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Dislocated Negative Feedback Control with Partial Replacement between Chaotic Lorenz Systems

Rosário D. Laureano

ISCTE-IUL Lisbon University Institute
Department of Mathematics, ISTA-School of Technology and Architecture
1649-026 Lisbon, PORTUGAL
maria.laureano@iscte.pt

Diana A. Mendes

ISCTE-IUL Lisbon University Institute
Department of Mathematics, ISTA-School of Technology and Architecture
1649-026 Lisbon, PORTUGAL
diana.mendes@iscte.pt

Manuel Alberto M. Ferreira

ISCTE-IUL Lisbon University Institute
Department of Mathematics, ISTA-School of Technology and Architecture
1649-026 Lisbon, PORTUGAL
manuel.ferreira@iscte.pt

Abstract

In order to obtain asymptotical synchronization, we combine negative feedback control and dislocated negative feedback control with partial replacement to the nonlinear terms of the response system, a coupling version that was less explored. All these unidirectional coupling schemes are applied between Lorenz systems where we consider some values for the control parameters that lead to chaotic behavior.

The sufficient conditions for global stable synchronization are obtained from a different approach of the Lyapunov direct method for the transversal system. In one of the coupling we apply a result based on the classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite, where $\mathbf{A}$ is characterizing the transversal system. In the other couplings presented here, the sufficient conditions are based on the derivative increase of an appropriate Lyapunov function. In fact, the effectiveness of the coupling between systems with equal dimension follows from the analysis of the synchronization error, $\mathbf{e}(t)$, and, if the system variables can be bounded by positive constants, then the derivative of an appropriate Lyapunov function can be increased as required by the Lyapunov direct method.

Mathematics Subject Classification: 34D06, 34D23, 34C60, 34H10

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1 Introduction

The ability of nonlinear oscillators to synchronize with each other is one of the basis for the explanation of several natural processes. Therefore, chaos synchronization is a robust property expected to hold in mademan devices and plays a quite significant role in science. However, the hypotesis that two or more chaotic systems can evolve in a coherent and synchronized way it is not an obvious phenomenon, since it is impossible exactly reproduce the initial conditions and any infinitesimal perturbations that lead to divergence of nearby starting orbits. Contrary to expectation, when ensembles of chaotic oscillators are coupled, the attractive effect of a suitable coupling can counterbalance the trend of the trajectories to diverge. In many cases there are parameters that control the strength of coupling between the systems, and the stability results from the synchronous chaotic state depending on them.

Coupled dynamical systems are constructed from simple low-dimensional dynamical systems and form new and more complex processes. The chaotic dynamics introduces new degrees of freedom in ensembles of coupled systems. However, when two or more chaotic oscillators are coupled and synchronization is achieved, in general the number of dynamic degrees of freedom for the coupled system effectively decreases.

In what follows we will always consider two chaotic dynamical systems, since they are sufficient to study the essential in the proposed coupling schemes.

Asymptotical Synchronization. Let X be a compact subset of $\mathbb{R}^m$ with $m \geq 3$ and consider two identical m-dimensional dynamical systems, S_1 and S_2, defined on X by the nonlinear autonomous ordinary differential equations $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$ and $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$, respectively, where $\mathbf{a}$ is a vector of real control parameters.

Let $\mathbf{u}_1(0)$ and $\mathbf{u}_2(0)$ be initial conditions for which, at certain value of $\mathbf{a}$, S_1 and S_2 evolve to an asymptotically stable chaotic attractor $\mathcal{A}$. The solutions $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ of the systems, starting at $\mathbf{u}_1(0) \neq \mathbf{u}_2(0)$ in the attraction basin $\mathcal{B}(\mathcal{A})$, are independent trajectories in $\mathcal{A}$ after some transient period of time. This evolution is characterized by a positive Lyapunov exponent.

Definition 1.1 *The dynamical systems S_1 and S_2 are asymptotically synchronized if*

$$\lim_{t\to+\infty} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| = 0. \tag{1}$$

The evolution of the difference $\mathbf{e}(t) = \mathbf{u}_2(t) - \mathbf{u}_1(t)$ between nearby starting orbits is described by

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{u}}_2(t) - \dot{\mathbf{u}}_1(t) = \mathbf{f}(\mathbf{u}_2(t);\mathbf{a}) - \mathbf{f}(\mathbf{u}_1(t);\mathbf{a}). \tag{2}$$

In the case of asymptotical synchronization, this difference gives the synchronization error and the system (2) is called a transversal system (or error system). From (1), we have that S_1 and S_2 achieve asymptotical synchronization if the transversal system (2) has an asymptotically stable equilibrium point at $\mathbf{e}(t) = \mathbf{0}$.

When asymptotical synchronization is achieved, the dynamics of $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ in $\mathcal{A}$, on the $2m$-dimensional phase space, are restricted to the m-dimensional smooth invariant manifold

$$\mathcal{M} \equiv \{(\mathbf{u}_1, \mathbf{u}_2) \in X \times X \mid \mathbf{u}_1 = \mathbf{u}_2\} \subset \mathbb{R}^{2m}, \tag{3}$$

where occurs the synchronized dynamics defined by the symmetric synchronous chaotic state.

Transversal Stability of the Coupled System. Roughly speaking the synchronization between two systems can be understood as a problem of asymptotical stability of the associated chaotic attractor, $\mathcal{A}$, in the $2m$-dimensional phase space of the coupled system [8], [6], [4].

It is important to distinguish between stability under tangent or under transversal perturbations of the synchronization manifold $\mathcal{M}$. As stated by Pecora *et al.*[5], the limit (1) must be satisfied for all initial conditions in a neighborhood of the equilibrium point $\mathbf{e}(t) = \mathbf{0}$. Since the system (2) characterizes the dynamics in the transversal direction to $\mathcal{M}$, it is necessary to analyze if small transversal perturbations to $\mathcal{M}$ are reduced or amplified by the evolution of S_1 and S_2. If they are reduced, then $\mathcal{M}$ is transversaly stable and the synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ it is also stable. So, in this case, the synchronization stability is designated as transversal stability.

Usually the following stability criteria are applied:

(i) Criterion based on the eigenvalues of the Jacobian matrix corresponding to the flow over $\mathcal{M}$, suggested by Fujisaka and Yamada [8],[7]; it requires that the largest eigenvalue should be negative in order to obtain early stable synchronization;

(ii) Criterion based on the construction and study of an appropriate Lyapunov function, $L(\mathbf{e}(t))$, for the vector field of transversal perturbations to $\mathcal{M}$, developed by He and Vaidya [2]; it requires that L must be positive definite in a neighborhood of $\mathcal{M}$, $L(\mathbf{e}(t)) \geq 0$, except in $\mathcal{M}$ where $L(\mathbf{0}) = 0$, and its derivative must be negative semidefinite, $\dot{L}(\mathbf{e}(t)) \leq 0$, and equal to 0 in $\mathcal{M}$;

(iii) Criterion based on the estimation of Lyapunov exponents, developed by Pecora and Carroll [4]; it is important to analyze if small transversal perturbations decrease or not and it requires that the largest transversal Lyapunov exponent should be negative.

The criterion *(ii)* allows us to prove a proposition about global asymptotical stability of the transversal system defined by (2).

Theorem 1.2 *Let A be the matrix characterizing the transversal system of a coupling between identical systems S_1 and S_2. If there is a constant $\delta < 0$ such that the symmetric matrix $A^T + A$ is negative definite and*

$$\mathbf{A^T} + \mathbf{A} \leq \delta \mathbf{I}$$

for any u_1 and u_2 in the phase space X, then the dynamics of the transversal system is globally stable and the systems S_1 and S_2 are in stable synchronization.

proof Consider the Lyapunov function defined by $L(\mathbf{e}(t)) = [\mathbf{e}(t)]^T \cdot \mathbf{e}(t)$. Its derivative is given by

$$\frac{dL}{dt}(\mathbf{e}) = \frac{d\left(\mathbf{e}^T\right)}{dt} \cdot \mathbf{e} + \mathbf{e}^T \cdot \frac{d\mathbf{e}}{dt} = \mathbf{e}^T \cdot \mathbf{A}^T \cdot \mathbf{e} + \mathbf{e}^T \cdot \mathbf{A} \cdot \mathbf{e}, \tag{4}$$

and verifies

$$\dot{L}(\mathbf{e}) = \mathbf{e}^T \left(\mathbf{A}^T + \mathbf{A}\right) \mathbf{e} \leq \delta \left(\mathbf{e}^T \cdot \mathbf{I} \cdot \mathbf{e}\right) = \delta \left(\mathbf{e}^T \cdot \mathbf{e}\right) < 0 \tag{5}$$

for all $\mathbf{e} \neq \mathbf{0}$. The Lyapunov direct method guaranties the global asymptotical stability of the transversal system

2 Unidirectional Coupling Schemes between Continuous Chaotic Dynamical Systems

Coupling by Partial Replacement. Consider a decomposition $\mathbf{u}_1 = (\mathbf{x}_1, \mathbf{y}_1)$ of the system $\mathbf{u}_1$ into two subsystems, that is,

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \wedge \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}), \tag{6}$$

with variables $\mathbf{x}_1 = (u_1, \ldots, u_k)$ and $\mathbf{y}_1 = (u_{k+1}, \ldots, u_m)$, respectively, for $1 \leq k \leq m$. Since $\mathbf{f}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_m(\mathbf{u}_1; \mathbf{a}))$, the vector fields $\mathbf{g}$ and $\mathbf{h}$ are defined by the component functions of the vector field $\mathbf{f}$ as

$$\mathbf{g}(\mathbf{u}_1; \mathbf{a}) = (f_1(\mathbf{u}_1; \mathbf{a}), \ldots, f_k(\mathbf{u}_1; \mathbf{a})) \tag{7}$$

and

$$\mathbf{h}(\mathbf{u}_1; \mathbf{a}) = (f_{k+1}(\mathbf{u}_1; \mathbf{a}), \ldots, f_m(\mathbf{u}_1; \mathbf{a})). \tag{8}$$

We take independent initial conditions $\mathbf{x}_1(0)$ and $\mathbf{y}_1(0)$ in the subsystems defined in equation (6). Let $\dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$ be a subsystem identical to $\dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ where the variable $\mathbf{x}_1$ it is replaced by $\mathbf{x}_2$, that is, $\mathbf{x}_2 = \mathbf{x}_1$ and

$$\dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}).$$

So, the equations

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \wedge \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}), \tag{9}$$

with $\mathbf{y}_2(0) \neq \mathbf{y}_1(0)$, will define a dynamical system $\dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a})$ which shares some of the variables with the system $\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a})$. Pecora and Carroll [4] formalized this unidirectional coupling between the systems (6) and (9) through the variable $\mathbf{x}_1$, $\dot{\mathbf{u}}_2 = \mathbf{f}_{x_2 \to x_1}(\mathbf{u}_2; \mathbf{a}) = \mathbf{f}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a})$, where the coupled system

$$\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \wedge \dot{\mathbf{y}}_1 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a}) \wedge \dot{\mathbf{y}}_2 = \mathbf{h}(\mathbf{x}_1, \mathbf{y}_2; \mathbf{a}) \tag{10}$$

is obtained by a complete replacement of the signal driver subsystem $\dot{\mathbf{x}}_1 = \mathbf{g}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{a})$ in the response (or slave) system (9).

Instead of completely replacing one of the variables in the response system by its correspondent from the drive (master or transport) system, a partial replacement can be used as suggested by Guemez and Matthias [1]. In this case, a variable of the response system gives rise to its correspondent in the drive system only in some of its equations. In general, the stability results in partial replacement differ from those in complete replacement. In what follows we will analyze the partial replacement in the nonlinear terms of the response system.

Coupling by Dislocated Negative Feedback Control. Consider the coupling between S_1 and S_2 through the linear term $\rho(\mathbf{u}_2 - \mathbf{u}_1)$, that is,

$$\dot{\mathbf{u}}_1 = \mathbf{f}(\mathbf{u}_1; \mathbf{a}) \wedge \dot{\mathbf{u}}_2 = \mathbf{f}(\mathbf{u}_2; \mathbf{a}) + \rho(\mathbf{u}_2 - \mathbf{u}_1), \tag{11}$$

where $\rho = (\rho_1, \rho_2, ..., \rho_m)$ is the coupling parameter vector, with $\rho_i > 0$ for all $i = 1, \ldots, m$. The unidirectional coupling in (11) is designated by negative feedback control through the damping term $\rho(\mathbf{u}_2 - \mathbf{u}_1)$.

Let $\mathbf{u}_1 = (u_1, u_2, \ldots, u_m) \in X$ and $\mathbf{u}_2 = (u'_1, u'_2, \ldots, u'_m) \in X$ be the variables of S_1 and S_2, respectively. Suppose that the dynamical variable $u_k(t)$, $1 \leq k \leq m$, and its corresponding $u'_k(t)$ can be measured. The addition of $\rho\,(u_k - u'_k)$, with $\rho > 0$, to the response system, that is,

$$\begin{cases} \dot{u}'_1 = f_{\mathbf{a},1}\,(u'_1, u'_2, \ldots, u'_m)\,, \\ \cdots \\ \dot{u}'_k = f_{\mathbf{a},k}\,(u'_1, u'_2, \ldots, u'_m) + \rho\,(u_k - u'_k) \\ \cdots \\ \dot{u}'_m = f_{\mathbf{a},m}\,(u'_1, u'_2, \ldots, u'_m) \end{cases}, \qquad (12)$$

leads to a particular case of (11) in which a single variable, u_k, makes the coupling. The term $\rho\,(u_k - u'_k)$ is used as a control signal (or perturbation signal) applied to the response system. In fact the control signal it is a negative feedback added to the system with no effects on its solution. The parameter ρ, known as coupling strength, it is experimentally adjustable and measures the perturbation intensity.

From initial conditions $\mathbf{u}_1\,(0)$ and $\mathbf{u}_2\,(0)$ such that $\mathbf{u}_1\,(0) \neq \mathbf{u}_2\,(0)$, the vector state of each systems S_1 and (12) are the same for certain value of ρ, after a certain time t_{sync}. When synchronization it is achieved, the control signal became zero, and the symmetric synchronous chaotic state $\mathbf{u}_1 = \mathbf{u}_2$ is established.

In this paper we study the dislocated negative feedback control. After choosing the driver variable u_k, the control signal $\rho\,(u'_k - u_k)$ is applied to the j-th equation of the response system S_2 with $j \neq k$. So, for $1 \leq j, k \leq m$, the response system is given by

$$\begin{cases} \dot{u}'_1 = f_{\mathbf{a},1}\,(u'_1, u'_2, \ldots, u'_m)\,, \\ \cdots \\ \dot{u}'_j = f_{\mathbf{a},j}\,(u'_1, u'_2, \ldots, u'_m) + \rho\,(u_k - u'_k) \\ \cdots \\ \dot{u}'_m = f_{\mathbf{a},m}\,(u'_1, u'_2, \ldots, u'_m) \end{cases}, \qquad (13)$$

for $j \neq k$. In what follows we apply this methodology to two coupled Lorenz systems.

3 Case Study: Unidirectional Couplings between Nonlinear Lorenz Systems

Consider the Lorenz system

$$\begin{cases} \dot{x} = \sigma\,(y - x) \\ \dot{y} = x\,(\alpha - z) - y \\ \dot{z} = xy - \beta z \end{cases} \qquad (14)$$

where σ, α and β are positive real parameters. This system of nonlinear ordinary differential equations, where σ is the Prandtl number, α is the Rayleigh number and β is a geometric factor, describes the flow of a fluid in a heated box along the base. In 1963, a research for the purpose of improving the climate prediction, the meteorologist and mathematician Edward Lorenz [3] introduced this system model as an approximate fluid flow from the atmosphere. He found that, for a wide range of parameters, the solutions of the system remain in a bounded region of phase space but oscillate in an irregular and aperiodic way. Lorenz detected unusual dynamical behavior with $\sigma = 10$, $\alpha = 28$ and $\beta = 8/3$. Starting computer simulations from slightly different initial conditions, he detected the sensitive dependence on initial conditions of this system, one of the main properties of chaotic behavior. In what follow we consider an unidirectional coupling scheme between two identical Lorenz systems.

Unidirectional Coupling by Dislocated Negative Feedback Control with Partial Replacement of x_2. Consider the driver variable x_1 by adding the control signal $\rho\,(x_1 - x_2)$, with $\rho > 0$, applied as dislocated negative feedback to the second equation of the response system. Furthermore, we introduce the partial replacement of variable x_2 by the corresponding x_1 only in the nonlinear terms $x_2 z_2$ and $x_2 y_2$ of the response system. We obtain the following

$$\left\{\begin{array}{l} \dot{x}_1 = \sigma\,(y_1 - x_1) \\ \dot{y}_1 = \alpha x_1 - x_1 z_1 - y_1 \\ \dot{z}_1 = x_1 y_1 - \beta z_1 \end{array}\right. \wedge \left\{\begin{array}{l} \dot{x}_2 = \sigma\,(y_2 - x_2) \\ \dot{y}_2 = \alpha x_2 - \underline{x_1} z_2 - y_2 + \rho\,(x_1 - x_2) \\ \dot{z}_2 = \underline{x_1} y_2 - \beta z_2 \end{array}\right. . \tag{15}$$

starting the coupled system from different initial conditions, that is, $x_1(0) \neq x_2(0)$, $y_1(0) \neq y_2(0)$ and $z_1(0) \neq z_2(0)$, the identical synchronization it is reached if the coupled system evolution is continuously confined to a hyperplane $\mathcal{M}$ in the phase space. The coordinates $e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$ of the synchronization error, $\mathbf{e}$, in the transversal subspace of $\mathcal{M}$ converge to 0 as $t \to +\infty$ if the point $(0, 0, 0)$ in the transversal subspace of $\mathcal{M}$ is an asymptotically stable equilibrium point. This resuming require that the dynamical system in $\mathbf{e} = (e_x, e_y, e_z)$ defining the transversal perturbations should be asymptotically stable at the equilibrium point $(0, 0, 0)$.

Consider now the function

$$\breve{\mathbf{f}} = \left(\sigma\,(y_2 - x_2)\,, \alpha x_2 - x_1 z_2 - y_2 + \rho\,(x_1 - x_2)\,, x_1 y_2 - \beta z_2\right) \tag{16}$$

obtained from the response in (15). For the whole values of ρ, the linearized equation which defines the transversal perturbations to $\mathcal{M}$ is given by

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} \approx D_{(x_2,y_2,z_2)}\breve{\mathbf{f}} \cdot \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \alpha - \rho & -1 & -x_1 \\ 0 & x_1 & -\beta \end{bmatrix} \cdot \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} . \tag{17}$$

By studying the eigenvalues of the Jacobian matrix $D_{(x_2,y_2,z_2)}\breve{\mathbf{f}}$, we conclude that we can reach locally stable synchronization if $\rho_{sync} = \alpha - 1$.

Now by taking the following values for the control parameters $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$ and the strength coupling $\rho = 27.1$, we verify that $x_2 \to x_1$, $y_2 \to y_1$ and $z_2 \to z_1$ when the systems evolve (Fig. 1a). After a certain period of time, the systems coordinates x, y and z verify the equalities $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ (Fig. 1b). So, the distances $|x_2 - x_1|$, $|y_2 - y_1|$ and $|z_2 - z_1|$ converge to 0 over time (Fig. 1c). The equations $x_2 = x_1$, $y_2 = y_1$ and $z_2 = z_1$ define a hyperplane $\mathcal{M}$ in the 6-dimensional phase space.

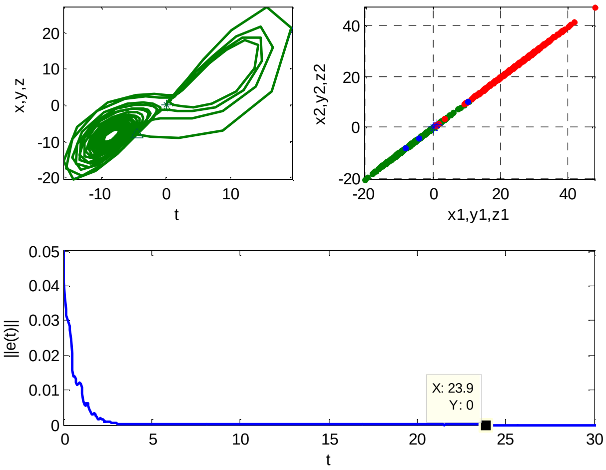

Figure 1: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\rho = 27.1$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

Applying the criterion *(ii)*, it is obtained the threshold of globally stable synchronization, $\tilde{\rho}_{sync}$. It is greater than the threshold ρ_{sync} obtained for local stability, that is,

$$\tilde{\rho}_{sync} = \alpha + \sigma > \alpha - 1 = \rho_{sync}, \tag{18}$$

leading to a more restrictive range of ρ values. In fact, consider the Lyapunov function $L(\mathbf{e}) = \left(e_x^2 + e_y^2 + e_z^2\right)/2$ which verifies $L(\mathbf{e}) > 0$ if $\mathbf{e} \neq \mathbf{0}$ and $L(\mathbf{0}) = 0$ for all $\rho > 0$. It is necessary to determine the coupling strength ρ such that the derivative of L satisfies $\dot{L}(\mathbf{e}) < 0$ if $\mathbf{e} \neq \mathbf{0}$ and $\dot{L}(\mathbf{0}) = 0$. Substituting the expression of $\dot{e}_x$, $\dot{e}_y$ and $\dot{e}_z$ in

$$\dot{L}(\mathbf{e}) = e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z \tag{19}$$

and simplifying, the derivative of L can be written as

$$\begin{aligned}\dot{L}(\mathbf{e}) &= -\sigma e_x^2 - e_y^2 - \beta e_z^2 + (\sigma+\alpha-\rho)\, e_x e_y \\ &\leq -\sigma e_x^2 - e_y^2 - \beta e_z^2 + (\sigma+\alpha-\rho)\, |e_x e_y| \,.\end{aligned}$$

Choosing a coupling strength satisfying $\tilde{\rho} > \alpha + \sigma$ the conditions required by the Lyapunov direct method are guaranteed. So, the globally stable synchronization it is achieved in the coupled system with a coupling strength $\tilde{\rho} = \tilde{\rho}(\sigma, \alpha)$ which do not depend on the control parameter β. In Figure 2(a,b,c) are taken the same values for the control parameters and the corresponding synchronization threshold $\tilde{\rho} = 38.1$. The time synchronization t_{sync} for $\tilde{\rho} = 38.1$ is lower than the one obtained for $\rho = 27.1 < 38.1$.

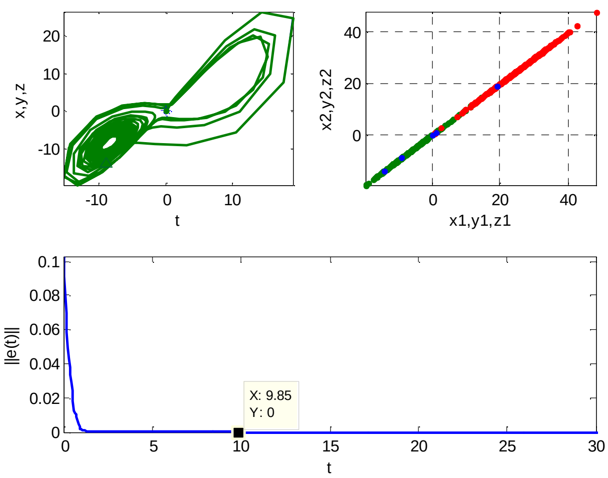

Figure 2: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\rho = 38.1$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

Table 1 presents sufficient conditions for globally stable synchronization obtained from the study of other similar cases. It was applied the dislocated control signal $\rho(x_1 - x_2)$ and, in some cases, also the partial replacement of x_2 by its corresponding x_1 in some nonlinear terms of the response system. The constants ξ and K represent the expressions $\rho - \sigma - \alpha$ and $K_x + K_x'$, respectively.

Unidirectional Coupling by Negative Feedback Control with Partial Replacement of x_2. Consider identical chaotic Lorenz systems coupled by negative feedback control, that is,

Disloc.	**Replac.**	**Synchronization sufficient condition**
to 2^{th} eq.	no	$\beta\left(\xi+K_z\right)^2<4\sigma\beta-K_y^2$
to 2^{th} eq.	on 3^{th} eq.	$\beta\left(\xi+K_z\right)^2<4\sigma\beta-\sigma K^2$
to 3^{th} eq.	on 2^{th} eq.	$\xi^2<4\sigma \;\wedge\; \beta\xi^2<4\sigma\beta-KK_y\xi+K_y^2+\sigma K^2$

Table 1: Unidirectional coupling by dislocated negative feedback control.

$$\left\{\begin{array}{l}\dot{x}_1=\sigma\left(y_1-x_1\right)\\ \dot{y}_1=\alpha x_1-x_1z_1-y_1\\ \dot{z}_1=x_1y_1-\beta z_1\end{array}\right. \wedge \left\{\begin{array}{l}\dot{x}_2=\sigma\left(y_2-x_2\right)+\rho\left(x_1-x_2\right)\\ \dot{y}_2=\alpha x_2-\underline{x_1}z_2-y_2+\rho\left(y_1-y_2\right)\\ \dot{z}_2=\underline{x_1}y_2-\beta z_2+\rho\left(z_1-z_2\right)\end{array}\right. \tag{20}$$

where it is also made a partial replacement of variable x_2 by x_1 only in the nonlinear terms x_2z_2 and x_2y_2 of the response system. Let $\breve{\mathbf{f}}$ be the function obtained from the response, whose components are $\breve{\mathbf{f}}_1=\sigma\left(y_2-x_2\right)+\rho\left(x_1-x_2\right)$, $\breve{\mathbf{f}}_2=\alpha x_2-x_1z_2-y_2+\rho\left(y_1-y_2\right)$ and $\breve{\mathbf{f}}_3=x_1y_2-\beta z_2+\rho\left(z_1-z_2\right)$. Consider the components $e_x=x_2-x_1$, $e_y=y_2-y_1$ and $e_z=z_2-z_1$ of the error term $\mathbf{e}$. For all values of ρ, the linearized equation which defines transversal perturbations to $\mathcal{M}$ is given by

$$\left[\begin{array}{c}\dot{e}_x\\ \dot{e}_y\\ \dot{e}_z\end{array}\right]\approx D_{(x_2,y_2,z_2)}\breve{\mathbf{f}}\cdot\left[\begin{array}{c}e_x\\ e_y\\ e_z\end{array}\right]=\left[\begin{array}{ccc}-\sigma-\rho & \sigma & 0\\ \alpha & -1-\rho & -x_1\\ 0 & x_1 & -\beta-\rho\end{array}\right]\cdot\left[\begin{array}{c}e_x\\ e_y\\ e_z\end{array}\right]. \tag{21}$$

The same can take the matrix form $\dot{\mathbf{e}}=\mathbf{A}\left(x_1\right)\cdot\mathbf{e}$ with

$$\mathbf{A}=\left[\begin{array}{ccc}-\sigma-\rho & \sigma & 0\\ \alpha & -1-\rho & -x_1\\ 0 & x_1 & -\beta-\rho\end{array}\right], \tag{22}$$

and it follows that the main determinants of the matrix

$$\mathbf{A}^T+\mathbf{A}=\left[\begin{array}{ccc}-2\left(\sigma+\rho\right) & \sigma+\alpha & 0\\ \sigma+\alpha & -2\left(1+\rho\right) & 0\\ 0 & 0 & -2\left(\beta+\rho\right)\end{array}\right] \tag{23}$$

are given by $\Delta_1=-2\left(\sigma+\rho\right)$, $\Delta_2=4\left(\sigma+\rho\right)\left(1+\rho\right)-\left(\sigma+\alpha\right)^2$ and

$$\Delta_3=\left[2\left(\sigma+\alpha\right)^2-8\left(\sigma+\rho\right)\left(1+\rho\right)2\left(\sigma+\alpha\right)^2\right]\left(\beta+\rho\right). \tag{24}$$

We have $-\Delta_1>0$ and the condition $-\Delta_3>0$ is satisfied when $\Delta_2>0$ (since $\beta+\rho>0$). So, we conclude by Theorem 1.2 that globally stable synchronization occurs if the control and coupling parameters verify the inequality

$$4\left(\sigma+\rho\right)\left(1+\rho\right)>\left(\sigma+\alpha\right)^2. \tag{25}$$

Taking $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$, we present the Figure 3(a,b,c) obtained for $\rho = 14.5$, which is the lowest value of ρ in a tenth step that verifies the previous inequality.

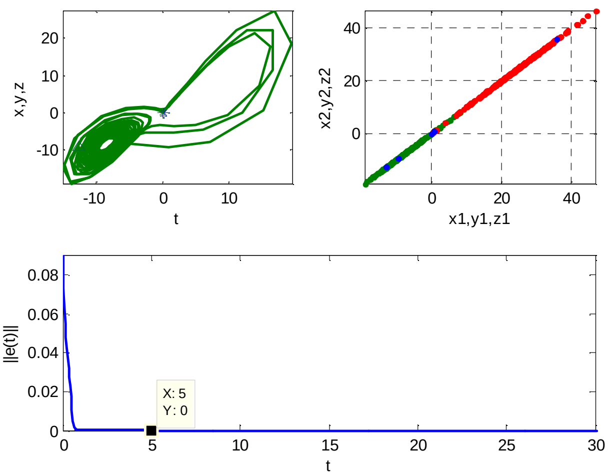

Figure 3: Parameter values $\sigma = 10$, $\alpha = 28$ and $\beta = 2.(6)$; coupling strength $\rho = 14.5$. (a) Coupled system attractor; (b) Synchronization manifold; (c) Evolution of synchronization error

Such an approach it is inconclusive if it is not assumed the partial replacement of the variable x_2 by x_1 in the nonlinear terms of the response system.

4 Conclusions

Either using an usual negative feedback control or applying a control signal as the dislocated negative feedback, the combination of each of these unidirectional couplings with replacement shows some advantages. Even when the replacement is partial (on the nonlinear terms of the response system) we obtained quite simple sufficient conditions for globally stable synchronization between identical chaotic Lorenz systems. These conditions result from the classification of the symmetric matrix $\mathbf{A}^T + \mathbf{A}$ as negative definite, where $\mathbf{A}$ is the matrix characterizing the transversal system of coupling, or are based on the increase of the derivative of an appropriate Lyapunov function. Theorem 1.2 it is not applicable without partial replacement. The approach based on derivative increase of an appropriate Lyapunov function also leads to a sufficient condition for globally stable synchronization in a coupling by active-passive decomposition for several driver signals.

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