DYNAMICS OF FRACTIONAL ORDER CHAOTIC SYSTEM

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Abstract

This paper deals with the dynamics of chaos and synchronization for fractional order chaotic system. For fractional order derivative Captuo definition is used here and numerical simulations are done using Predictor-Correctors scheme by Diethlm based on the Adams-Baseforth-Moulton algorithm. Stability analysis is discussed here for non linear fractional order chaotic system and synchronization is achieved between two non identical fractional order chaotic systems: Finance chaotic system(driving system)and Lorenz system(response system)via active control.Numerical simulations are performed to show the effectiveness of these approaches.

Key words: Fractional calculus;Chaos;Bifurcation; Non linear finance chaotic system; Lorenz chaotic system;stability;Synchronization;Active control technique.

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1. Introduction

As the wide ranging applications of chaos and synchronization became increasingly clear over the past few decades, the research community witnessed much activity in the areas of chaos control and synchronization. Several papers have been published on chaos control and synchronization by mathematicians, engineers and physicists. Pecora and Carroll [1] first gave the new dimension to the chaos synchronization for integer order system. After that many types of synchronization such as adaptive control, active control, time delayed control, feedback control etc. [2]-[5] have been adopted in this literature. However, the motivating problems and examples in these areas have exclusively been integer order dynamical systems.

On the other hand, the last few decades witnessed a resurrection of the three hundred year old subject of fractional calculus. Using fractional calculus (FC), it is possible to depict many physical problems using mathematical models, that have been impossible within the ambit of classical calculus. In particular, the problems which exhibits the property of memory (time or location or both), for example diffusion process in non uniform medium have elegant description in terms of fractional calculus. Thus the generalization of classical calculus with the help of fractional calculus enhances the reach and power of calculus. As a result, it has been applied in control theory[6]-[7], viscoelasticity [8], diffusion [9], electromagnetism [10], signal processing [11]-[12] and bio engineering [13]-[14]. A theory on controlling a fractional order system can be thought of as a useful generalization of the traditional control theory. The same is true for synchronization of fractional order chaotic systems. In the recent years, some important research has been conducted at the interface of chaos theory and fractional calculus. Various authors have worked on the stability, chaos and control of chaos on popular fractional order chaotic systems such as the Lorentz system [15], Chen system [16], Rossler system [17], Newton-Leipnic system [18] and Liu system [19].

In this paper, our target is to analyze the synchronization phenomenon between two different fractional order chaotic systems. This abstract formulation has important applications in secure communication using chaos. We motivate the discussion by comparing the dynamics of nonlinear fractional order system with that of the corresponding integer order system. The paper has been arranged as follows. Section 2 recapitulates the basic

ideas of fractional calculus. Section 3 describes the stability analysis of chaos for integer order chaotic system while Section 4 discusses the same for fractional order chaotic system. In Section 5, the fractional order system is stabilized at its equilibrium points using feedback control. Synchronization is achieved between the fractional order non linear finance chaotic system and fractional order Lorentz system using active control method in Section 6. The range of the system parameters for the non linear fractional order finance system. Numerical simulations are carried in Section 7 to illustrate these theoretical results. The appendix summarizes the numerical algorithm used for solving fractional order differential equations.

2. Basics of Fractional calculus

The idea of "Fractional calculus" at first came to the knowledge of L-Hospital in 1695. In "Fractional calculus", the generalized operator "differenintegral" was introduced which is now denoted by $_aD_t^q$, where a and t are bounds of the operator with q as order for any real number. It is also naively written as $_aD_t^q(f) = \dfrac{d^q f(t)}{d(t-a)^q}$. However, the most used definitions of differenintegral operator are the definitions proposed by Riemann- Liouville, Grunwald-Letnikov and Caputo. Weyl, Fourier, Cauchy and Abel, among many others, had also proposed other definitions for such operators. The three most widely used definitions are as follows :

Definition 1. *The three equivalent definitions of fractional order derivatives are as follows :*

(1) **Grunwald-Letnikov definition**

$$_aD_t^q(f) = \lim_{N \to \infty} \left[\frac{t-a}{N} \right]^{-q} \sum_{j=0}^{N-1} (-1)^j (q) f \left(t - j \left[\frac{t-a}{N} \right] \right)$$

(2) **Riemann-Liouville definition**

$$_aD_t^q(f) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau, \ for \ n-1 < q < n.$$

(3) **Caputo definition**

$$_aD_t^q(f) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{q-n+1}} d\tau, \ for \ n-1 < q < n.$$

The Laplace transform of Riemann-Liouville fractional derivative is

$$\mathscr{L}\{_aD_t^q\}(f) = s^q \mathscr{L} f(t) - \sum_{k=0}^{n-1} s^k D^{(q-k-1)} f(0)$$

and that of Caputo's fractional derivative is

$$\mathscr{L}\{_aD_t^q(f)\} s^q \mathscr{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{(q-k-1)} f^{(k)}(0).$$

It is seen that the Laplace transform of fractional order derivative for Caputo's definition does not require any initial condition involving fractional order derivatives, although it is necessary for Riemann's definition. As most of the physical problems have initial conditions with integer order derivatives only, it is more convenient to use Caputo's definition in mathematical models involving fractional calculus.

Using Riemann-Liouville definition, we find that the derivative of a constant c is $\frac{ct^{-\alpha}}{\Gamma(1-\alpha)}$ where as by Caputo's definition it is zero. Thus, Caputo's defintion treats fractional differentiation as a generalization of the classical ineteger order differentiation. Thus it is more convenient to use Caputo's definition while modelling real world problems using fractional order differential equations. Hence, in this paper, we use Caputo's definition of fractional order derivatives.

3. Chaos and stability analysis of integer order chaotic system

In order to discuss the stability of a fractional order chaotic system, it is instructive to begin with the corresponding integer order system. Let us consider the non linear finance chaotic system as an example, which is given by

(1)
$$\begin{aligned}
\dot{x}_1 &= x_3 + (x_2 - a)x_1 \\
\dot{x}_2 &= 1 - bx_2 - x_1^2 \\
\dot{x}_3 &= -x_1 - cx_3
\end{aligned}$$

The system (1) exhibits chaos for two sets of parameter values $\{a = 3.6, b = 0.1, c = 1.0\}$ and $\{a = 0.00001, b = 0.1, c = 1.0\}$.

The system (1) admits three equilibrium points

$$E_1(0, \tfrac{1}{b}, 0)$$

$$E_2\left(\sqrt{\tfrac{c-b-abc}{c}}, \tfrac{1+ac}{c}, -\tfrac{1}{c}\sqrt{\tfrac{c-b-abc}{c}}\right)$$

$$E_3\left(-\sqrt{\tfrac{c-b-abc}{c}}, \tfrac{1+ac}{c}, \tfrac{1}{c}\sqrt{\tfrac{c-b-abc}{c}}\right)$$

assuming the conditions $(c - b - abc) \geq 0$ with $b, c \neq 0$ on the parameters. The Jacobian matrix for the above system of equations computed at the point $P(x_1, x_2, x_3)$ is

$$J = \begin{pmatrix} x_2 - a & x_1 & 1 \\ -2x_1 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}$$

To study the stability of the system at the equilibrium point E_i, we consider the characteristic equation of the Jacobian matrix J at E_i. We choose the above said sets of parameter values $\{a = 3.6, b = 0.1, c = 1.0\}$ and $\{a = 0.00001, b = 0.1, c = 1.0\}$ for which the system is chaotic.

3.1. **Case 1.** For the first set of parameter values $\{a = 3.6, b = 0.1, c = 1.0\}$, the equilibrium points are
$E_1(0, 10, 0)$,
$E_2(0.7348, 4.6, -0.7348)$ and $E_3(-0.7348, 4.6, 0.7348)$.

The eigenvalues at the equilibrium point $E_1(0, 10, 0)$ are $\lambda_1 = -0.1$, $\lambda_2 = -0.8623$ and $\lambda_3 = 6.2623$ which indicates that the equilibrium point E_1 is a saddle point and unstable.

At E_2 and E_3, the Jacobian matrix have the eigenvalues $\lambda_1 = -0.7123$, $\lambda_2 = 0.3062 + 1.1927i$ and $\lambda_3 = 0.3062 - 1.1927i$, implying that the equilibrium points are saddle focus and unstable.

3.2. **Case 2.** For the second set of parameter values $\{a = 0.00001, b = 0.1, c = 1.0\}$, the equilibrium points are $E_1(0, 10, 0)$, $E_2(0.94868, 1.00001, -0.94868)$ and $E_3(-0.94868, 1.00001, 0.94868)$.

The eigenvalues at the equilibrium point $E_1(0, 10, 0)$ are $\lambda_1 = 9.9083$, $\lambda_2 = -0.9083$ and $\lambda_3 = -0.1$ which indicates that the equilibrium point E_1 is a saddle point and unstable.

At E_2 and E_3, the Jacobian matrix have the eigenvalues $\lambda_1 = -0.7749$, $\lambda_2 = 0.3374 + 1.4863i$ and $\lambda_3 = 0.3374 - 1.4863i$, implying that the equilibrium points are saddle focus and unstable.

4. Chaos and stability analysis of fractional order chaotic system

Now consider the fractional order non linear finance chaotic system which is described by

$$(2) \qquad \begin{aligned} \frac{d^{q_1} x_1}{dt^{q_1}} &= x_3 + (x_2 - a)x_1 \\ \frac{d^{q_2} x_2}{dt^{q_2}} &= 1 - bx_2 - x_1^2 \\ \frac{d^{q_3} x_3}{dt^{q_3}} &= -x_1 - cx_3 \end{aligned}$$

As before same sets of parameter values $\{a = 3.6, b = 0.1, c = 1.0\}$ and $\{a = 0.00001, b = 0.1, c = 1.0\}$ are considered here for discussion and studied case wise.

The equilibrium points of the above fractional order system are same as integer order chaotic system and for this case we denote them as $E_i^*(i = 1, 2, 3)$
The stability region of the fractional order system is enhanced using lemma [21]-[23] for fractional order systems.

Lemma 2. *If the eigenvalues of the Jacobian matrix (for fractional order system)satisfy $|arg(\lambda)| \geq \pi\frac{q}{2}$, where $q = q_1 = q_2 = q_3$ then the system is asymptotically stable at the equilibrium points.*

4.1. **Case 1.** For the parameter values $a = 3.6, b = 0.1$ and $c = 1$, the equilibrium points are $E_1^*(0, 10, 0)$, $E_2^*(0.7348, 4.6, -0.7348)$ and $E_3^*(-0.7348, 4.6, 0.7348)$ same as integer order chaotic system.

Result 3. *The equilibrium point E_1^* is unstable for any $q \in (0, 1)$.*

Proof. The eigenvalues of the Jacobian matrix for the fractional order financial system at the equilibrium point E_1^* are $\lambda_1 = 6.2623, \lambda_2 = -0.8623$ and $\lambda_3 = -0.1$. Since one of the eigenvalues is positive real number, the equilibrium point E_1^* could not be stabled for any $q \in (0, 1)$. □

Result 4. *The equilibrium points E_2^* and E_3^* are stable for any $q < 0.84$.*

Proof. The eigenvalues of the Jacobian matrix for the fractional order financial system at the equilibrium points E_2^* and E_3^* are $\lambda_1 = -0.7123$, $\lambda_2 = 0.3062 + 1.1927i$ and $\lambda_3 = 0.3062 - 1.1927i$. Though the complex eigenvalues have the positive real part, according to the above Lemma the equilibrium points are controllable for

$$|arg(\lambda_1, \lambda_2, \lambda_3)| > \frac{\pi q}{2}$$
$$i.e. \ q < 0.84$$

Thus the fractional order financial system is stable at the equilibrium points E_2^* and E_3^* for any $q < 0.84$. □

4.2. **Case 2.** For the parameter values $a = 0.00001, b = 0.1$ and $c = 1$, the equilibrium points are $E_1^*(0, 10, 0)$, $E_2^*(0.9487, 1.00001, -0.9487)$ and $E_3^*(-0.9487, 1.00001, 0.9487)$

Result 5. *The equilibrium point E_1^* is unstable for any $q \in (0, 1)$.*

Proof. The eigenvalues of the Jacobian matrix for the fractional order financial system at the equilibrium point E_1^* are 9.9083,-0.9083,-0.1. Since one of the eigenvalues is positive real number, the equilibrium point E_1^* could not be stabled for any $q \in (0, 1)$. □

Result 6. *The equilibrium points E_2^* and E_3^* are stable for any $q < 0.85$.*

Proof. The eigenvalues of the Jacobian matrix for the fractional order financial system at the equilibrium points E_2^* and E_3^* are $\lambda_1 = -0.7749$, $\lambda_2 = 0.3374 + 1.4863i$ and $\lambda_3 = 0.3374 - 1.4863i$. Though the complex eigenvalues have the positive real part, according to the above Lemma the equilibrium points are controllable for

$$|arg(\lambda_1, \lambda_2, \lambda_3)| > \frac{\pi q}{2}$$
$$i.e. \ q < 0.8578$$

Thus the fractional order financial system is stable at the equilibrium points E_2^* and E_3^* for any $q < 0.85$. □

5. FEEDBACK CONTROL OF THE UNSTABLE EQUILIBRIUM POINTS

After chaos, the natural phenomena comes for controlling chaos. Here the feedback control method is adopted for the controlling chaos on fractional order finance chaotic system. The controlled fractional order finance chaotic system is described as

$$\frac{d^{q_1} x_1}{dt^{q_1}} = x_3 + (x_2 - a)x_1 - k_1(x_1 - x_1^*)$$
$$\frac{d^{q_2} x_2}{dt^{q_2}} = 1 - bx_2 - x_1^2 - k_2(x_2 - x_2^*)$$
$$\frac{d^{q_3} x_3}{dt^{q_3}} = -x_1 - cx_3 - k_3(x_3 - x_3^*)$$

where k_1, k_2 and k_3 are positive feedback control parameters.
The Jacobian matrix for the above controlled system is

$$J^* = \begin{pmatrix} x_2 - (a + k_1) & x_1 & 1 \\ -2x_1 & -(b + k_2) & 0 \\ -1 & 0 & -(c + k_3) \end{pmatrix}.$$

The discriminant of the characteristic equation of the above Jacobian matrix J^* is

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2$$

where a_1, a_2 and a_3 are given by

$$a_1 = a + b + c + k_1 + k_2 + k_3 - x_2^*$$

$$a_2 = -bx_2^* - k_2 x_2^* + ab + ak_2 + bk_1 + k_1 k_2 + 2x_1^{*2} + bc + bk_3 + ck_2 + k_2 k_3$$
$$- cx_2^* - k_3 x_2^* + ac + ak_3 + ck_1 + k_1 k_3 + 1$$

$$a_3 = (a + k_1 - x_2^*)(bc + bk_3 + ck_2 + k_2 k_3) + 2x_1^{*2}(c + k_3) + (b + k_2)$$

We choose $k_1 = k_2 = k_3 = \alpha$ and term α as the feedback control parameter.

Routh-Hurwitz criteria for fractional order system:
If $P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$ be the characteristic polynomial of the Jacobian matrix at the equilibrium point of the fractional order system and the discriminant be

$$D(P) = 18a_1 a_2 a_3 + (a_1 a_2)^2 - 4a_3 a_1^3 - 4a_2^3 - 27a_3^2$$

then the following conditions hold.[24]-[25]

  i) The necessary and sufficient condition that the corresponding equilibrium point of the fractional order system is locally asymptotically stable if $D(P) > 0$ and $a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0$.

  ii) If $D(P) < 0$, and $a_1 > 0, a_2 > 0, a_3 > 0$ the corresponding equilibrium point of the fractional order system is locally asymptotically stable for $q < \frac{2}{3}$.

  (iii) If $D(P) < 0$, and $a_1 < 0, a_2 < 0$ the corresponding equilibrium point of the fractional order system is locally asymptotically stable for $q > \frac{2}{3}$.

  (iv) If $D(P) < 0$, and $a_1 > 0, a_2 > 0, a_1 a_2 - a_3 = 0$, then the corresponding equilibrium point of the fractional order system is locally asymptotically stable for $q \in [0, 1)$.

  (v) $a_3 > 0$ is the necessary condition for the stability of the corresponding equilibrium point of the fractional order system.

5.1. **Case 1.** For the set of parameter values $a = 3.6, b = 0.1$ and $c = 1.0$, the results are discussed here.

Result 7. *For the feedback control parameter $\alpha > 10$, the equilibrium point $E_1^*(0, 10, 0)$ is stable for any fractional order q.*

Proof. If we choose the feedback control parameter $\alpha > 10$, we see $a_1 > 0, a_2 > 0, a_3 > 0$ and the discriminant $D(p)$ of the characteristic equation for the controlled financial chaotic system is also positive. Consequently the equilibrium point E_1^* can be stabilized for any fractional order q using Routh-Hurwitz criteria for fractional order system. □

Result 8. *The equilibrium points $E_2^*(0.7348, 4.6, -0.7348)$ and $E_3^*(-0.7348, 4.6, 0.7348)$ are stable for $q < \frac{2}{3}$ for feedback control parameter $\alpha > 0$.*

Proof. As the coefficients a_1, a_2 and a_3 are all positive and the discriminant $D(p)$ is negative when $\alpha > 0$, The equilibrium points E_2^* and E_3^* are controllable with $q < \frac{2}{3}$ by Routh-Hurwitz criteria. □

Result 9. *The equilibrium points E_2^* and E_3^* are stable for any $q \in (0, 1)$ with feedback control parameter $\alpha = 0.3$*
.

Proof. When $k_1 = k_2 = k_3 = \alpha = 0.3$, the discriminant $D(p)$ of the characteristic equation is negative whereas $a_1, a_2 > 0$ and $a_1 a_2 - a_3 = 0$. By Routh-Hurwitz criteria the equilibrium points E_2^* and E_3^* become stable for any $q \in (0, 1)$. □

5.2. **Case 2.** In this case we study the control analysis with the parameter values $a = 0.00001, b = 0.1, c = 1.0$

Result 10. *The equilibrium point E_1^* is stable for any fractional order q with feedback control parameter $\alpha > 11$.*

Proof. If $\alpha > 11$, the coefficients of the characteristic equation of the Jacobian matrix for the controlled finance chaotic system $a_1 > 0, a_2 > 0$ and $a_3 > 0$ and the discriminant $D(p)$ of the characteristic equation is also positive. By Routh-Hurwitz criteria, the equilibrium point E_1^* is stable for any fractional order q. □

Result 11. *The equilibrium points E_2^* and E_3^* are stable for $q < \frac{2}{3}$ for feedback control parameter $\alpha > 0$.*

Proof. The coefficients of the characteristic equation of the Jacobian matrix for the controlled finance chaotic system a_1, a_2, a_3 are all positive and the discriminant $D(p)$ is negative when $\alpha > 0$. By Routh-Hurwitz criteria, the equilibrium points E_2^* and E_3^* are stable for $q < \frac{2}{3}$. $\square$

Result 12. *The equilibrium points E_2^* and E_3^* are stable for any $q \in (0,1)$ with feedback control parameter $\alpha = 0.335$. .*

Proof. When $k_1 = k_2 = k_3 = \alpha = 0.335$, the discriminant $D(p)$ of the characteristic equation is negative whereas $a_1, a_2 > 0$ and $a_1 a_2 - a_3 = 0$. By Routh-Hurwitz criteria the equilibrium point E_2^* and E_3^* become stable for any $q \in (0,1)$. $\square$

5.3. **Numerical Investigation of the feedback control problem:** In this section, we numerically investigate several aspects of the feedback control problem which are difficult to address through theoretical computations. The two main questions that will be discussed here are :

 (i) Determination of a critical value for the control parameter α, that is, a critical value below which the feedback controller is ineffective

 (ii) Relation between the control parameter α and the order of the fractional order derivative q

We first work with **Case 1**. In figure 1, we address our question (i). It plots bifurcation diagram of the feedback controlled finance chaotic system of order $q = 0.98$. The diagram proves the existence of a critical control factor α_0 around 0.25. The figure clearly illustrates the transition to asymptotic stability as α increases past α_0 (refer to Section 5.1). Figures 2(a) and (b) present the local behaviour of the system around E_2^* in the absence and presence of feedback control. The two variable bifurcation diagram (figure 3) with respect to the feedback control parameter α and fractional order of the system q explores our question (ii). The blue region of this diagram is the set of $\alpha - q$ values for which the finance system remains stable. The red region represents the respective values for which the system approaches a high periodic orbit or becomes chaotic. The intermediate colours indicate a transition from stability towards chaos through period-doubling bifurcation. The gradient of the colour from blue towards red indicates an increase in periodicity. It is observable that for lower values of q, the system has a greater tendency to stabilize - relatively small value of the control parameter α stabilizes the system in this situation. For values of q close to 1, the system does not stabilize inspite of large values of α. This numerical computation indicates a complex relationship that exists between α and q.

Now we work with **Case 2**. Figure 4 illustrates the existence of a critical feedback control parameter α_0^* close to 0.25 (like in Case 1) and answers question (i). Thus, the change of the parameter set (a, b, c) does not have a marked effect on the critical value of the control parameter. Figures 5 and 6 illustrate the local behaviour of the finance system around the equilibrium points E_2^* and E_3^* both in presence and absence of the feedback controllers. Figure 7, the two variable bifurcation diagram between α and q, seeks the answer to (ii). As in the previous discussion for case 1, here also blue regions indicate stability, red and dark red regions indicate chaotic and high periodic behaviour while the light blue, green, yellow and orange indicate a increase in periodicity (the colours have been named in the increasing order of periodicity). This figure indicates a marked change from what was observed for Case 1. Here we restricted α between 0 and 0.1 while q was larger than 0.8. It was observed that whatever be the value of q, depending on the value of α, the stability behaviour keeps switching. As α is varying in a range that lies below the critical coupling α_0^*, one has no reason to expect stable regions. But then, the numerical investigation suggests the existence of certain very narrow windows for the values of α which stabilizes the system inspite of lying below the critical value. We believe that the existence of these windows is reminiscent of the stable windows arising in the bifurcation diagram of chaotic systems and implies a complex dependence of the system dynamics on α and q.

6. SYNCHRONIZATION OF FRACTIONAL ORDER CHAOTIC SYSTEMS

The classical synchronization problem considered by Pecorra and Carroll [1] involved two identical systems which when coupled through a weak, diffusive coupling, ended up being in perfect synchrony although the original systems might have been chaotic themselves. This spontaneous appearance of apparent order in a pair of chaotic systems arising out of the simple diffusive coupling is perhaps the most fascinating aspect of chaos synchronization.

6.1. **Synchronization of structurally equivalent systems with parameter mismatch:** Let us consider a pair of structurally equivalent systems (X, Y) with parameter mismatch. In the following, we will denote the original parameter by $\theta \in \mathbb{R}^n$ and the mismatched parameter by $\hat{\theta} \in \mathbb{R}^n$. The pair of structurally equivalent systems with parameter mismatch can then be represented by

$$D^q X = f(X) + g(X)\theta$$

(3)

$$D^q Y = f(Y) + g(Y)\hat{\theta}$$

where $D^q \equiv \begin{pmatrix} D^{q_1} \\ D^{q_2} \\ \vdots \\ D^{q_n} \end{pmatrix}$ and $D \equiv \frac{d}{dt}$.

We consider the fractional order nonlinear finance system as

$$\begin{pmatrix} D^{q_1} x_1 \\ D^{q_2} x_2 \\ D^{q_3} x_3 \end{pmatrix} = \begin{pmatrix} x_3 + x_2 x_1 \\ 1 - x_1{}^2 \\ -x_1 \end{pmatrix} + \begin{pmatrix} -x_1 & 0 & 0 \\ 0 & -x_2 & 0 \\ 0 & 0 & -x_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

that is,

$$D^q X = f(X) + g(X)\theta,$$

which is the master system and the corresponding slave system is

$$D^q Y = f(Y) + g(Y)\hat{\theta}.$$

Here $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $f(X) = \begin{pmatrix} x_3 + x_2 x_1 \\ 1 - x_1{}^2 \\ -x_1 \end{pmatrix}, g(X) = \begin{pmatrix} -x_1 & 0 & 0 \\ 0 & -x_2 & 0 \\ 0 & 0 & -x_3 \end{pmatrix}$, $\theta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \hat{\theta} = \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}$.

We will design a suitable control input to bring about a synchronization of the pair (X, Y) with X as the master and Y as the slave system. In order to achieve this, we introduce a external force input $U(X, Y)$ to the slave system.

Theorem 13. *For the control input*

$$U(X, Y) = g(Y)\theta - g(X)\hat{\theta} - k(Y - X),$$

there exists a diffusive coupling factor $k > k_0 = \max\{2M - (a + \hat{a}), M - (b + \hat{b})\}$ such that the systems X and Y attain globally stable synchronization.

Proof. Let us define the synchronization error as $e = Y - X$. Then, with the form of $U(X, Y)$ as in the theorem, the error dynamics of the coupled system becomes

$$\begin{pmatrix} D^{q_1} e_1 \\ D^{q_2} e_2 \\ D^{q_3} e_3 \end{pmatrix} = (f(Y) - f(X)) + (g(Y) - g(X))(\theta + \hat{\theta}) - k(Y - X)$$

$$= \begin{pmatrix} y_2 - (a + \hat{a} + k) & x_1 & 1 \\ -(x_1 + y_1) & -(b + \hat{b} + k) & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$= Q \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

where $Q = \begin{pmatrix} y_2 - (a + \hat{a} + k) & x_1 & 1 \\ -(x_1 + y_1) & -(b + \hat{b} + k) & 0 \\ -1 & 0 & -(c + \hat{c} + k) \end{pmatrix}$

If λ is an eigenvalue of the matrix Q and the eigenvector corresponding to this λ is $\omega = (\omega_1, \omega_2, \omega_3)^T$, then

$$Q\omega = \lambda\omega$$

It is clear that the eigenvalues are functions of time as the matrix Q is a function of the state variables. We emphasize this by rewriting the above equation as,

(4) $$Q(t)\omega(t) = \lambda(t)\omega(t)$$

Let Q^H denote the conjugate transpose of the matrix Q. Then, we have,

(5) $$\omega(t)^H \frac{Q(t) + Q(t)^H}{2} \omega(t) = \frac{1}{2}(\lambda(t) + \overline{\lambda(t)})(\omega(t)^H \omega(t))$$

We know that the chaotic attractor is a compact set. So, any continuous function defined on the chaotic attractor will be bounded. Restricting our discussion to the chaotic trajectories of the finance system, we can thus find a constant $M \in \mathbb{R}$ such that $||X(t)||_\infty \leq M$ and $||Y(t)||_\infty \leq M$ for all $t \in \mathbb{R}$. Here, we use the notation $||X||_\infty$ for the uniform norm in $\mathbb{R}^l$, that is, $||X||_\infty = \max_{1 \leq i \leq l} X_i$.

Thus we can show that

$$
\begin{aligned}
\omega(t)^H &\frac{Q(t) + Q(t)^H}{2} \omega(t) \\
&= [\omega_1, \ \omega_2, \ \omega_3] \begin{pmatrix} y_2 - (a + \hat{a} + k) & -\frac{y_1}{2} & 0 \\ -\frac{y_1}{2} & -(b + \hat{b} + k) & 0 \\ 0 & 0 & -(c + \hat{c} + k) \end{pmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \\
&\leq \ [|\omega_1| \ , \ |\omega_2| \ , \ |\omega_3|] \ L \ \begin{bmatrix} |\omega_1| \\ |\omega_2| \\ |\omega_3| \end{bmatrix}
\end{aligned} \tag{6}
$$

where

$$
L = \begin{pmatrix} M - (a + \hat{a} + k) & -\frac{M}{2} & 0 \\ -\frac{M}{2} & -(b + \hat{b} + k) & 0 \\ 0 & 0 & -(c + \hat{c} + k) \end{pmatrix}.
$$

L is negative definite if and only if,

$$
\begin{aligned}
M - (a + \hat{a}) &< k \\
(k + a + \hat{a})(k + b + \hat{b}) - M(k + b + \hat{b}) - \frac{M^2}{4} &> 0 \\
\det(L) &< 0
\end{aligned} \tag{7}
$$

If we choose the diffusive coupling factor $k > k_0$, then k must satisfy the conditions in (7) and the matrix L becomes negative definite and in fact, $L \leq -\varepsilon I$ for some positive real constant ε.

Thus, from the relation (5) that

$$
\begin{aligned}
\frac{\lambda(t) + \overline{\lambda(t)}}{2} &= \frac{\omega(t)^H \left(\frac{Q(t) + Q(t)^H}{2} \right) \omega(t)}{\omega(t)^H \omega(t)} \\
&\leq \frac{1}{\omega^H \omega} [|\omega_1^H|, |\omega_2^H|, |\omega_3^H|] L [|\omega_1||\omega_2|, |\omega_3|]^T \\
&\leq -\varepsilon \frac{\omega(t)^H \omega(t)}{\omega(t)^H \omega(t)} = -\varepsilon
\end{aligned} \tag{8}
$$

Thus, the real parts of the the eigenvalues of the matrix Q are negative whenever the diffusive coupling factor $k > k_0$. Thus, the theorem is proved and we get the above suitable conditions on the diffusive matrix K such that the two finance chaotic system becomes synchronized.

$$\square$$

In particular, while analyzing our results numerically, we choose the mismatched parameters as the two parameters sets we have previously used in this paper, namely $\theta = (a, b, c)^T = (3.6, .1, 1)^T$ and $\hat{\theta} = (\hat{a}, \hat{b}, \hat{c})^T = (.00001, .1, 1)^T$.

6.2. **Synchronization between structurally inequivalent systems:** Synchronization between two different fractional order chaotic system is important mainly because of its applications in secure communication. In this paper, we consider the synchronization scenario where the fractional order Finance

system is the driving system and Lorentz system acts as a response system. Active control inputs have been designed to attain the desired synchronization. The fractional order driving system is

(9)
$$\frac{d^{q_1} x_1}{dt^{q_1}} = z_1 + (y_1 - a)x_1$$
$$\frac{d^{q_2} y_1}{dt^{q_2}} = 1 - by_1 - x_1^2$$
$$\frac{d^{q_3} z_1}{dt^{q_3}} = -x_1 - cz_1$$

and the fractional response system is,

(10)
$$\frac{d^{q_1} x_2}{dt^{q_1}} = \sigma(y_2 - x_2) + u_1$$
$$\frac{d^{q_2} y_2}{dt^{q_2}} = x_2(\rho - z_2) - y_2 + u_2$$
$$\frac{d^{q_3} z_2}{dt^{q_3}} = x_2 y_2 - \beta z_2 + u_3$$

where $U = (u_1, u_2, u_3)^T$ is the active control input.

Result 14. *The active controller*
$$u_1 = (\sigma - a)x_1 - \sigma y_2 + x_1 y_1 + z_1$$
$$u_2 = 1 - x_1^2 - (b-1)y_1 - (\rho - z_2)x_2$$
$$u_3 = (\beta - c)z_1 - (x_1 + x_2 y_2)$$

leads to globally stable synchronization between the driving system (9) *and the response system* (10).

Proof. Let us define the error states as $e_1 = x_2 - x_1, e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$, so that the error dynamics becomes as

(11)
$$\frac{d^{q_1} e_1}{dt^{q_1}} = -\sigma e_1 + (a - \sigma)x_1 + \sigma y_2 - x_1 y_1 - z_1 + u_1$$
$$\frac{d^{q_2} e_2}{dt^{q_2}} = -e_2 + (b-1)y_1 + x_2(\rho - z_2) + x_1^2 - 1 + u_2$$
$$\frac{d^{q_3} e_3}{dt^{q_3}} = -\beta e_3 + (c - \beta)z_1 + x_2 y_2 + x_1 + u_3$$

With the active controller U as in (11), the error dynamics are governed by

(12)
$$\frac{d^{q_1} e_1}{dt^{q_1}} = -\sigma e_1$$
$$\frac{d^{q_2} e_2}{dt^{q_2}} = -e_2$$
$$\frac{d^{q_3} e_3}{dt^{q_3}} = -\beta e_3$$

and the new error system becomes as negative definite linear system. It is easy to observe that in this linear system, the eigenvalues are negatives and consequently the error system is globally asymptotically stable at the origin [20]. Thus the synchronization is achieved between two different fractional order chaotic systems.

□

7. NUMERICAL SIMULATION

The projection of the phase portrait of the fractional order ($q = 0.98$) finance chaotic system for the parameter set $a = 3.6, b = 0.1, c = 1$ on x-y plane is shown in figure 8. In figure 9(a)-9(d) we present the Poincare sections of the finance chaotic system by the fixed plane $x_3 = 4$ for different fractional orders. It shows that the system becomes chaotic as the order of the system increases towards 1. Figure 10 illustrates sensitive dependence of the finance chaotic system on the initial conditions. It plots the
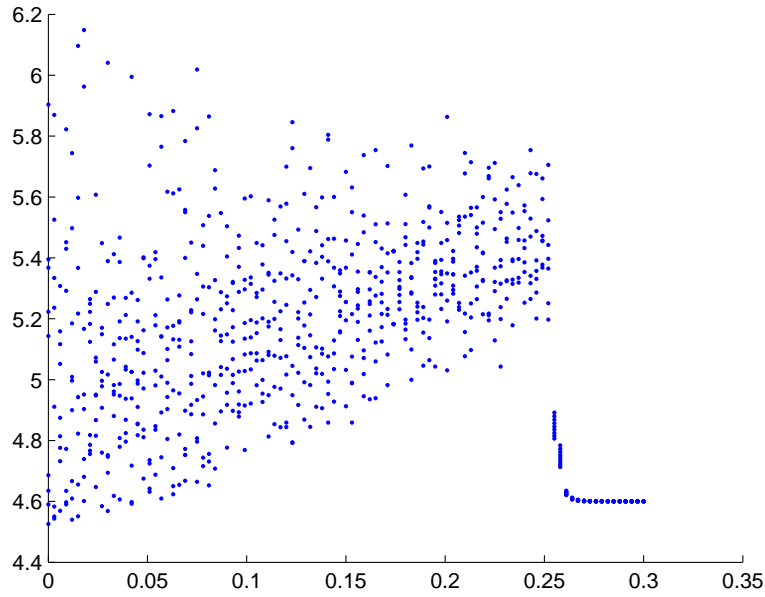
FIGURE 1. Bifurcation diagram of controlled finance system with order $q = 0.98$
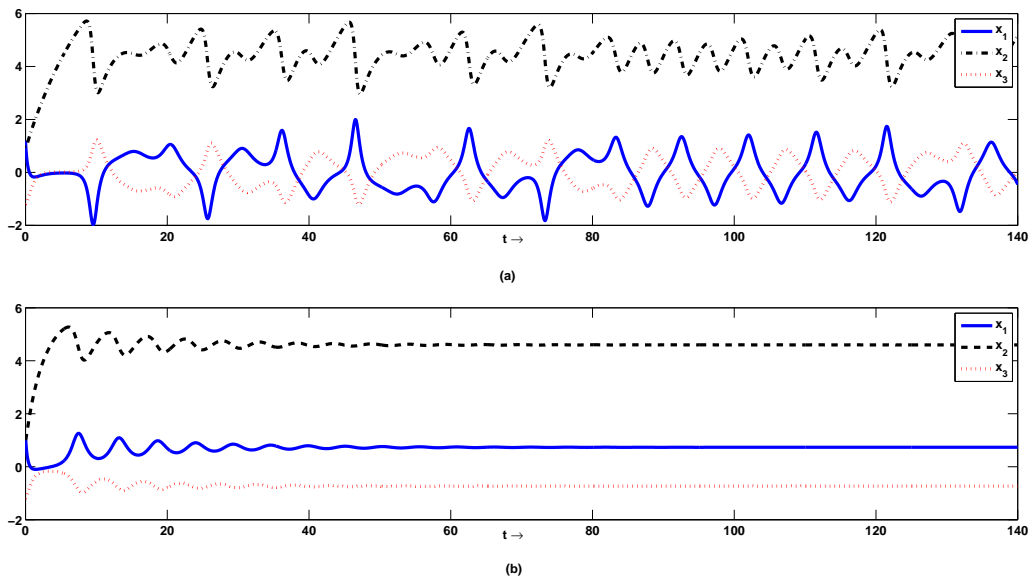
FIGURE 2. Behaviour of the system around E_2^* in the absence and presence of feedback control

distance between two points which were at a distance of 10^{-7} at $t = 0$ and are then allowed to evolve on the attractor. Figure 11 plots the bifurcation diagram of finance chaotic system of order $q = 0.98$ with c as the bifurcation parameter. From the figure, it is observed that there exists a critical value of c above which the system becomes highly periodic and possibly chaotic. For values below this value of c, the system remains stable. Similar behaviour is observed for the other two parameters a and b is increased, it becomes chaotic after a certain value. Figure 12 shows the globally stable synchronization of two fractional order finance chaotic system with different set of parameters for a sufficiently large value of the coupling factor k. The bound M needs to be guessed by letting the fractional order finance system evolve on the chaotic attractor for a long enough period. Figure 13 shows the synchronization between two structurally inequivalent fractional order systems: the finance chaotic system and Lorenz system. The attractor on the Lorentz system is deformed to the attractor of the finance chaotic system in the process.
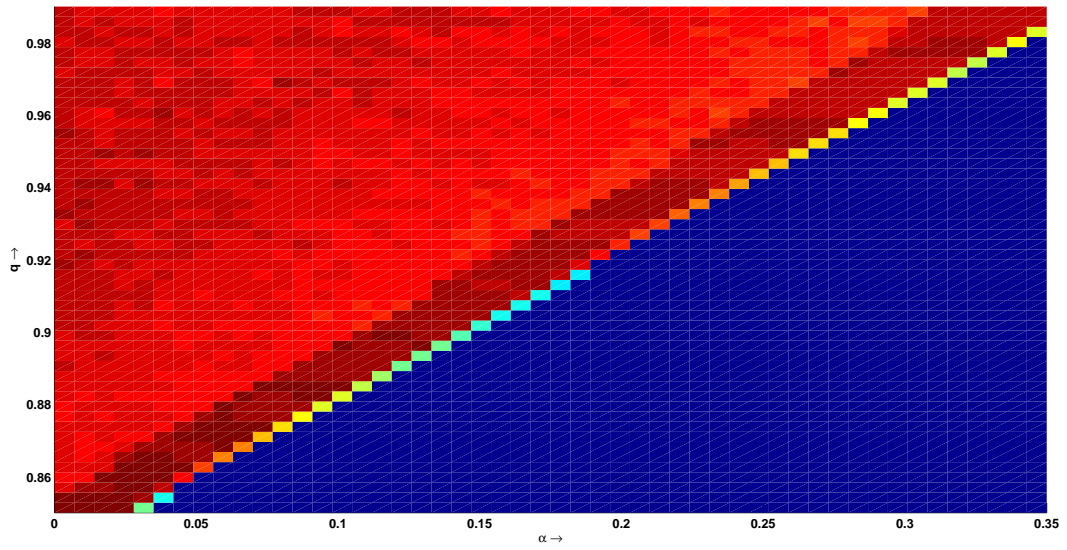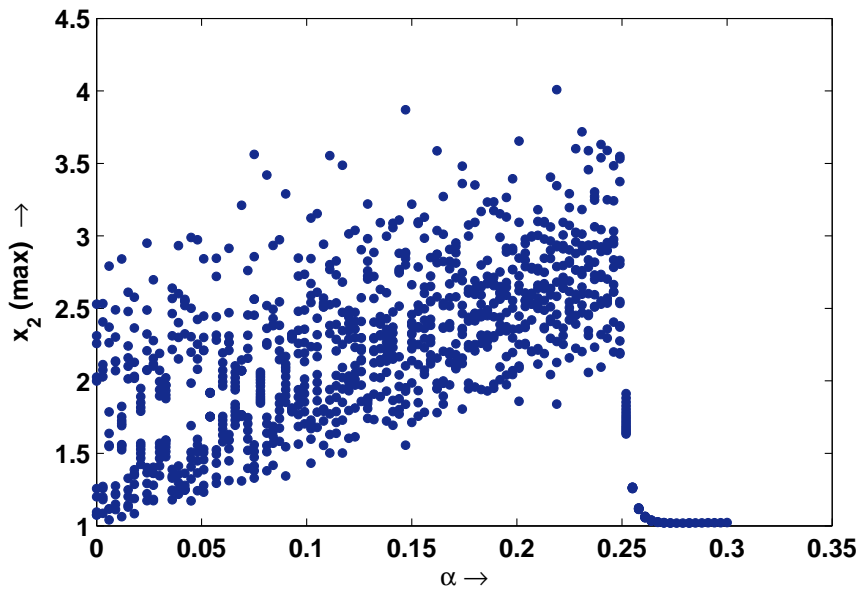
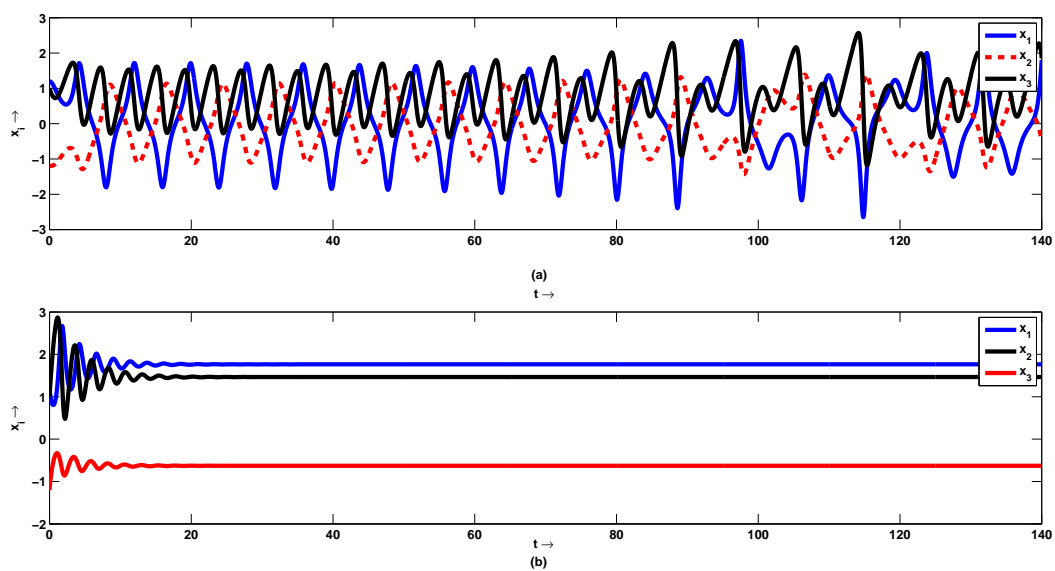FIGURE 3. Two variable bifurcation diagram between α and q

FIGURE 4. Bifurcation diagram of controlled finance system with parameter values of case 2

Appendix

APPENDIX A. NUMERICAL APPROACH FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATION

To solve fractional order differential equations numerically, the algorithm proposed by Diethlm [26] is considered here which is modified method of Adams-Bashforth-Moulton algorithm. The method is described as follows: Consider the following fractional-order differential equation:

$$(13) \qquad D^q y(x) = f(t, y(t)), 0 \leqslant t \leqslant T$$
$$y^{(k)}(0) = y_0^{(k)}, \ k = 0, 1, ...m - 1, m = [q]$$
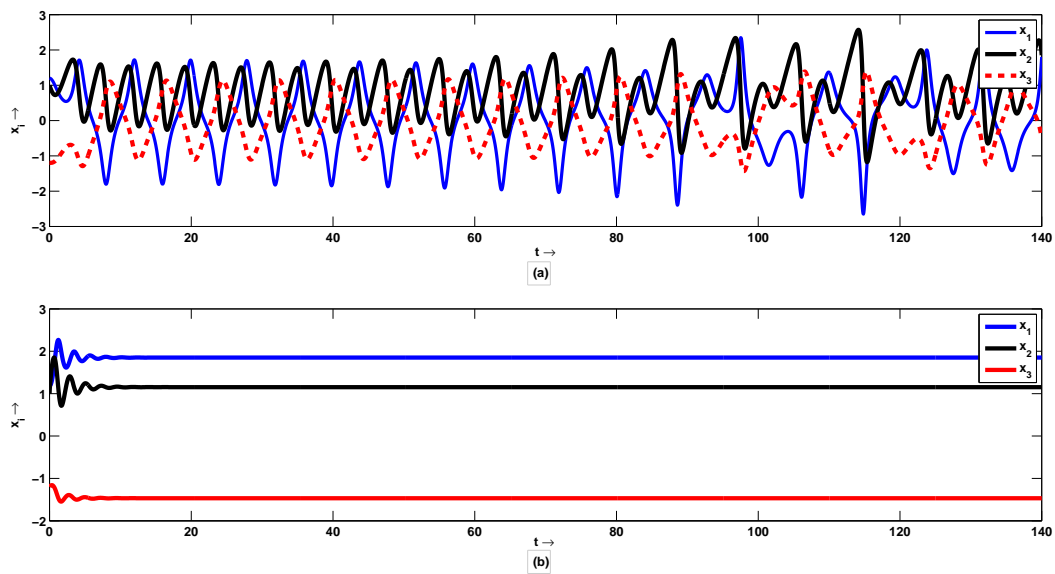
FIGURE 5. Behaviour of the system around E_2^* without and with feedback control

FIGURE 6. Behaviour of the system around E_3^* in the absence and presence of feedback control

It is equivalent to the Volterra integral equation

$$(14) \qquad y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds$$
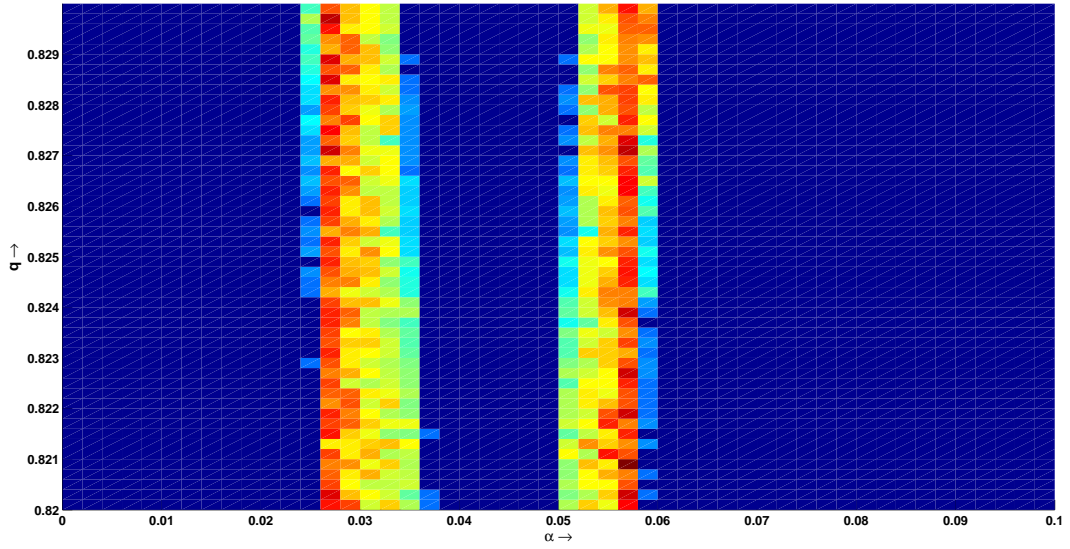
FIGURE 7. Two variable bifurcation diagram between α and q
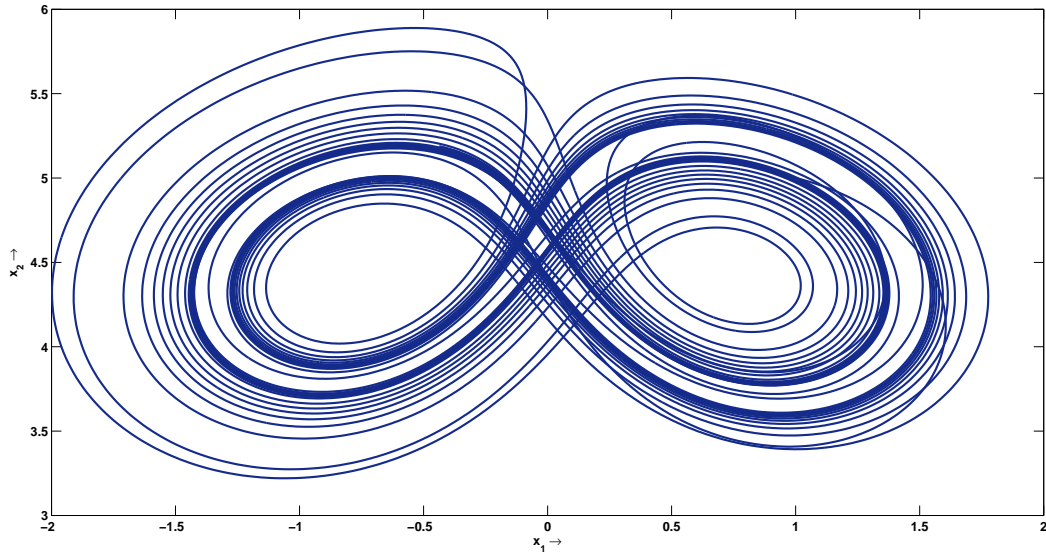
FIGURE 8. The phase portrait of the fractional order($q = 0.98$)finance chaotic system for the parameter set $a = 3.6, b = 0.1, c = 1$

Taking $h = \frac{T}{N}$ and $t_n = nh, n = 0, 1, 2...., N$, the system (13) can be discretized as

$$x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-ax_{n+1}^p - ey_{n+1}^p z_{n+1}^p) + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^{n} \alpha_{1,j,n+1}(-ax_j - ey_j z_j)$$

$$y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}(by_{n+1}^p + kx_{n+1}^p z_{n+1}^p) + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^{n} \alpha_{2,j,n+1}(by_j + kx_j z_j)$$

$$z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}(-cz_{n+1}^p - mx_{n+1}^p z_{n+1}^p) + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^{n} \alpha_{3,j,n+1}(-cz_j - mx_j y_j)$$
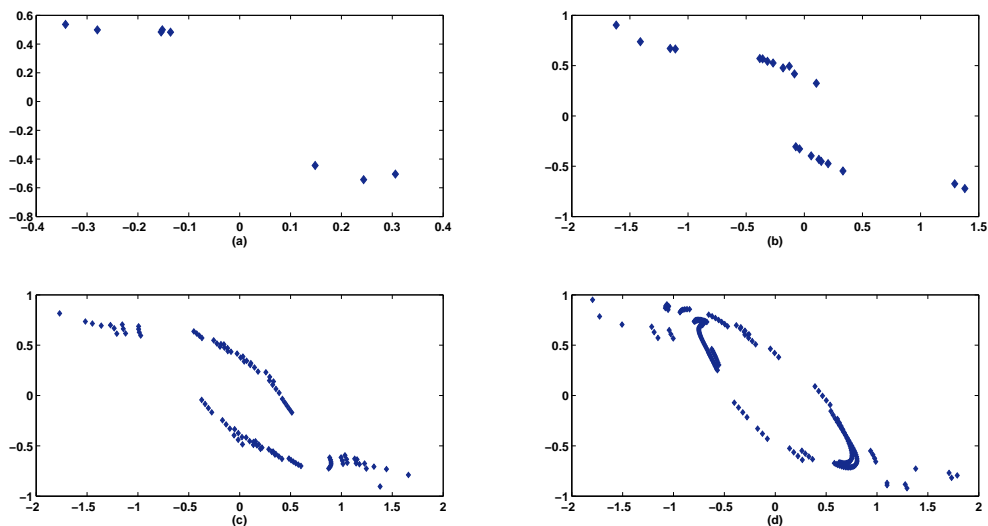
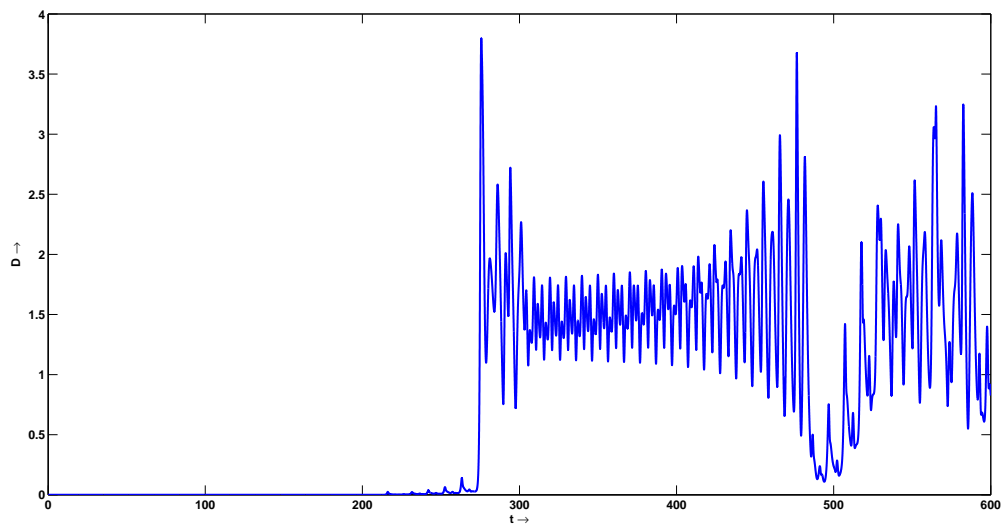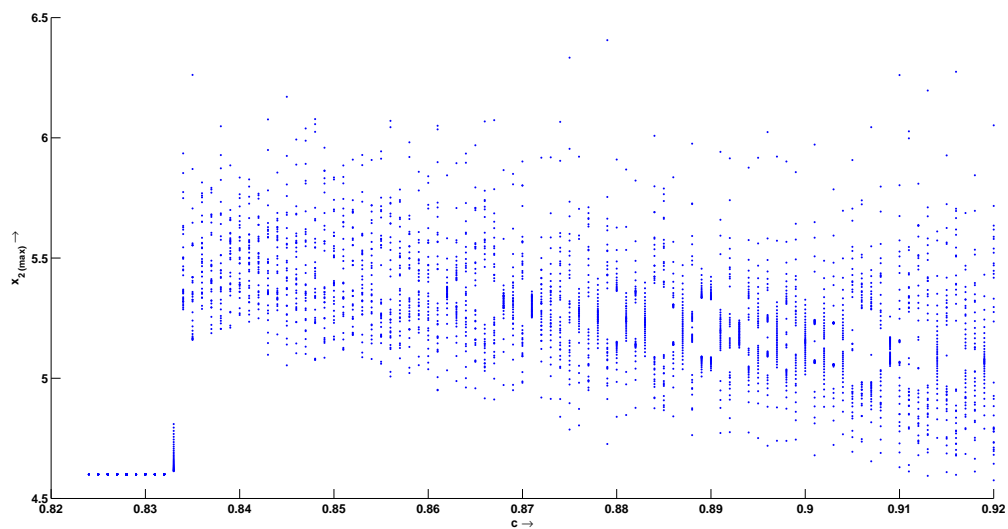FIGURE 9. Poincare sections of the Finance chaotic system

FIGURE 10. Chaotic nature presented by the distance between two points of the chaotic system

FIGURE 11. Bifurcation diagram of the finance chaotic system of order $q = 0.98$ with parameter c
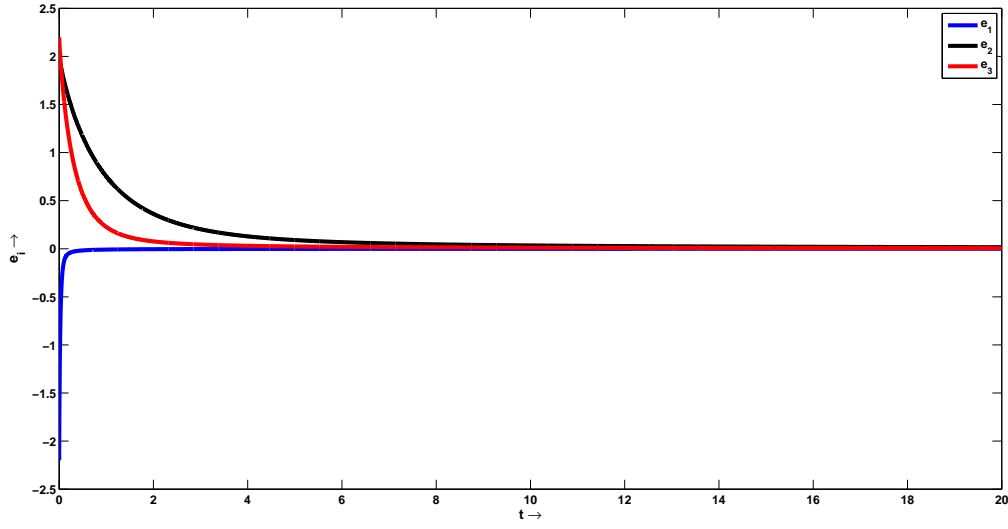
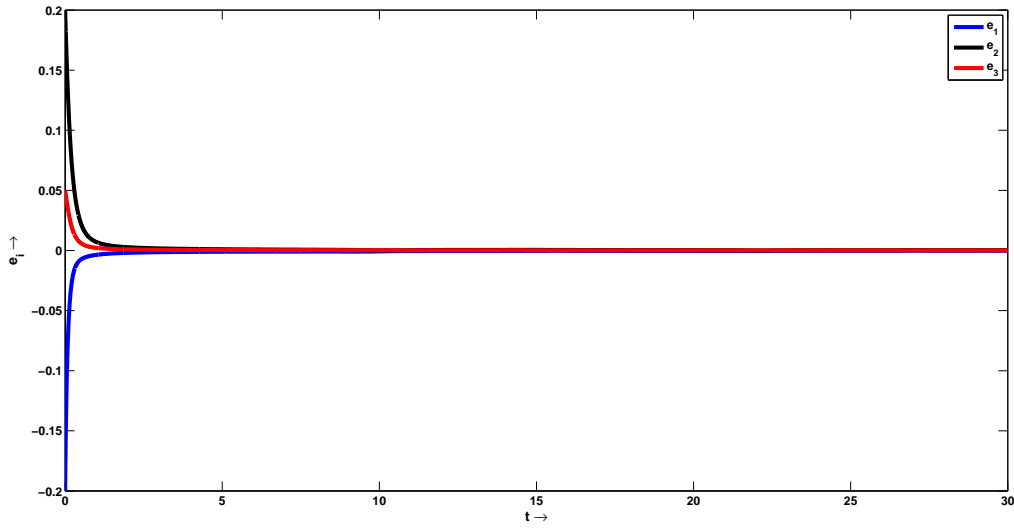FIGURE 12. Synchronization between two fractional order finance chaotic systems with different set of parameters

FIGURE 13. Synchronization between the Finance chaotic system and Lorentz system

where

$$x^p_{n+1} = x_0 + \frac{1}{\Gamma q_1} \sum_{j=0}^{n} \beta_{1,j,n+1}(-ax_j - ey_j z_j)$$

$$y^p_{n+1} = y_0 + \frac{1}{\Gamma q_2} \sum_{j=0}^{n} \beta_{2,j,n+1}(by_j + kx_j z_j)$$

$$z^p_{n+1} = z_0 + \frac{1}{\Gamma q_3} \sum_{j=0}^{n} \beta_{3,j,n+1}(-cz_j - mx_j y_j)$$

and

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n - n^{q_i})(n + 1^{q_i}), & \text{j=0;} \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, & 1 \le j \le n; \\ 1, & \text{j=n+1} \end{cases}$$

$$\beta_{i,j,n+1} = \left\{ \begin{array}{l} \frac{h^{q_i}}{q_i}(n-j+2)^{q_i} + (n-j)^{q_i}, \quad 0 \le j \le n. \end{array} \right.$$

The error estimate of the above scheme is $\max_{j=0,1,\ldots,N}\{|y(t_j - y_h(t_j)| = (h^p)$, in which $p = \min(2, 1 + q_i)$ and $q_i > 0, i = 1, 2, 3$.

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