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Exterior direct sum n-Lie algebras

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Abstract

For a given n-Lie algebra A, n-Lie algebra structures on the vector space $A^n = \{ (x_1, \cdots, x_n) \mid x_i \in A, 1 \le i \le n \}$ are constructed. For each $s \ge 2$, the n-Lie product $[, \cdots,]_s$ is defined on the vector space A^n, which is called the exterior direct sum n-Lie algebra of the n-Lie algebra A. And it is proved that, n-Lie algebra A can be embedded into its exterior direct sum n-Lie algebras. And some ideals and subalgebras of $(A^n, [, \cdots,]_s)$ are obtained.

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1 Introduction

n-Lie algebra [2, 3] is a kind of multiple algebraic system appearing in many fields in mathematics and mathematical physics [4, 5, 6]. Especially, the structure of 3-Lie algebras is applied to the study of supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra; the n-Jacobi identity can be regarded as a generalized Plucker relation in the physics literature, and so on (cf. [1, 7]). In this paper, we pay our main attention to construct n-Lie algebras from a given n-Lie algebra. For a given n-Lie algebra A, we construct an n-Lie algebra structure on the vector space A^n, which is the direct sum vector space

$A^n = \{(x_1, \cdots, x_n) \mid x_i \in A, 1 \le i \le n\}$, satisfying that for all $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n) \in A^n$ and $\lambda \in F$,

$$X + Y = (x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n),$$

$$\lambda X = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n).$$

The vector space A^n is called the exterior direct sum vector space of A.

An n-Lie algebra [3] is a vector space A over a field F endowed with an n-ary multilinear skew-symmetric multiplication satisfying that for all $x_1, \cdots, x_n$, $y_2, \cdots, y_{n-1} \in A$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n}[x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \qquad (1)$$

The identity (1) is usually called the n-Jacobi identity.

Let A be an n-Lie algebra and V be a subspace of A. If V satisfies that $[V, \cdots, V] \subseteq V$, then V is a subalgebra of the n-Lie algebra A. If V satisfies that $[V, A, \cdots, A] \subseteq V$, then V is a called an ideal of the n-Lie algebra A. If V satisfies that $[V, \cdots, V] = 0$ ($[V, V, A, \cdots, A] = 0$), then V is called an abelian subalgebra (an abelian ideal).

2 The exterior direct sum n-Lie algebra A^n

Theorem 2.1 *Let A be an n-Lie algebra. Then A^n is an n-Lie algebra in the multiplication $[, \cdots, ,]_2$, where for all $X_j = (x_1^j, \cdots, x_n^j) \in A^n$, $j = 1, \cdots, n$,*

$$[X_1, \cdots, X_n]_2 = (\sum_{i=1}^{n}[x_2^1, \cdots, x_1^i, \cdots, x_2^n], [x_2^1, \cdots, x_2^n], \cdots, [x_n^1, \cdots, x_n^n]). \qquad (2)$$

Proof It is clear that the multiplication $[, \cdots,]_2$ defined by Eq.(2) is n-ary linear and skew-symmetric. Now we prove it satisfies identity (1). For all $X_j = (x_1^j, \cdots, x_n^j) \in A^n$, $j = 1, \cdots, n$, $Y_k = (y_1^k, \cdots, y_n^k) \in A^n$, $k = 2, \cdots, n$. Suppose

$(z_1, \cdots, z_n) = [[X_1, \cdots, X_n]_2, Y_2, \cdots, Y_n]_2$
$= [[(x_1^1, \cdots, x_n^1), \cdots, (x_1^n, \cdots, x_n^n)]_2, (y_1^2, \cdots, y_n^2), \cdots, (y_1^n, \cdots, y_n^n)]_2.$

By Eq.(2), for all $2 \le k \le n$, $z_k = [[x_k^1, \cdots, x_k^n], y_k^2, \cdots, y_k^n]$, and

$$z_1 = \sum_{i=1}^{n}[[x_2^1, \cdots, x_1^i, \cdots, x_2^n], y_2^2, \cdots, y_2^n] + \sum_{l=2}^{n}[[x_2^1, \cdots, x_2^n], y_2^2, \cdots, y_1^l, \cdots, y_2^n]$$
$$= \sum_{i=1}^{n}([x_2^1, \cdots, x_2^{(i-1)}, [x_1^i, y_2^2, \cdots, y_2^n], \cdots, x_2^n]$$
$$+ \sum_{s \neq i}^{n}[x_2^1, \cdots, x_2^{(s-1)}[x_2^s, y_2^2, \cdots, y_2^n], \cdots, x_1^i, \cdots, x_2^n])$$
$$+ \sum_{l=2}^{n}\sum_{t=1}^{n}[x_2^1, \cdots, x_2^{(t-1)}, [x_2^t, y_2^2, \cdots, y_1^l, \cdots, y_2^n], x_2^{(t+1)}, \cdots, x_2^n]$$

$= \sum_{i=1}^{n}[x_2^1, \cdots, x_2^{(i-1)}, [x_1^i, y_2^2, \cdots, y_2^n], x_2^{(i+1)}, \cdots, x_2^n]$
$+ \sum_{i,s=1}^{n} \sum_{s\neq i}^{n}[x_2^1, \cdots, x_2^{(s-1)}, [x_2^s, y_2^2, \cdots, y_2^n], \cdots, x_1^i, \cdots, x_2^n]$
$+ \sum_{l=2}^{n} \sum_{t=1}^{n}[x_2^1, \cdots, x_2^{(t-1)}, [x_2^t, y_2^2, \cdots, y_1^l, \cdots, y_2^n], \cdots, x_2^n].$

Now suppose that

$$(u_1, \cdots, u_n)= \sum_{i=1}^{n}[(x_1^1, \cdots, x_n^1), \cdots, [(x_1^i, \cdots, x_n^i), (y_1^2, \cdots, y_n^2), \cdots, (y_1^n, \cdots, y_n^n)]_2, \cdots, (x_1^n, \cdots, x_n^n)]_2.$$

By Eq.(2), for $2 \leq k \leq n$, we have

$$u_k = \sum_{i=1}^{n}[x_k^1, \cdots, [x_k^i, y_k^2, \cdots, y_k^n], \cdots, x_k^n] = [[x_k^1, \cdots, x_k^n], y_k^2, \cdots, y_k^n] = z_k.$$

$u_1 = \sum_{i=1}^{n}[x_2^1, \cdots, x_2^{(i-1)}, [x_1^i, y_2^2, \cdots, y_2^n]$
$+ \sum_{l=2}^{n}[x_2^i, y_2^2, \cdots, y_1^l, \cdots, y_2^n], x_2^{i+1} \cdots, x_2^n]$
$+ \sum_{i=1}^{n} \sum_{s\neq i}^{n}[x_2^1, \cdots, [x_2^i, \cdots, y_2^n], \cdots, x_1^s, \cdots, x_2^n]$
$= \sum_{i=1}^{n}[x_2^1, \cdots, x_2^{(i-1)}, [x_1^i, y_2^2, \cdots, y_2^n], x_2^{(i+1)}, \cdots, x_2^n]$
$+ \sum_{i=1}^{n} \sum_{l=2}^{n}[x_2^1, \cdots, x_2^{(i-1)}, [x_2^i, y_2^2, \cdots, y_1^l, \cdots, y_2^n], \cdots, x_2^n]$
$+ \sum_{i=1}^{n} \sum_{s\neq i}^{n}[x_2^1, \cdots, x_2^{(i-1)}[x_2^i, y_2^2, \cdots, y_2^n], \cdots, x_1^s, \cdots, x_2^n] = z_1.$

Therefor $[, \cdots,]_2$ satisfies Eq.(2).The proof is complete.

Example Let A be a 3 dimension 3-Lie algebra with a basis e_1, e_2, e_3, and the multiplication is $[e_1, e_2, e_3] = e_1$. Then A^3 is a 27 dimensional 3-Lie algebra with a basis $\{(e_i, e_j, e_k) \mid 1 \leq i, j, k \leq 3\}$. For all

$X_1 = (\sum_{i=1}^{3} a_{1i}e_i, \sum_{i=1}^{3} b_{1i}e_i, \sum_{i=1}^{3} c_{1i}e_i)$, $X_2 = (\sum_{j=1}^{3} a_{2j}e_j, \sum_{j=1}^{3} b_{2j}e_j, \sum_{j=1}^{3} c_{1j}e_j)$,

$X_3 = (\sum_{k=1}^{3} a_{3k}e_k, \sum_{k=1}^{3} b_{3k}e_k, \sum_{k=1}^{3} c_{3k}e_k)$, where $a_{ij}, b_{ij}, c_{ij} \in F$, $1 \leq i, j \leq 3$. Suppose $(u_1, u_2, u_3) = [X_1, X_2, X_3]_2$. Then

$$u_1 = [\sum_{i=1}^{3} a_{1i}e_i, \sum_{j=1}^{3} b_{2j}e_j, \sum_{k=1}^{3} b_{3k}e_k] +[\sum_{i=1}^{3} b_{1i}e_i, \sum_{j=1}^{3} a_{2j}e_j, \sum_{k=1}^{3} b_{3k}e_k] + \sum_{i=1}^{3} b_{1i}e_i, \sum_{j=1}^{3} b_{2j}e_j, \sum_{k=1}^{3} a_{3k}e_k]$$

$$= \{det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} + det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} +det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\}e_1,$$

$$u_2 = [\sum_{i=1}^{3} b_{1i}e_i, \sum_{j=1}^{3} b_{2j}e_j, \sum_{k=1}^{3} b_{3k}e_k] = det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} e_1,$$

$$u_3=[\sum_{i=1}^{3}c_{1i}e_i,\sum_{j=1}^{3}c_{2j}e_j,\sum_{k=1}^{3}c_{3k}e_k]=det\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{pmatrix}e_1.$$

Specially, $[(e_1,e_1,e_1),(e_2,e_2,e_2),(e_3,e_3,e_3)]_2=(3e_1,e_1,e_1)$,
$[(e_1,e_1,e_1),(e_2,e_1,e_3),(e_3,e_3,e_3)]_2=(e_1,0,0)$.

Theorem 2.2 *Let A be an n-Lie algebra. Then for any $3\le s\le n$, A^n is an n-Lie algebra in the multiplication $[,\cdots,]_s$, where for all $x_i^j\in A$, $i,j=1,\cdots,n$,*

$$[(x_1^1,\cdots,x_n^1),\cdots,(x_1^n,\cdots,x_n^n)]_s$$
$$=(\sum_{i=1}^{n}[x_s^1,\cdots,x_1^i,\cdots,x_s^n],[x_2^1,\cdots,x_2^n],\cdots,[x_n^1,\cdots,x_n^n]).\qquad(3)$$

Proof The proof is similar to Theorem 2.1, we omit the computation process.

Corollary 2.3 *Let A be an n-Lie algebra. For any $2\le s,k\le n$, $s\ne k$, the vector space A^n is an n-Lie algebra in the multiplication $[,\cdots,]_k^s$, where for all $X_j=(x_1^j,\cdots,x_n^j)\in A^n$, $j=1,\cdots,n$,*

$$[X_1,\cdots,X_n]_k^s=([x_1^1,\cdots,x_1^n),\cdots,\underbrace{\sum_{i=1}^{n}[x_k^1,\cdots,x_s^i,\cdots,x_k^n]}_{s},\cdots,[x_n^1,\cdots,x_n^n]).\ (4)$$

Proof The proof is similar to Theorem 2.1.

Denote $A^j=\{(0,\cdots,0,\underbrace{x}_{j},0,\cdots,0)\ |x\in A\}$, $j=1,\cdots,n$, and $B_i=(A,\cdots,A,\underbrace{0}_{i},A,\cdots,A)$, for $i=1,\cdots,n$.

Then we have the following result.

Theorem 2.4 *Let A be an n-Lie algebra. Then*

1) For each $j\ne 2$, and $1\le j\le n$, A_j is an ideal of the exterior direct sum n-Lie algebra $(A^n,[,\cdots,]_2)$, A_2 is a subalgebra.

2) For $j\ge 2$, A_j is isomorphic to the n-Lie algebra A, therefore, n-Lie algebra A can be embedded in its exterior direct sum n-Lie algebra $(A^n,[,\cdots,]_2)$.

3) A_1 is an abelian subalgebra of $(A^n,[,\cdots,]_2)$, that is, $[A_1,\cdots,A_1]_2=0$.

4) The subspace B_1 is a subalgebra of the n-Lie algebra $(A^n,[,\cdots,]_2)$, and B_i for $i=2,\cdots,n$ are ideals.

Proof For all $x_k\in A$, $1\le k\le n$, and $j\ge 2$, by Eq.(2),

$$[(0,\cdots,0,\underbrace{x_1}_{j},0,\cdots,0),\cdots,(0,\cdots,0,\underbrace{x_n}_{j},0,\cdots,0)]_2$$
$$=(0,\cdots,0,\underbrace{[x_1,\cdots,x_n]}_{j},0,\cdots,0)\in A_j.$$ And for $j\ne 2$,

$$[(0,\cdots,0,\underbrace{x_1}_{j},0,\cdots,0),A^n,\cdots,A^n]_2$$
$$=(0,\cdots,0,\underbrace{[x_1,A,\cdots,A]}_{j},0,\cdots,0)\subseteq A_j.$$

It follows the result 1). The result 2) follows from the result 1).

By Eq.(2), we have $[A_1, A_1, A^n, \cdots, A^n]_2 = 0$, the result 3) holds.

By the similar discussion, we obtain the result 4).

Corollary 2.5 *For* $3 \leq s \leq n$, *the subspace* A_j *is a subalgebra of the* n*-Lie algebra* $(A^n, [, \cdots,]_s)$, $j = 1, \cdots, n$, *and in the case* $j \neq 1$, A_j *is isomorphic to the* n*-Lie algebra* A. *For* $i \neq s$, $1 \leq i \leq n$, A_i *are ideals of* $(A^n, [, \cdots,]_s)$. *And The subspace* B_1 *is a subalgebra of the* n*-Lie algebra* $(A^n, [, \cdots,]_s)$, *and* B_i *for* $i = 2, \cdots, n$ *are ideals.*

Proof The prof is similar to Theorem 2.4. We omit the computation process.

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