali *et al.* [2] determined the estimates of β for which the subordination $1+\beta zp'(z)/p^j(z) \prec$

$(1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies $p(z) \prec (1+Az)/(1+Bz)$ where $A, B, D, E \in [-1,1]$. Recently, Omar and Halim [8] discussed the condition on β in terms of complex number D and real number E with $-1 < E < 1$ and $|D| \leq 1$ such that $1 + \beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies $p(z) \prec \sqrt{1+z}$. We can see [3, 9, 10, 11] for more details.

Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be an analytic function and $p_*(z)$ be a function with positive real part like $\sqrt{1+z}$, $(1+Az)/(1+Bz)$, e^z, $\Phi_0(z) = 1 + \frac{z}{k}((k+z)/(k-z))$ $(k = \sqrt{2}+1)$, $\Phi_1(z) = 1 + \sin z$, $\Phi_2(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ and $\Phi_3(z) = z + \sqrt{1+z^2}$. We determine the sharp bounds on β such that $p(z) \prec P(z) = \frac{e^{-i\theta_0}p_*(z) + i\sin\theta_0}{\cos\theta_0}$, whenever

$$1 + \beta zp'(z) \prec \frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0},$$

where $\theta_0 \in [0, 2\pi]$. Obviously, Re $e^{i\theta_0}P(z) > 0$. Our result is sharp and a generalization of the corresponding one in [1].

2 Main Results

Our first result gives bounds of β such that

$$1 + \beta zp'(z) \prec \frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0}$$

implies that the function $p(z)$ is subordinate to some well-known starlike functions. First, we give the following lemma.

Lemma 2.1 [6, P_{132}, Theorem 3.4] *Let $q(z)$ be analytic in $\mathbb{D}$ and let $\psi(\omega)$ and $\upsilon(\omega)$ be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \upsilon(q(z)) + Q(z)$. Suppose that* (i) *either $h(z)$ is convex, or $Q(z)$ is starlike univalent in $\mathbb{D}$ and* (ii) Re $(zh'(z)/Q(z)) > 0$ *for $z \in \mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq U$ and*

$$\upsilon(p(z)) + zp'(z)\psi(p(z)) \prec \upsilon(q(z)) + zq'(z)\psi(q(z)),$$

then $p(z) \prec q(z)$,and $q(z)$ is best dominant.

Theorem 2.2 *Suppose that the function $p(z)$ is analytic in $\mathbb{D}$, $p(0) = 1$ and*

$$1 + \beta zp'(z) \prec \frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0}$$

with $\theta_0 \in [0, 2\pi]$. Then the following results of subordination hold:

(1)*If* $\beta \geq \frac{2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2})}{\sqrt{2}-1} \approx 1.09116$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0}.$$

(2)*If* $\beta \geq \frac{2(1-\log 2)}{3-2\sqrt{2}} \approx 3.57694$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}\Phi_0(z)+i\sin\theta_0}{\cos\theta_0}.$$

(3)*If* $\beta \geq \frac{2(1-\log 2)}{\sin(1)} \approx 0.729325$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}\Phi_1(z)+i\sin\theta_0}{\cos\theta_0}.$$

(4)*If* $\beta \geq 3(1-\log 2) \approx 0.920558$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}\Phi_2(z)+i\sin\theta_0}{\cos\theta_0}.$$

(5)*If* $\beta \geq (2+\sqrt{2})(1-\log 2) \approx 1.044766$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}\Phi_3(z)+i\sin\theta_0}{\cos\theta_0}.$$

(6)*Let* $-1 < B < A < 1$ *and* $B_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2})} \approx 0.151764$. *If either* (i) $B < B_0$ *and* $\beta \geq \frac{2(1-B)(1-\log 2)}{A-B} \approx 0.613706\frac{1-B}{A-B}$ *or* (ii) $B > B_0$ *and* $\beta \geq \frac{2(1+B)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2})}{A-B} \approx 0.451974\frac{1+B}{A-B}$, *then*

$$p(z) \prec \frac{e^{-i\theta_0}(1+Az)/(1+Bz)+i\sin\theta_0}{\cos\theta_0}.$$

The bounds on β *are best possible.*

Proof. The function $q_\beta(z)$: $\bar{\mathbb{D}} \to C$ defined by

$$q_\beta(z) = 1 + \frac{2e^{-i\theta_0}}{\beta\cos\theta_0}(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1)$$

is analytic and is a solution of the differential equation

$$1 + \beta z q'_\beta(z) = \frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0}.$$

Suppose that the functions $\upsilon(\omega)=1$ and $\psi(\omega)=\beta$. The function $Q: \bar{\mathbb{D}}\to C$ is defined by $Q(z):=zq_\beta'(z)\psi(q_\beta(z))=\beta zq_\beta'(z)$. Since

$$\frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0}-1$$

is starlike function in $\mathbb{D}$, it follows that function $Q(z)$ is starlike. Also note that the function $h(z):=\upsilon(q_\beta(z))+Q(z)$ satisfies $\mathrm{Re}\,(zh'(z)/Q(z))>0$ for $z\in\mathbb{D}$. Therefore, by using Lemma 2.1, it follows that $1+\beta zp'(z)\prec 1+\beta zq_\beta'(z)$ implies $p(z)\prec q_\beta(z)$.

Let

$$q_\beta^*(z)=1+\frac{2}{\beta}(\sqrt{1+z}-\log(1+\sqrt{1+z})+\log 2-1).$$

By the proof of [1, Theorem 2.1], we have $q_\beta^*(z)\prec p_*(z)$ which is equivalent to

$$q_\beta(z)=\frac{e^{-i\theta_0}}{\cos\theta_0}(q_\beta^*(z)-1)+1\prec\frac{e^{-i\theta_0}}{\cos\theta_0}(p_*(z)-1)+1=P(z).$$

The remaining part of the proof can be obtained by using the same arguments as that of the proof of [1, Theorem 2.1].

Theorem 2.3 *Suppose that the function $p(z)$ is analytic in $\mathbb{D}$, $p(0)=1$, $\theta_0\in[0,2\pi]$.*

(1)*If*

$$1+\beta zp'(z)/p(z)\prec\frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z)\prec q_\beta(z)=\exp\left(\frac{2e^{-i\theta_0}}{\beta\cos\theta_0}(\sqrt{1+z}-\log(1+\sqrt{1+z})+\log 2-1)\right).$$

(2)*If*

$$1+\beta zp'(z)/p(z)\prec\frac{e^{-i\theta_0}(1+Az)/(1+Bz)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z)\prec q_\beta(z)=\exp\left(\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(\log|1/Bz|+(A/B)\log|1+Bz|)\right).$$

(3)*If*

$$1+\beta zp'(z)/p(z)\prec\frac{e^{-i\theta_0}\Phi_0(z)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \exp\left(\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(-z/k - 2\log|z/k-1|)\right).$$

(4)*If*

$$1 + \beta zp'(z)/p(z) \prec \frac{e^{-i\theta_0}\Phi_2(z) + i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \exp\left(\frac{e^{-i\theta_0}}{\beta\cos\theta_0}((4/3)z + (1/3)z^2)\right).$$

(5)*If*

$$1 + \beta zp'(z)/p(z) \prec \frac{e^{-i\theta_0}\Phi_3(z) + i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \exp\left(\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(z + \sqrt{z^2+1} - \log(1+\sqrt{z^2+1}) + \log 2 - 1)\right).$$

Proof. The function $q_\beta(z)$ in $\bar{\mathbb{D}}$ defined by

$$q_\beta(z) = \exp\left(\frac{2e^{-i\theta_0}}{\beta\cos\theta_0}(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1)\right)$$

is analytic and is a solution of the differential equation

$$1 + \beta zq'_\beta(z)/q_\beta(z) = \frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0}.$$

Suppose that the functions $v(\omega) = 1$ and $\psi(\omega) = \beta/\omega$. The function $Q : \bar{\mathbb{D}} \to C$ is defined by $Q(z) := zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z)/q_\beta(z)$. Since

$$\frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0} - 1$$

is starlike function in $\mathbb{D}$, it follows that function $Q(z)$ is starlike. Note that the function $h(z) := v(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies Re $(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by using Lemma 2.1, it follows that $1 + \beta zp'(z)/p(z) \prec 1 + \beta zq'_\beta(z)/q_\beta(z)$ implies $p(z) \prec q_\beta(z)$. By using the similar arguments, we have the other results of Theorem 2.3.

The proof of Theorem 2.3 is completed.

Theorem 2.4 *Suppose that the function $p(z)$ is analytic in $\mathbb{D}$, $p(0) = 1$, $\theta_0 \in [0, 2\pi]$.*

(1)*If*

$$1+\beta zp'(z)/p^2(z) \prec \frac{e^{-i\theta_0}\sqrt{1+z}+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \left[1-\frac{2e^{-i\theta_0}}{\beta\cos\theta_0}(\sqrt{1+z}-\log(1+\sqrt{1+z})+\log 2-1)\right]^{-1}.$$

(2)*If*

$$1+\beta zp'(z)/p^2(z) \prec \frac{e^{-i\theta_0}(1+Az)/(1+Bz)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \left[1-\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(\log|1/Bz|+(A/B)\log|1+Bz|)\right]^{-1}.$$

(3)*If*

$$1+\beta zp'(z)/p^2(z) \prec \frac{e^{-i\theta_0}\Phi_0(z)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \left[1-\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(-z/k-2\log|z/k-1|)\right]^{-1}.$$

(4)*If*

$$1+\beta zp'(z)/p^2(z) \prec \frac{e^{-i\theta_0}\Phi_2(z)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \left[1-\frac{e^{-i\theta_0}}{\beta\cos\theta_0}((4/3)z+(1/3)z^2)\right]^{-1}.$$

(5)*If*

$$1+\beta zp'(z)/p^2(z) \prec \frac{e^{-i\theta_0}\Phi_3(z)+i\sin\theta_0}{\cos\theta_0},$$

then

$$p(z) \prec q_\beta(z) = \left[1-\frac{e^{-i\theta_0}}{\beta\cos\theta_0}(z+\sqrt{z^2+1}-\log(1+\sqrt{z^2+1})+\log 2-1)\right]^{-1}.$$

Proof. The function $q_\beta(z)$ in $\bar{\mathbb{D}}$ defined by

$$q_\beta(z) = \left[1-\frac{2e^{-i\theta_0}}{\beta\cos\theta_0}(\sqrt{1+z}-\log(1+\sqrt{1+z})+\log 2-1)\right]^{-1}$$

is analytic and is a solution of the differential equation

$$1 + \beta z q_\beta'(z)/q_\beta^2(z) = \frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0}.$$

Suppose that the functions $\upsilon(\omega) = 1$ and $\psi(\omega) = \beta/\omega^2$. The function $Q : \bar{\mathbb{D}} \to C$ is defined by $Q(z) := zq_\beta'(z)\psi(q_\beta(z)) = \beta z q_\beta'(z)/q_\beta^2(z)$. Since

$$\frac{e^{-i\theta_0}\sqrt{1+z} + i\sin\theta_0}{\cos\theta_0} - 1$$

is starlike function in $\mathbb{D}$, it follows that function $Q(z)$ is starlike. Note that the function $h(z) := \upsilon(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies Re $(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. So, by using Lemma 2.1, it follows that $1 + \beta zp'(z)/p^2(z) \prec 1 + \beta z q_\beta'(z)/q_\beta^2(z)$ implies $p(z) \prec q_\beta(z)$. By using the similar arguments, we have the other results of Theorem 2.4.

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