Explicit building the nonlinear coherent states associated to weighted shift $z^p \frac{d^{p+1}}{dz^{p+1}}$ of order p in classical Bargmann representation

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Abstract

This article is devoted to build the nonlinear coherent states associated to the some specific bakward shift unbounded operators $\mathbb{H}_p = \mathbb{A}^{*p}\mathbb{A}^{p+1}; p = 0, 1, ...$ realized as differential operators in classical Bargmann space where $\mathbb{A}$ and $\mathbb{A}^*$ are the standard Bose annihilation and creation operators such that $[\mathbb{A}, \mathbb{A}^*] = \mathbb{I}$. We use the Gazeau-Klauder formalism to construct the coherent states of this operator $\mathbb{H}_p$ and investigate some properties of these coherent states.

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1 Introduction

The coherent states play an important role in the context of Hermitian quantum mechanics see for example [13, 24, 25]. In recent years non Hermitian quantum mechanics have been extensively studied from various stand points see for example [20, 21, 22] and recently the concept of coherent states was also introduced to non Hermitian quantum systems see for example [1] or in the quantization of a nonrenormalizable scalar quantum field theory by affine techniques see for example [16]. However, as in Hermitian models, coherent

states corresponding to arbitrary non Hermitian potential are not easy to construct.

It is know that coherent states can be constructed by different techniques (e.g., a coherent state may be defined as a minimum uncertainly state, annihilation operator eigenstate, etc ..) and usually they have different properties

In the Gazeau-Klauder formalism, a coherent state should satisfy the following criteria [5]:

i) continuity of labelling, ii) temporel stability, iii)resolution of identity and iv) action identity.

An essential ingredient in the definition of coherent states is the completeness property (or the resolution of unity condition) and as sufficiently many eigenvectors of an unbounded operator is connected to its hypercyclity or its chaoticity. More recently in [6] [Advances in Mathematical Physics (2011)], we have established that the backward shift unbounded operators $\mathbb{H}_p = \mathbb{A}^{*p}\mathbb{A}^{p+1} = z^p \frac{d^{p+1}}{dz^{p+1}}; p = 0, 1, ...$ are non-wandering and hypercyclic operators on classical Bargmann space, the space of entire functions with Gaussian measure.In this way,the aim of this paper is to construct nonlinear coherent states corresponding to $\mathbb{H}_p$, where $\mathbb{A}$ and $\mathbb{A}^*$ are the standard Boson annihilation and creation operators satisfying the commutation relation $[\mathbb{A}, \mathbb{A}^*] = \mathbb{I}$.

At this point, we note that coherent states can be constructed following any of three methods

(i) By applying the unitary displacement operator to the ground state or (ii) Defining coherent states as eigenstate of the lowering operator or (iii) Defining coherent states as minimum uncertainty states.

These three methods are generally not equivalent and only in the case of standard harmonic oscillator, where the commutator of the raising and lowering operator is the unit operator,these three methods are equivalent.

In this work we shall follow the second approch to give an explicit construction of coherent states corresponding to $\mathbb{H}_p$.

We now describe briefly the contents of this paper, section by section. In Section 2, We make a review of some basic properties of coherent states and their connections with classical Bargmann space within our necessity and we recall some preliminaries results on the operator $\mathbb{H}_p$. The section 3 deals with the construction the non linear coherent states of $\mathbb{H}_p$.

2 Pedagogic review of some basic properties of coherent states and their connections with Bargmann space

Of special interest is a representation of operators $\mathbb{A}$ and $\mathbb{A}^*$ as linear operators in a separable Hilbert $\mathbb{B}$ (Fock space)spanned by eigenvectors (Dirac notation) $\mid n >; n = 0, 1, ...$ of the positive semi-definite number operator $\mathbb{N} = \mathbb{A}^* \mathbb{A}$ (number operator).One has the well-known relations:

$$[\mathbb{N}, \mathbb{A}^*] = \mathbb{A}^*, \quad [\mathbb{N}, \mathbb{A}] = -\mathbb{A}, \quad [\mathbb{A}, \mathbb{A}^*] = \mathbb{I}. \tag{2.1}$$

The actions of $\mathbb{A}$ and $\mathbb{A}^*$ on $\mathbb{B}$ are given by

$$\mathbb{A} \mid n >= \sqrt{n} \mid n-1 > \text{ and }, \mathbb{A}^* \mid n >= \sqrt{n+1} \mid n+1 > \tag{2.2}$$

Where $\mid 0 >$ is a normalized vacuum ($\mathbb{A} \mid 0 >= 0$) and $< 0 \mid 0 >= 1$

From (2.2) state $\mid n >$ for $n \geq 1$ are given by

$$\mid n >= \frac{(A^*)^n}{\sqrt{n!}} \mid 0 > \tag{2.3}$$

These states satisfy the orthogonality and completeness conditions

$$< m \mid n >= \delta_{mn}, \sum_{n=0}^{\infty} \mid n >< n \mid = 1 \tag{2.4}$$

For the normalized state $\mid z >\in \mathbb{B}; z \in \mathbb{C}$ the following three conditions are equivalent :

(i) $\mathbb{A} \mid z >= z \mid z >$ and $< z \mid z >= 1$ $\tag{2.5}$

(ii) $\mid z >= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n >$ $\tag{2.6}$

(iii) $\mid z >= e^{z\mathbb{A}^* - \overline{z}\mathbb{A}} \mid 0 >$ $\tag{2.7}$

This equivalence is based on the famous elementary Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \tag{2.8}$$

whenever $[A, [A, B]] = [B, [A, B]] = 0$

Definition 2.1. *(coherent state) The state $|z >$ that satisfies one of (i) or (ii) or (iii) above is called the coherent state.*

Remark 2.2. *(resolution of unity) 1) It is not easy to derive (iii) from (ii) without knowing Baker-Campbell-Hausdorff formula. The coherent states (iii) are called a Perelomov's type and the coherent states (ii) are called a Barut-Girardello's type.*
2) Let $< z \mid =< 0 \mid e^{zA^*} e^{-\frac{1}{2}|z|^2}$ the adjoint vector of coherent state $\mid z >$ then the important property of coherent states is the following resolution (partition) of unity.

$$\frac{1}{\pi} \int_{\mathbb{C}} \mid z >< z \mid dxdy = \sum_{n=0}^{\infty} \mid n >< n \mid = 1 \text{ where } z = x + iy \tag{2.9}$$

Definition 2.3. *(coherent states space)   The space of coherent state vectors $\mathbb{B}$ is the space spanned by the set $\{\mid z >\}$ where*

$$\mid z >= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n >; z = x + iy \tag{2.10}$$

From the above properties, we have $\mathbb{A}^* \mid z > = (\frac{\overline{z}}{2} + \frac{\partial}{\partial z}) \mid z >$ and $A \mid z > = (\frac{z}{2} - \frac{\partial}{\partial \overline{z}}) \mid z >$ where $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are linear partial differential operators on $\mathbb{R}^2$ given by $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}$; $z = x + iy$ and $\dfrac{\partial}{\partial \overline{z}} = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$; $\overline{z} = x - iy$

Now, we recall that the space $\mathbb{B}$ of coherent state vectors is closely related to classical Bargmann's space which was used in [**2**] for the canonical commutation rules as representation space of quantum mechanics.

For any coherent state $\mid \phi >$, we can define an entire analytic function by

$$\phi(z) = e^{\frac{1}{2}|z|^2} < \phi \mid z > = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} < \phi \mid n > \tag{2.11}$$

As $\displaystyle\int_{\mathbb{C}} |< \phi \mid z >|^2 \, dxdy < \infty$ then

$$\int_{\mathbb{C}} \mid \phi(z) \mid^2 e^{-|z|^2} dxdy < \infty \tag{2.12}$$

We denote the classical Bargmann space by :

Definition 2.4. *(Bargmann space)   The classical Bargmann space is a subspace of the space $O(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$ given by*
$\mathbb{B} = \{\phi \in O(\mathbb{C}); < \phi, \phi > < \infty\}$ *where the pairing $<,>$ is defined by*
$$< \phi, \psi > = \int_{\mathbb{C}} \phi(z)\overline{\psi(z)} e^{-|z|^2} dxdy \tag{2.13}$$
for all $\phi, \psi \in O(\mathbb{C})$ and Lebesgue measure $dxdy$ on $\mathbb{C}$.

It is easy to verify that the pairing (2.13) defined on the classical Bargmann space is an inner product and the associated norm is

$$\parallel \phi \parallel = \sqrt{\int_{\mathbb{C}} \mid \phi(z) \mid^2 e^{-|z|^2} dxdy} \tag{2.14}$$

Now, we can used a theorem of Weierstrass to show that any Cauchy sequence in $\mathbb{B}$ has a limit $\phi \in O(\mathbb{C})$ and we check that $\phi \in \mathbb{B}$ and indeed is the limit of the Cauchy sequence in the norm $\parallel . \parallel$ of $\mathbb{B}$ induced by the inner product. These steps show that the space $\mathbb{B}$ is complete and we have

Lemma 2.5.   *i) The classical Bargmann space is a Hilbert space.*
ii) An orthonormal basis of $\mathbb{B}$ is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; n = 0, 1, \dots.$$ (2.15)

Suppose that we have a space of holomorphic functions in the variables z, multiplication operators
$\mathbb{W} : \phi(z) \to \mathbb{W}\phi(z) = z\phi(z)$ and differential operators $\overline{\mathbb{W}} : \phi(z) \to \mathbb{W}\phi(z) = \frac{d}{dz}\phi(z)$ so that the commutation relation $[\mathbb{W}, \overline{\mathbb{W}}] = \mathbb{I}$ is satisfied.
Furthermore, suppose that there exists an inner product on this space of holomorphic functions that is of the form
$$< \phi, \psi > = \int_{\mathbb{C}} \phi(z)\overline{\psi(z)}\rho(z, \overline{z})dxdy$$ (2.16)
for some weight function $\rho(z, \overline{z})$. The following lemma explains why the space of holomorphic functions is equipped with a Gaussian measure:

Lemma 2.6. *(Gaussian measure)   The Gaussian measure that is used in the definition of the inner product (2.13) follows from the properties of the operators $\mathbb{W}$ and $\overline{\mathbb{W}}$. In particular the requirement that $\mathbb{W}^* = \overline{\mathbb{W}}$ where $\mathbb{W}^*$ is the adjoint of $\mathbb{W}$.*

Proof : The requirement that $\mathbb{W}^* = \overline{\mathbb{W}}$ then gives that $< \frac{d}{dz}\phi, \psi > = < \phi, z\psi >$, i.e. $\int_{\mathbb{C}} (\frac{d}{dz}\phi(z))\overline{\psi(z)}\rho(z, \overline{z})dxdy = \int_{\mathbb{C}} \phi(z)\overline{z\psi(z)}\rho(z, \overline{z})dxdy$
As $(\frac{d}{dz}\phi(z))\overline{\psi(z)}\rho(z, \overline{z}) = \frac{d}{dz}(\phi(z)\overline{\psi(z)}\rho(z, \overline{z})) - \phi(z)(\frac{d}{dz}\overline{\psi(z)})\rho(z, \overline{z}) - \phi(z)\overline{\psi(z)}\frac{d}{dz}\rho(z, \overline{z})$
then $\int_{\mathbb{C}} \frac{d}{dz}(\phi(z))\overline{\psi(z)}\rho(z, \overline{z})dxdy =$
$\int_{\mathbb{C}} \frac{d}{dz}(\phi(z)\overline{\psi(z)}\rho(z, \overline{z}))dxdy - \int_{\mathbb{C}} \phi(z)(\frac{d}{dz}\overline{\psi(z)})\rho(z, \overline{z})dxdy - \int_{\mathbb{C}} \phi(z)\overline{\psi(z)}\frac{d}{dz}\rho(z, \overline{z})dxdy$
In the right hand side, the first term of the integrand vanishes if we assume that the inner product between ϕ and ψ is finite, so that $\phi\overline{\psi}\rho \to 0$ sufficiently fast as $| z | \to \infty$. The second term also vanishes, because ψ is holomorphic, so that $\overline{\psi}$ is anti-holomorphic and hence $\frac{d}{dz}\overline{\psi(z)} = 0$ This gives
$$\int_{\mathbb{C}} \phi(z)\overline{z\psi(z)}\rho(z, \overline{z})dxdy + \int_{\mathbb{C}} \phi(z)\overline{\psi(z)}\frac{d}{dz}\rho(z, \overline{z})dxdy = 0$$ (2.17)
which is solved for arbitrary ϕ and ψ if
$$\overline{z}\rho(z, \overline{z}) + \frac{d}{dz}\rho(z, \overline{z}) = 0$$ (2.18)
giving $\rho(z, \overline{z}) = Ce^{-|z|^2}$. The constant C is chosen to be $\frac{1}{\pi}$, so that the norm of the constant function $\phi(z) = 1$ is one. This explains why the space of holomorphic functions is equipped with a Gaussian measure.

Remark 2.7.   *Consider the following variant of classical Bargman space. Let us consider functions $f : \mathbb{R} \to \mathbb{C}$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{\sqrt{n!}}; \qquad \sum_{n=0}^{\infty} \mid a_n \mid^2 < \infty \tag{2.19}$$

the a_n being complex coefficients. Here our functions f are complex-valued analytic functions of one real variable x. Call $\mathbb{B}V$ the space of all functions satisfying (2.19). A basis for $\mathbb{B}V$ is given by the set of all real monomials

$$f_n(x) := \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N} \tag{2.20}$$

We can define a scalar product on $\mathbb{B}V$ by declaring these monomials to be orthonormal,

$$((f_n, f_m)) = \delta_{nm} \qquad n, m \in \mathbb{N} \tag{2.21}$$

and extending the above to all elements of $\mathbb{B}V$ by complex linearity. This scalar product makes $\mathbb{B}V$ a complex Hilbert space. The difference with respect to classical Bargmann space $\mathbb{B}$ is that, the functions $f \in \mathbb{B}$ depending on the real variable x instead of the complex variable z, the scalar product on $\mathbb{B}V$ is no longer given by (2.13), nor by its real analogue.

Indeed, given any two $f; g \in \mathbb{B}V$, the analogue of (2.13) for $\mathbb{B}V$ would be the integral

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}e^{-x^2} dx \tag{2.22}$$

Although this integral does define a scalar product on $\mathbb{B}V$, this scalar product does not make the basis (2.20) orthogonal, as one readily verifies. Therefore one, and only one, of the following properties can be satisfied:

i) the space $\mathbb{B}V$ is Hilbert with respect to the scalar product (2.22), but the monomial basis (2.20) is not orthogonal with respect to it

ii) the space $\mathbb{B}V$ is Hilbert with respect to the scalar product (2.21), and the monomial basis (2.20) is indeed orthonormal with respect to it, but this scalar product is not given by the integral (2.22).

Now, after the above pedagogic analysis, we come back to classical Bargmann representation, in this representation the annihilator and creator operators are defined by

$\mathbb{A}\phi(z) = \frac{d}{dz}\phi(z)$ with domain $D(\mathbb{A}) = \{\phi \in \mathbb{B}; \frac{d}{dz}\phi \in \mathbb{B}\}$ $\tag{2.23}$

$\mathbb{A}^*\phi(z) = z\phi(z)$ with domain $D(\mathbb{A}^*) = \{\phi \in \mathbb{B}; z\phi \in \mathbb{B}\}$ $\tag{2.24}$

$\mathbb{B}$ is closed in $L_2(\mathbb{C}, d\mu(z))$ where the measure $d\mu(z) = e^{-|z|^2}dxdy$ and it is closed related to $L_2(\mathbb{R})$:

The Schrödinger equation of harmonic oscillator in one dimension is:

$$\mathbb{H}f(x) = \sigma f(x) \tag{2.25}$$

and

$$\mathbb{H} = \frac{1}{2m}(p_x^2 + m^2\omega^2 x^2), \qquad [x, p_x] = i\hbar \tag{2.26}$$

put $x = \sqrt{\frac{\hbar}{m\omega}}$ then we obtain

$$\mathbb{H} = \hbar\omega(p^2 + q^2) \tag{2.27}$$

the solution of Schrödinger equation is

$$\mathbb{H}u_n(q) = \sigma_n u_n(q), \sigma_n = \hbar\omega(n + \tfrac{1}{2}) \tag{2.28}$$

with

$$u_n(q) = (\sqrt{\pi}2^n n!)^{-\frac{1}{2}} e^{-\frac{q^2}{2}} H_n(q), u_n(x) = (\tfrac{m\omega}{\hbar})^{\frac{1}{4}} u_n(q) \tag{2.29}$$

$\mathbb{H}_n(q)$ is the Hermite polynomial with

$$\mathbb{H}_n(-q) = (-1)^n H_n(q) \text{ and } u_n(-q) = (-1)^n u_n(q) \tag{2.30}$$

we find then the generating function(see :[**18**])

$$G(z,q) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} u_n(q) = \pi^{-\frac{1}{4}} e^{-\frac{q^2}{2} - \frac{z^2}{2} + \sqrt{2}qz} \tag{2.31}$$

In (2.31), we note that $G(z,q) = \sum_{n=0}^{\infty} u_n(q)e_n(z)$ where $e_n(z) = \frac{z^n}{\sqrt{n!}}$ is orthonomal basis of classical Bargmann space.We have also

1) $\int_{\mathbb{C}} e^{\alpha\bar{z}} e^{\beta z} d\mu(z) = e^{\alpha\beta}$ and $\phi(z) = \int_{\mathbb{C}} e^{z\bar{\xi}} \phi(\xi) d\mu(\xi)$ (2.32)

where $d\mu(z)$ is the cylindrical measure and ϕ is in classical Bargmann space.
2) Let $\widehat{q}$ the multiplication operator with respect the variable q and $\mid q >$ is the eigenfunction which verify $\widehat{q} \mid q >= q \mid q >$ then we can define the Dirac transform by:

$< q \mid: u_n \to u_n(q) =< q \mid n >$ where $\mid n >= \frac{A^{*n}}{\sqrt{n!}} \mid 0 >$ and to get:

(i) $G(z,q) =< q \mid z >$ where $\mid z >= e^{-\frac{|z|^2}{2}} e^{z\mathbb{A}^*} \mid 0 >$ (2.33)

(ii) $< q \mid= \int_{\mathbb{C}} G(z,q) d\mu(z) < z \mid$ where $< z \mid$ is the adjoint vector of coherent state $\mid z >$. (2.34)

In [**2**], Bargmann have defined an integral transform $\mathbb{I}$ of $L_2(\mathbb{R})$ onto $\mathbb{B}$ by

$$[\mathbb{I}(f)](z) = \phi(z) = \int_{\mathbb{R}} G(z,q)f(q)dq, f \in L_2(\mathbb{R}) \tag{2.35}$$

Theorem 2.8. *(Bargmann)* *1) If $f \in L_2(\mathbb{R})$ the integral (2.35) converges absolutely.*
2) The Bargmann transform $\mathbb{I} : L_2(\mathbb{R}) \to \mathbb{B} : f \to \phi$ defined by

$$\phi(z) = \int_{\mathbb{R}} G(z,q)f(q)dq, f \in L_2(\mathbb{R}) \text{ and } G(z,q) = \pi^{-\frac{1}{4}} e^{-\frac{q^2}{2} - \frac{z^2}{2} + \sqrt{2}qz} \text{ is a } \boldsymbol{surjective} \text{ isometry.}$$

Now we define

$$\mathbb{B}_p = \{\phi \in \mathbb{B}; \frac{d^j}{dz^j}\phi(0) = 0, 0 \le j \le p\} \tag{2.36}$$

An orthonormal basis of $\mathbb{B}_p$ is given by

$$e_n(z) = \frac{z^n}{\sqrt{n!}}; n = p+1, p+2, ... \tag{2.37}$$

Hence a family of weighted shifts $\mathbb{H}_p$ as following

$$\mathbb{H}_p = \mathbb{A}^{*p}\mathbb{A}^{p+1} \text{ with domain } D(\mathbb{H}_p) = \{\phi \in \mathbb{B}; \mathbb{H}_p\phi \in \mathbb{B}\} \bigcap \mathbb{B}_p \tag{2.38}$$

Remark 2.9.   *(i) For $p = 0$, the operator $\mathbb{H}_0 = \mathbb{A}$ is the derivation in classical Bargmann space and it is the celebrated quantum annihilation operator.*

(ii) $\mathbb{H}_0^* e_n = \sqrt{n+1} e_{n+1}$ is weighted shift with weight $\omega_n = \sqrt{n+1}$ for $n = 0, 1, ...$

(iii) It is known that $\mathbb{H}_0$ with its domain $D(\mathbb{H}_0)$ is an operator chaotic in classical Bargmann space.

(iv)$\mathbb{H}_0 \phi_\lambda(z) = \lambda \phi_\lambda(z)$ for all $\lambda \in \mathbb{C}$ where $\phi_\lambda(z) = \sum_{n=0}^{\infty} \dfrac{\lambda^n}{\sqrt{n!}} e_n(z)$ and

$\| \phi_\lambda \|^2 = e^{|\lambda|^2}$

(v) The function $e^{-|\lambda|^2} \phi_\lambda(z)$ is called a coherent normalized quantum optics (see [15] , [19] and [22]).

Remark 2.10.   *(i) For $p = 1$, the operator $\mathbb{H}_1 = \mathbb{A}^* \mathbb{A}^2 = z \dfrac{d^2}{dz^2}$ has as adjoint the operator $\mathbb{H}_1^* = z^2 \dfrac{d}{dz}$*

(ii) $\mathbb{H}_1^* e_n = n\sqrt{n+1} e_{n+1}$is weighted shift with weight $\omega_n = n\sqrt{n+1}$ for $n = 1, ...$ and it is known that $\mathbb{H}_1 + \mathbb{H}_1^*$ is a not selfadjoint operator and it is chaotic in classical Bargmann space [4]. This operator play an essential role in Reggeon field theory (see [8] and [9])

(iii) The operators $\mathbb{H}_p$ arising also in the interaction picture the Jaynes-Cummings models, see for example a model introduced by Obada-Abd Al-Kader in [23] the interaction Hamiltonian for the model is :

$$\mathbb{H}_I = -\sum_{p=0}^{\infty} \{\Omega_1 e^{i\phi_1} e^{-\eta_1^2/2} \frac{(i\eta_1)^{2p+1}}{p!(p+1)!} H_p^* + \Omega_2 e^{i\phi_2} e^{-\eta_2^2/2} \frac{(i\eta_2)^{2p+1}}{p!(p+1)!} H_p\}\sigma_- + h.c (2.39)$$

Where Ω_j are the Rabi frequencies and η_j^2 are the Lamb-Dicke ; j = 1,2.

The operators σ_- and σ_+ act on the ground state $\mid g >$ and excited state $\mid e >$ as follow $\sigma_\pm \mid g >= \frac{1 \pm 1}{2} \mid e >$ and $\sigma_\pm \mid e >= \frac{1 \mp 1}{2} \mid g >$.

(iv) On $\mathbb{B}_p, p = 0, 1, ...,$ which is the orthogonal of span $\{e_n; n \leq p - 1\}$ in classical Bargmann space, the adjoint of $\mathbb{H}_p$ is $\mathbb{H}_p^* = z^{p+1} \dfrac{d^p}{dz^p}$ such that

$\mathbb{H}_p^* e_n = \omega_n e_{n+1}$ with weight $\omega_n = \sqrt{n+1} \dfrac{n!}{(n-p)!}$ for $n \geq p \geq 0$ (2.40).

We point out that a generalization of the coherent states was done by q-deforming the basic commutation relation $[\mathbb{A}, \mathbb{A}^*] = \mathbb{I}$. A further generalization is to define states that are eigenstates of the operator $f(\mathbb{A}^*\mathbb{A})\mathbb{A}$ where $f(\mathbb{A}^*\mathbb{A})$ is a operator valued function of the number operator $\mathbb{N} = \mathbb{A}^*\mathbb{A}$. These eigenstates are called as non-linear coherent states [12] and they are nonclassical. In the linear limit, $f(\mathbb{A}^*\mathbb{A}) = \mathbb{I}$, the non-linear coherent states become the usual coherent states $\mid z >$.

We make contact with the combinatorial sequences appearing in the solution of the boson normal ordering [7] for an explicit construction of operator valued

function of the number operator $\mathbb{N} = \mathbb{A}^*\mathbb{A}$ *and nonlinear coherent states related to our operator* $\mathbb{H}_p$

In next section we verify that $[\mathbb{N}, \mathbb{H}_p^*] = \mathbb{H}_p^*$, $[\mathbb{N}, \mathbb{H}_p] = \mathbb{H}_p$ and $[\mathbb{H}_p, \mathbb{H}_p^*] = f_p(n+1) - f_p(n)$

where f_p is an entire function such that $f_p(p) = 0$ and $f_p(n) > 0$ as $n > p$.

If we consider the Fock space on which $\mathbb{H}_p$ and $\mathbb{H}_p^*$ act is $\{ | \ |n>; n \geq p \}$ whose actions are

$$\mathbb{H}_p^* |n> = \sqrt{f_p(n+1)} |n+1>, n \geq p \qquad (2.41)$$

$$\mathbb{H}_p |n> = \sqrt{f_p(n)} |n-1>, n \geq p \qquad (2.42)$$

From (2.41) states $|n>$ are given by

$$| \ n> = \frac{(\mathbb{H}_p^*)^n}{\sqrt{\prod_{j=p+1}^{n} f_p(j)}} \ | \ p> \qquad (2.43)$$

On a generalized oscillator algebra $\{\mathbb{I}, \mathbb{H}_p, \mathbb{H}_p^*, N\}$, we are interesting to study in detail the coherent states of Barut-Girardello's type but we are not concerned in this work to study

$| \ z> = e^{z\mathbb{H}_p^* - \bar{z}H_p}$ the coherent states of Perelomov's type.

By using the commutation relation $[\mathbb{A}, \mathbb{A}^*] = \mathbb{I}$ of the Boson operators $\mathbb{A}$ and $\mathbb{A}^*$, it is easy to deduce the following lemmas:

Lemma 2.11. *Let* $\mathbb{N} = \mathbb{A}^*\mathbb{A}$ *then*

i) $\mathbb{A}^*\mathbb{N}^m = (\mathbb{N} - \mathbb{I})^m \mathbb{A}^*; m = 0, 1, ...$

ii) $\mathbb{A}^*\mathbb{N}^m\mathbb{A} = (\mathbb{N} - \mathbb{I})^m \mathbb{N}; m = 0, 1, ...$

iii) $\mathbb{A}^{*m}\mathbb{N} = (\mathbb{N} - m\mathbb{I})\mathbb{A}^{*m}; m = 0, 1, ...$

iv) For $1 \leq p \leq m+1$ *we have*

$$\mathbb{A}^{*m+1}\mathbb{A}^p = \prod_{j=0}^{p-1}(\mathbb{N} - (m-j)\mathbb{I})\mathbb{A}^{*m-(p-1)}$$

v) $\mathbb{A}^{*m+1}\mathbb{A}^{m+1} = \prod_{j=0}^{m}(\mathbb{N} - jI)$

Lemma 2.12. *(Blasiak-Penson-Solomon [3]) Let* $\mathbb{A}$ *and* $\mathbb{A}^*$ *the Boson operators then we have*

i) $(A^*A)^m = \sum_{k=1}^{m} S(m, k)\mathbb{A}^{*k}\mathbb{A}^k$ *where* $S(m, k)$ *are Stirling numbers of second kind with corresponding numbers*

$B(m) = \sum_{k=1}^{m} S(m, k)$ *called Bell numbers.*

ii) $(\mathbb{A}^{*r}\mathbb{A}^s)^m = \mathbb{A}^{*m(r-s)}\sum_{k=s}^{ms} S_{r,s}(m, k)A^{*k}A^k$ *where* $S_{r,s}(m, k)$ *are generalized Stirling numbers of second kind with corresponding generalized numbers*

$$B_{r,s}(m) = \sum_{k=1}^{m} S_{r,s}(m,k) \ \text{called generalized Bell numbers.}$$

From the above lemmas we deduce the below properties

Lemma 2.13.   *Let* $\mathbb{N} = \mathbb{A}^*\mathbb{A}$ *and* $\mathbb{H}_p = \mathbb{A}^{*p}\mathbb{A}^{p+1}$ $p = 0, 1, \dots..$ *then*
i) $[\mathbb{A}, \mathbb{A}^{*p}] = p\mathbb{A}^{*p-1}$, $[\mathbb{A}^*, \mathbb{A}^p] = -p\mathbb{A}^{p-1}$, $[\mathbb{A}, \mathbb{A}^{*p}] = p\mathbb{A}^{*p}$ *and* $[\mathbb{N}, \mathbb{A}^p] = -p\mathbb{A}^p$
ii) $[\mathbb{N}, \mathbb{H}_p] = -\mathbb{H}_p$ *and* $[\mathbb{N}, \mathbb{H}_p^*] = \mathbb{H}_p^*$
iii) $[\mathbb{H}_0, \mathbb{H}_0^*] = \mathbb{I}$, $[\mathbb{H}_1, \mathbb{H}_1^*] = 3\mathbb{N}^2 - \mathbb{N}$
and

$$[\mathbb{H}_p, \mathbb{H}_p^*] = [(2p+1)\mathbb{N}^2 - p^2\mathbb{N}]\prod_{j=1}^{p-1}(\mathbb{N} - j\mathbb{I})^2$$

We end this section by recalling some spectral properties of $\mathbb{H}_p$ established in [6] under theorem form

Theorem 2.14.   *Let* $\mathbb{H}_p = z^p \dfrac{d^{p+1}}{dz^{p+1}} = \mathbb{A}^{*p}\mathbb{A}^{p+1}$ *with domain*
$D(\mathbb{H}_p) = \{\phi \in \mathbb{B}; \mathbb{H}_p\phi \in \mathbb{B}\} \bigcap \mathbb{B}_p$ *and for each positive integer* m, *the operator*
$(\mathbb{H}_p)^m$, *with domain* $D((\mathbb{H}_p)^m) = \{\phi \in \mathbb{B}; (\mathbb{H}_p)^m\phi \in \mathbb{B}\} \bigcap \mathbb{B}_p$, *where*
$\mathbb{B}_p = \{\phi \in \mathbb{B}; \frac{d^j}{dz^j}\phi(0) = 0, 0 \le j \le p\}$. *Then we have*
i) $(\mathbb{H}_p)^m$ *is a closed operator*
ii) The spectrum of $\mathbb{H}_p$ *fills the entire complex plane.*
iii) $\mathbb{H}_p$ *is hyper-cyclic operator*
iv) $\mathbb{H}_p$ *is non wandering operator*
v) $\mathbb{H}_p$ *is chaotic operator.*

In the next section, we present the aim work of this paper, which can be considered as a natural continuation of study doing in [**6**].

3   Construction the non-linear coherent states of the operator $\mathbb{H}_p = z^p \dfrac{d^{p+1}}{dz^{p+1}}$ on Bargmann space

We shall now construct nonlinear coherent states corresponding to the operator $\mathbb{H}_p$ as eigenstates of this operator for $p \ge 1$. We make the ansatz

$$| z >_p = N^{-1/2}(| z |^2) \sum_{n=p}^{\infty} \frac{z^{n-p}}{\omega_p.\omega_{p+1}.......\omega_{n-1}} | n > \qquad (3.1)$$

The constant $N(| z |^2$ is determined via the normalization of $| z >_p$

$$_p < z | z >_p = N^{-1}(| z |^2) \sum_{n=p}^{\infty} \frac{| z |^{2(n-p)}}{(\omega_p.\omega_{p+1}.......\omega_{n-1})^2} = 1 \qquad (3.2)$$

Then $N(| z |^2)$ can be expressed in terms of generalized hypergeometric functions which are defined by

$$_pF_q(\alpha_1, \alpha_2, ..., \alpha_p, \beta_1; \beta_2, ..., \beta_p; \xi) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \alpha_2)_m ......\alpha_p)_m}{(\beta_1)_m(\beta_2)_m.......}.(\beta_q)_m \frac{\xi^m}{m!} \qquad (3.3)$$

where $(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$ is the Pochhammer's symbol and $\Gamma(\alpha)$ is the usual Gauss function.

Lemma 3.1. *The normalization constant $N(| z |^2)$ is*
$N(| z |^2) = \gamma_p^2 \cdot_0 F_{2p}(\beta_1, \beta_2, ..., \beta_{2p}, | z |^2)$ *where*
$$\gamma_p = \sqrt{p!} \prod_{j=1}^{p} (p - j)! \text{ and } \beta_1 = p + 1, \beta_{p+1} = 1 \text{ and}$$
$$\beta_j = \beta_{p+j} = p + 2 - j; 2 \le j \le p$$

Proof : As $\omega_n = \sqrt{n+1} \prod_{j=0}^{p-1} (n - j)$ then

$$\omega_p.\omega_{p+1}......\omega_{n-1} = \frac{\sqrt{n!} \prod_{j=1}^{p} (n - j)!}{\sqrt{p!} \prod_{j=1}^{p} (p - j)!} \qquad (3.4)$$

If we put $\gamma_p = \sqrt{p!} \prod_{j=1}^{p} (p - j)!$ and apply the condition of the normalization we obtain

$$N^{-1}(| z |^2)\gamma_p^2 \sum_{n=p}^{\infty} \frac{| z |^{2(n-p)}}{n! \prod_{j=1}^{p} (n - j)!^2} = 1 \qquad (3.5)$$

Let $m = n - p$ and from the above equation we deduce that

$$N^{-1}(| z |^2)\gamma_p^2 \sum_{m=0}^{\infty} \frac{| z |^{2m}}{(m + p)! \prod_{j=1}^{p} (m + p - j)!^2} = 1 \qquad (3.6)$$

i.e.

$$N^{-1}(|z|^2)\gamma_{p,0}^2 F_{2p}(\beta_1, \beta_2, ..., \beta_{p-1}, |z|^2) = 1 \qquad (3.7)$$

where $\beta_1 = p+1, \beta_{p+1} = 1, \alpha_j = \beta_{p+j} = p+2-j; 2 \leq j \leq p$

Another important property, namely, the resolutions of unity can also be obtained for the non-linear coherent states via the Meijer's G-function in the case where the normalization constant is hypergeometric function.

The Meijer's G-function is defined by a Mellin-Barnes type integral

$$G_{p,q}^{m,n}(z \mid_{b_1,b_2,........,b_q}^{a_1,a_2,......,a_p}) = \frac{1}{2i\pi} \int_L \frac{\prod_{j=1}^{m}\}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds \qquad (3.8)$$

where m, n, p, q are integers with q with $q \geq 1, 0 \leq n \leq p, 0 \leq m \leq q$.

The parameters a_j and b_j are such that no pole of $\Gamma(b_j - s), j = 1, ..., m$ coincides with any pole of $\Gamma(1 - b_j + s), j = 1, ..., n$.

The poles of the integrand must be simple and those of $\Gamma(b_j - s), j = 1, ..., m$ lie on one side of the contour L and those of $\Gamma(1 - a_j + s), j = 1, ..., n$ must lie on the other side.

The connexion between the generalized hypergeometric functions and the Meijer's G-functions [17] is given by

$$_pF_q(\alpha_1, \alpha_2, ..., \alpha_p, \beta_1; \beta_2, ..., \beta_p; z)$$

$$= \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{j=1}^{p}\Gamma(\alpha_j)} G_{p,q+1}^{1,p}(-z \mid_{0;1-\beta_1,1-\beta_2,........,1-\beta_q}^{1-\alpha_1,1-\alpha_2,......,1-\alpha_p}) \qquad (3.9)$$

Now let

$$\sqrt{\rho(m)} = \sqrt{(m+p)!}\prod_{j=1}^{p}(m+p-j)! \qquad (3.10)$$

we have $\sqrt{\rho(0)} = \gamma_p$.

By choosing

$$\tilde{\rho}(m) = \frac{1}{\gamma_p^2}\rho(m) \qquad (3.11)$$

and

$$|p; z> = \tilde{N}^{-1/2}(|z|^2)\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{\tilde{\rho}(m)}} \qquad (3.12)$$

Where

$$\tilde{N}(|z|^2) = \sum_{m=0}^{\infty}\frac{|z|^{2m}}{\tilde{\rho}(m)} \qquad (3.13)$$

and following the nice article of Klauder-Penson-Sixdeniers [14] we shall show that the sequence $\tilde{\rho}(m)$ is adapted to satisfy a completeness relation for our

specific Hamiltonian. Then we have

Theorem 3.2. *There exists a probability density σ on the positive half-line which verify the power moments $\int_0^\infty x^m \sigma(x) dx = \tilde{\rho}(m); m = 0, 1, ...$ and there exists a positive measure $\mu(z, \overline{z})$ such that*
$$\int_{\mathbb{C}} d\mu(z, \overline{z}) \mid p; z >< p; z \mid = 1 \text{ where } d\mu(z, \overline{z}) = \frac{1}{2\pi} \sigma(x) \tilde{N}(x) dx d\theta \text{ with } x = \mid z \mid^2$$
and $z = \mid z \mid e^{i\theta}$

Proof : For $\mid p; z >$ the resolution of unity 1 takes the form
$$\int_{\mathbb{C}} dz d\overline{z} \mu(\mid z \mid^2) \mid p; z >< p; z \mid = 1 \tag{3.14}$$
with a positive weight function $\mu(x)$ depending on $x = \mid z \mid^2$.
Introducing the states (3.12) into Eq.(3.14) and performing the angular integration, the following conditions result
$$\int_0^\infty x^m \sigma(x) dx = \tilde{\rho}(m); m = 0, 1, ... \tag{3.15}$$
where $\sigma(x) = \dfrac{\mu(x)}{\tilde{N}(x)}$
which is given by the solution of a Stieltjes moment problem with the moments given by $\tilde{\rho}(m); m = 0, 1, ...$
As shown by Klauder and al.[**14**], the solution can be obtained by using Mellin transform techniques. Thus, replacing the discrete variable m by the complex variable (s-1), the distribution $\sigma(x)$ and the parameter function $\tilde{\rho}(s-1)$ become a Melling transform related pair. There are well known references where such pairs are tabulated [**16**] we find $\sigma(x)$ in terms of the Meijer G-function
$$\int_0^\infty x^{s-1} G_{0,2p+1}^{2p+1,0}(x \mid_{\beta_1, \beta_2, ..., \beta_{2p}, 0}) dx = \tilde{\rho}(s-1) \tag{3.16}$$
it follows that the weight function is
$$\mu(\mid z \mid^2) = G_{0,2p+1}^{2p+1,0}(\mid z \mid^2 \mid_{\beta_1, \beta_2, ..., \beta_{2p}, 0}) \sum_{m=0}^\infty \frac{\mid z \mid^{2m}}{\tilde{\rho}(m)} \tag{4.16}$$
We denote E_p the space spanned by
$$\mid p; z >= \tilde{N}^{-1/2}(\mid z \mid^2) \sum_{m=0}^\infty \frac{z^m}{\sqrt{\tilde{\rho}(m)}} \tag{3.17}$$
for each complex $z = x + iy$
The vectors $\mid p; z >$ are eigenvectors of our operator $\mathbb{H}_p$, one has
$$\mathbb{H}_p \mid p; z >= z \mid p; z > \tag{3.18}$$
This space is closely related to Bargmann's hypergeometric representation, for any $\mid \phi_p >$ we can define an entire analytic function by

$$\phi_p(z) = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\tilde{\rho}(m)}} < \phi_p \mid m > \tag{3.19}$$

The resolution of unity can be used for define a scalar product

$$< \phi_p, \psi_p >= \frac{1}{\pi} \int_{\mathbb{C}} \phi_p(z) \bar{\psi_p}(z) \lambda(\mid z \mid^2) dz d\bar{z} \tag{3.20}$$

Where $\lambda(\mid z \mid^2) = \dfrac{\mu(\mid z \mid^2)}{\tilde{N}(\mid z \mid^2)}$

The usual Bargmann representation based on the coherent states recalled in the introduction is recovered for p = 0.

The state $\mid p; z >$ have the Fock representation :

$< p; z \mid m >= \dfrac{z^m}{\sqrt{\tilde{\rho}(m)\tilde{N}(\mid z \mid^2)}}$, from which the photon number distribution

follows

$$P(m) = \frac{1}{\tilde{\rho}(m)} \frac{\mid z \mid^{2m}}{\tilde{N}(\mid z \mid^2)} \tag{3.21}$$

Comparing the state $\mid z >_p$ with the state $\mid p; z >$, it is clear that

$$\mathbb{H}_p \mid m >= 0 = \mathbb{H}_p^* \mid m > \text{ for } 0 \le m \le p-1 \text{ with } p = 1, 2, ... \tag{3.22}$$

We note that the states $\mid z >_p$ are not complete because of absence of the states $\mid 0 >, \mid 1 >, ..., \mid p-1 >$. We can, however, call this set of states the photon-added states of order p.

We conclude that the results of this work can be improved to construct non-linear coherent states associated to the operators defined in [**10**] on generalized Fock-Bargmann spaces or to operators defined in [**11**] on $(\Gamma; \chi)$-theta Fock-Bargmann spaces.

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