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Description of a wave-wave interaction model by Variational and Decomposition methods

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Abstract

In this paper, a wave-wave interaction model is proposed considering coupled Schrödinger–Korteweg–de Vries (or Sch–KdV) equation with appropriate initial values. The model is described with the help of Variational Iteration Method (VIM) and Adomian's Decomposition Method (ADM). The exact and numerical solutions of Sch–KdV equation are obtained by VIM and ADM. The solutions are found to be in agreement to each other in both the approaches. The comparison of solution plots infers the superiority of VIM over ADM.

MSCs : 5A15, 70H03, 49M27, 35A15, 35C07, 34K28

Key words: Coupled Schrödinger–KdV equation; Variational iteration method; Lagrange multiplier, Adomian decomposition method; Traveling wave solution; Numerical solution

1. Introduction:

A wave-wave interaction model can be described by a system of coupled partial differential equations. Here the Schrödinger–KdV (Sch–KdV) equation is considered as a wave-wave interaction model, which has the following forms (1) and (2):

$$iu_t = u_{xx} + uv \qquad (1)$$

$$v_t = -6uv_x - v_{xxx} + \left(|u|^2\right)_x \qquad (2)$$

with initial data

$$u(x,0) = g_1(x), \qquad v(x,0) = g_2(x)$$

Calculation of exact and numerical solutions of above equation, in particular, traveling wave solutions, play an important role in wave-wave interaction and soliton theory [1,2].

Many explicit exact methods have been introduced in literature [3-6] for various equations. Some of them are: Backlund transformation, Generalized Miura Transformation, Darboux transformation, Cole–

Hopf transformation, tanh method, sine–cosine method, Painleve method, homogeneous balance method, similarity reduction method and so on. Recently, Fan et al. [7] applied extended tanh method and found some new explicit solutions for the Sch–KdV equations.

In recent years a lot of attention has been devoted to the study of Adomian's decomposition method to investigate various scientific models. The Adomian decomposition method, which accurately computes the series solution, is of great interest in applied sciences [2,12,13] as it provides the solution in a rapidly convergent series with components that can be elegantly computed. The literature reports [2,14,15] that the power series obtained, converges rapidly and introduces the exact solution in a closed form which is reliable and effective.

The model equations can also be solved by variational iteration method which was first proposed by He [8,9,10] and was successfully applied to autonomous ordinary and partial differential equations [19]. This method has many merits and has much advantage over the Adomian method [11].

The theme of this paper is to find the analytical and numerical solutions of the Sch-KdV equations using both VIM and ADM approaches. As both the methods have their own characteristics and significances, a comparison of solutions has been made. However VIM proved to be better then ADM

2. Description of the Variational iteration and Adomian's decomposition methods

2.1. Variational iteration Method:

To illustrate the basic concepts of variational iteration method, we consider the following differential equation (3)

$$Lu + N\,u = g(x) \tag{3}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ an inhomogeneous term. According to the variational iteration method, we can construct a correct functional as equation (4)

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}\mathrm{d}\tau \tag{4}$$

where λ is a general Lagrangian multiplier [8,9,16], which can be identified optimally via the variational theory. The subscript n denotes the nth order approximation whereas $\tilde{u}_n$ is considered as a restricted variation [8,9,16], *i.e.* $\delta\,\tilde{u}_n = 0$.

To solve equations (1) and (2) by means of Variational Iteration Method, we construct a correction functional can be represented as equations (5) and (6)

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(u_{nt} + \mathrm{i}(u_{nxx} + u_n\tilde{v}_n))\,\mathrm{d}\tau \tag{5}$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(v_{nt} + 6u_n\tilde{v}_{nx} + v_{nxxx} - \tilde{u}^2)\,d\tau \tag{6}$$

where $\delta u_n \tilde{v}_{nx}$, $\delta u_n \tilde{v}_n$ and $\tilde{u}^2$ are considered as restricted variations. The stationary conditions for equations (5) and (6) are obtained as follows

$$\lambda_1'(\tau) = 0 \tag{7a}$$

$$1 + \lambda_1(\tau)\rfloor_{\tau=t} = 0 \tag{7b}$$

$$\lambda_2'(\tau) = 0 \tag{8a}$$

$$1 + \lambda_2(\tau)\rfloor_{\tau=t} = 0 \tag{8b}$$

The Lagrange multipliers, therefore, can be identified as $\lambda_1 = \lambda_2 = -1$, subsequently, using these values equations (9) and (10) obtained as variational iteration formula for equations (5) and (6) respectively.

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (u_{nt} + \mathrm{i}(u_{nxx} + u_n v_n))\,d\tau \tag{9}$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t (v_{nt} + 6u_n v_{nx} + v_{nxxx} - u^2)\,d\tau. \tag{10}$$

The implementation of the method is shown in section (3.1).

2.2. Adomian's decomposition method

To start with Sch–KdV equation (1) and (2) are written in operator form (11) and (12)

$$Ł_t u = -iŁ_{xx}u - iN(u,v) \tag{11}$$

$$Ł_t v = -6M(u,v) - Ł_{xxx}v + R(u,v) \tag{12}$$

where

$Ł_t = \frac{\partial}{\partial t}$, $Ł_{xx} = \frac{\partial^2}{\partial x^2}$ and $Ł_{xxx} = \frac{\partial^3}{\partial x^3}$ symbolize the linear differential operators and the notations $N(u, v) = uv$, $M(u,v) = uv_x$ and $R(u, v) = (|u|^2)_x$ symbolize the nonlinear operators. Applying the inverse operator $Ł_t^{-1} = \int_0^t (\cdot)\,dt$ to the equations (11) and (12), equations (13)1 and (14) are derived

$$u(x,t) = g_1(x) - iŁ_t^{-1}[L_{xx}u + N(u,v)] \tag{13}$$

$$v(x,t) = g_2(x) - Ł_t^{-1}[6M(u,v) + L_{xxx}v - R(u,v)] \tag{14}$$

where $g_1(x) = u(x, 0)$ and $g_2(x) = v(x, 0)$ are given functions for initial conditions. The Adomian decomposition method [16,17] assumes an infinite series solution for unknown functions $u(x, t)$ and $v(x,t)$ in the following form (15)

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \tag{15}$$

and nonlinear operators $N(u, v) = uv$, $M(u,v) = uv_x$ and $R(u, v) = (|u|^2)_x$ by the infinite series of Adomian polynomials are given by

$$N(u, v) = \sum_{n=0}^{\infty} A_n, \qquad M(u, v) = \sum_{n=0}^{\infty} B_n, \qquad R(u, v) = \sum_{n=0}^{\infty} C_n \tag{16}$$

where A_n, B_n and C_n are the appropriate Adomian's polynomials which are generated according to algorithms determined in [18]. For nonlinear operator $N(u, v)$ these polynomials can be defined by equation (17)

$$A_n(u_0, \ldots, u_n; v_0, \ldots, v_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left[\sum_{k=0}^{n} \lambda^k u_k, \sum_{k=0}^{n} \lambda^k v_k\right]\right]_{\lambda=0}, \quad n \geqslant 0. \tag{17}$$

This formula is easy to be set in a computer code to get as many polynomial as we need in the calculation of the numerical as well as explicit solutions. Following the decomposition method, the nonlinear system (13) and (14) is constructed in a form of the recursive relations given by equation 18(a-d), for where $n \geq 1$.

$$u_0(x,t) = 0, \qquad u_1(x,t) = g_1(x) - i Ł_t^{-1}[Ł_{xx} u_0 + A_0] \tag{18a}$$

$$u_{n+1}(x,t) = -i Ł_t^{-1}[Ł_{xx} u_n + A_n] \tag{18b}$$

$$v_0(x,t) = 0, \qquad v_1(x,t) = g_2(x) - Ł_t^{-1}[6B_0 + Ł_{xxx} v_0 - C_0] \tag{18c}$$

$$v_{n+1}(x,t) = -Ł_t^{-1}[6B_n + Ł_{xxx} v_n - C_n] \tag{18d}$$

The functions $g_1(x)$ and $g_2(x)$ are obtained from the initial conditions. It is worth noting if the zeroth components u_0 and v_0 are defined then the remaining components u_n and v_n for $n \geq 1$, can be determined using the previous terms. As a result, the components $u_0, u_1, u_2, \ldots,$ and $v_0, v_1, v_2, \ldots,$ are identified and the series solutions are entirely determined. For numerical comparison purposes, we construct the solution $u(x, t)$ and $v(x, t)$

$$u(x, t) = \lim_{n\to\infty} \phi_n \qquad v(x, t) = \lim_{n\to\infty} \varphi_n \tag{19}$$

where

$$\phi_n(x,t) = \sum_{k=0}^{n-1} u_k(x,t), \qquad \varphi_n(x,t) = \sum_{k=0}^{n-1} v_k(x,t), \quad n \geqslant 1$$

and the recurrence relation is given as in (17). The implementation of the scheme to solve the Sch-KdV equation is given in section (3.2). The theoretical treatment of convergence of the decomposition method has already considered [20-23]. Some results about the speed of convergence are provided to solve linear and nonlinear functional equations.

3. Implementation of the methods

For the solutions of Sch–KdV equations, using VIM and ADM, the following initial conditions are considered

$$u(x,0) = 6\sqrt{2}\,e^{i\alpha x} k^2 \operatorname{sech}^2(kx) \tag{20}$$

$$v(x,0) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(kx) \tag{21}$$

where α and k are arbitrary constants.

3.1. Variational Iteration Method:

Using iteration formula (9) and (10) with initial condition (20) and (21) for the functional coupled equations (1) and (2), the components u_0 and v_0 are derived as equations (22) and (23)

$$u_0 = 6\sqrt{2}\,e^{i\alpha x} k^2 \operatorname{sech}^2(kx) \tag{22}$$

$$v_0 = \frac{\alpha + 16k^2}{3} - 16k^2 \tanh^2(kx) \tag{23}$$

The other components are obtained as equations (24) to (27)

$$u_1(x,t) = 6\sqrt{2}\,e^{i\alpha x} k^2 \operatorname{sech}^2(kx) + 6it\sqrt{2}\,e^{i\alpha x} k^2 \operatorname{sech}^2(kx)[\alpha^2 - 2k^2 \operatorname{sech}^2(kx) - 4i\sqrt{2}\alpha k \tanh(kx) + 4\sqrt{2}k^2 \tanh^3(kx)] \tag{24}$$

$$v_1(x,t) = \frac{\alpha + 16k^2}{3} - 16k^2 \tanh^2(kx) - 48tk^5 \tanh(kx) \operatorname{sech}^2(kx)[2\operatorname{sech}^2(kx) - \tanh^2(kx)] \tag{25}$$

$$\begin{aligned} u_2(x,t) = {} & u_1(x,t) - it\sqrt{2}\,e^{i\alpha x} k^2 \operatorname{sech}^4(kx)[\alpha + 34k^2 + \alpha\cosh(2kx) - 2k^2 \cosh(2kx) \\ & + \tfrac{3}{8} t \operatorname{sech}^2(kx)\{-3i\alpha^4 - 72i\alpha^2 k^2 - 528ik^4 - 4i\alpha^4 \cosh(2kx) - 48i\alpha^2 k^2 \cosh(2kx) \\ & + 416ik^4 \cosh(2kx) - i\alpha^4 \cosh(4kx) + 24i\alpha^2 k^2 \cosh(4kx) - 16ik^4 \cosh(4kx) \\ & + 16\alpha^3 k \sinh(2kx) + 320\alpha k^3 \sinh(2kx) + 8\alpha k \sinh(4kx)(\alpha^2 - 4k^2)\}] \end{aligned} \tag{26}$$

$$\begin{aligned} v_2(x,t) = {} & v_1(x,t) - 6tk^4 \operatorname{sech}^5(kx)[k^4 t \operatorname{sech}^3(kx)(1208 - 1191\cosh(2kx) + 120\cosh(4kx) \\ & - \cosh(6kx)) + 24\,e^{2i\alpha x}(-i\alpha \cosh(kx) + 2k \sinh(kx))], \end{aligned} \tag{27}$$

and so on, in the same manner the rest of components of the iteration formulae (9) and (10) can be obtained. The solutions of $u(x, t)$ and $v(x, t)$ in a closed form are readily found to be equations (28) and (29) respectively

$$u(x,t) = 6\sqrt{2}\,e^{i\theta} k^2 \operatorname{sech}^2(k\xi) \tag{28}$$

$$v(x,t) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(k\xi) \tag{29}$$

Where $\theta = \left(\frac{\alpha t}{3} + \alpha^2 t - \frac{10k^2 t}{3} + \alpha x\right), \qquad \xi = x + 2\alpha t$

3.2. Adomian's decomposition method

Using (15) with (16) for the functional coupled equation (1) and (2) and initial conditions (18) we obtain

$$u_0=0 \tag{30}$$

$$v_0=0 \tag{31}$$

The other components are obtained as equations (32) to (37)

$$u_1 = 6\sqrt{2}e^{i\alpha x}k^2 \operatorname{sech}(kx)^2 \tag{32}$$

$$v_1 = \frac{\alpha+16k^2}{3} - 6k^2\tanh(kx)^2 \tag{33}$$

$$u_2 = 6it\left[\left(\sqrt{2}\alpha^2 e^{i\alpha x}k^2\operatorname{sech}^2(kx)\right) - 2\sqrt{2}e^{i\alpha x}k^4\operatorname{sech}^4(kx) - 4i\sqrt{2}\alpha e^{i\alpha x}k^3\operatorname{sech}^2(kx)\tanh(kx) + 4\sqrt{2}e^{i\alpha x}k^4\operatorname{sech}^2(kx)\tanh^3(kx)\right] \tag{34}$$

$$v_2 = -48t\left(2k^5\operatorname{sech}(kx)^4\tanh(kx) - k^5\operatorname{sech}(kx)^2\tanh(kx)^3\right) \tag{35}$$

$$u_3 = -i\Bigg[\sqrt{2}e^{i\alpha x}k^2t\left(\alpha+34k^2+\alpha\cosh(2kx)-2k^2\cosh(2kx)\right)\operatorname{sech}^4(kx) + \frac{3e^{i\alpha x}k^2t^2\operatorname{sech}(kx)^6}{4\sqrt{2}}\big[-3i\alpha^4-72i\alpha^2k^2-528ik^4-4i\alpha^4\cosh(2kx)-48i\alpha^2k^2\cosh(2kx) + 416ik^4\cosh(2kx)-i\alpha^4\cosh(4kx)+24i\alpha^2k^2\cosh(4kx)-16ik^4\cosh(4kx) + 16\alpha^3k\sinh(2kx)+320\alpha k^3\sinh(2kx)+8\alpha^3k\sinh(4kx)-32\alpha k^3\sinh(4kx)\big]\Bigg] \tag{36}$$

$$v_3 = -6k^8t^2\left(1208-1191\cosh(2kx)+120\cosh(4kx)-\cosh(6kx)\right)\operatorname{sech}^8(kx) - 144e^{2i\alpha x}k^4t\operatorname{sech}^5(kx)\left(-i\alpha\cosh(kx)+2k\sinh(kx)\right), \tag{37}$$

and so on, the other components of the decomposition series (15) can be determined in a similar way. Substituting equations (30)–(37) into (15) and using the decomposition series (15) which is a Taylor series we got the exact solution. In the same manner and for the higher value of n, we obtain the closed form solutions which are same as obtained by VIM.

$$u(x,t) = 6\sqrt{2}e^{i\theta}k^2\operatorname{sech}^2(k\xi) \tag{38}$$

$$v(x,t) = \frac{\alpha+16k^2}{3} - 6k^2\tanh^2(k\xi) \tag{39}$$

where $\theta = \left(\frac{\alpha t}{3}+\alpha^2 t-\frac{10k^2t}{3}+\alpha x\right), \qquad \xi = x+2\alpha t$

and α, k are arbitrary constants

4. Numerical simulation

4.1. Results by the Variational Iteration Method:

The behavior of the two solutions obtained by variational iteration methods with the exact solutions for different values of time are plotted in Figure1

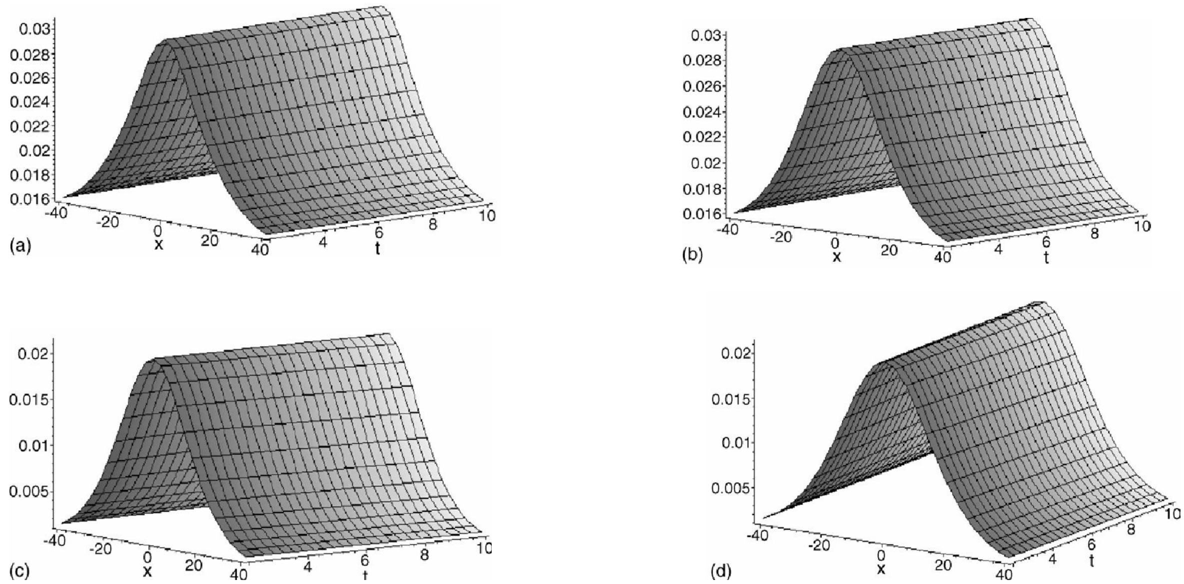
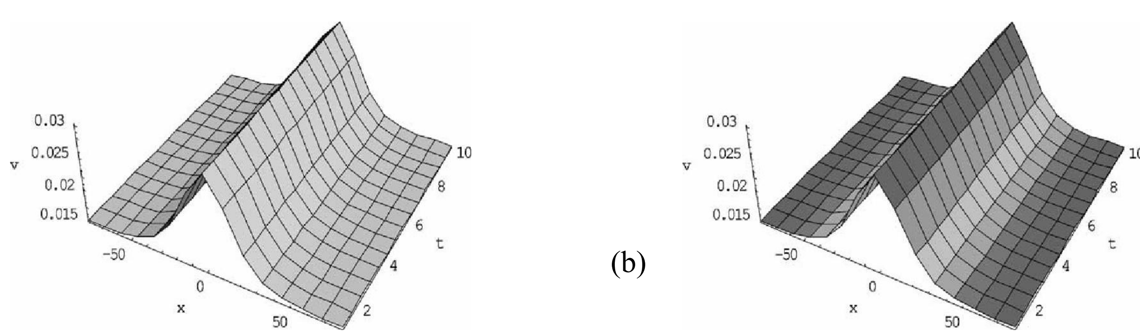


Fig 1. The plots of results for solution of Sch-KdV equations with a fixed values of $\alpha = 0.05$, $k = 0.05$ and for different values of time (a) Numerical results for $v(x, t)$ by means of variational iteration method (b) Exact solution for $v(x, t)$ (c) Numerical results for $u(x, t)$ by means of variational iteration method (d) Exact solution for $u(x, t)$

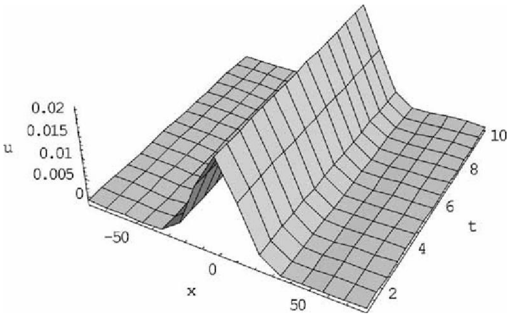
4.2 Results by Adomain Decomposition Method:

The Adomian's Decomposition Method is used for finding the exact and approximate traveling-waves solutions of the Sch–KdV equation. Both the exact and approximate solutions obtained for $n = 4$ by using ADM are plotted in Fig. 2. It is evident that when compute more terms for the decomposition series the numerical results are getting much more closer to the corresponding analytical solutions with the initial conditions (20) and (21) of equations (1) and (2).

(a) (b)



(c)



(d)

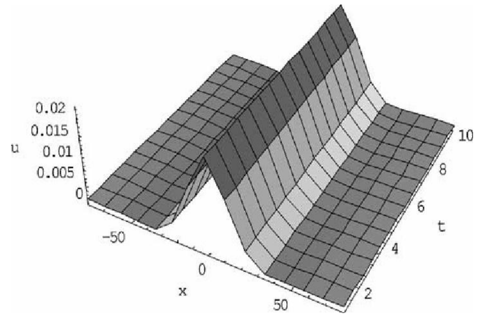


Fig. 2. The plots of results for solution of Sch-KdV equations with a fixed values of $\alpha = 0.05$, $k = 0.05$ and for different values of time (a) Analytical solutions for $v(x, t)$ (b) Numerical results for $\varphi_4(x, t)$ by means of ADM (c) Analytical solutions for $u(x, t)$ (d) Numerical results for $\psi_4(x, t)$ by means of ADM

5. Conclusions

The main goal of this work is to conduct a comparative study between Variational Iteration Method (VIM) and the Adomian decomposition method. The two methods are powerful and efficient methods that both give approximations of higher accuracy and closed form solutions if exist. An important conclusion can made here. The Variational Iteration Method gives several successive approximations using the iteration of the correction functional. However, Adomian decomposition method provides the components of the exact solution, where these components should follow the summation given in (15). Moreover, the VIM requires the valuation of the Lagrangian multiplier λ, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. It is interesting to point out that unlike the successive approximations obtained by the VIM, the ADM provides the solution in successive components that will be added to get the series solution. More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials; hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For nonlinear equations that arise frequently to express nonlinear phenomenon, variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

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