Mathematica Aeterna, Vol. 6, 2016, no. 4, 515 - 520

Constructions of Hom-Jordan algebras

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Abstract

The purpose of this paper is to study Hom-Jordan algebras. We discuss some of its properties. For further study, we also give some new definitions and the examples.

Mathematics Subject Classification: 17A40, 17B10, 17B56

Keywords: Hom-Jordan algebras, morphism

1 Introduction

The notion and some properties of Jordan algebras were introduced by A. A. Albert in [1]. And in [5], we know there is a relationship between Jordan algebras and associative algebras. The definition of Hom-Jordan algebra was introduced by A. Makhlouf in [2]. It is clear that the Hom-Jordan algebra (V, μ, id) is the Jordan algebra V itself. More applications of the Jordan algebras and Hom-algebras can be found in [1, 3].

In section 2 we give the definition of Hom-Jordan algebras and some examples about Hom-Jordan algebra. We also show that the direct sum of two Hom-Jordan algebras is still a Hom-Jordan algebra. And we have proved that a linear map between Hom-Jordan algebras is a morphism if and only if its graph is a Hom subalgebra.

2 Preliminary Notes

Definition 2.1 [5] *A linear space J over a field $\mathbb{F}$ is called a Jordan algebra when we define a bilinear operation satisfying for any $x, y \in J$*

$$x \cdot y = y \cdot x, \tag{1}$$

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x), \quad x^2 = x \cdot x. \tag{2}$$

Definition 2.2 [2] *(1) A Hom-Jordan algebra is a triple* (V, μ, α) *consisting of a linear space* V*, a multiplication* $\mu : V \times V \to V$ *which is commutative and a homomorphism* $\alpha : V \to V$ *satisfying for any* $x, y \in V$

$$\mu(\alpha^2, \mu(y, \mu(x, x))) = \mu(\mu(\alpha(x), y), \alpha(\mu(x, x))) \tag{3}$$

where $\alpha^2 = \alpha \circ \alpha$.

(2) A Hom-Jordan algebra is multiplicative if α *is an algebra morphism, i.e. for any* $x, y, \in V$*, we have* $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$.

(3) A Hom-Jordan algebra is regular if α *is an algebra automorphism.*

(4) A subvector space $W \subseteq V$ *is a Hom subalgebra of* (V, μ, α) *if* $\alpha(W) \subseteq W$ *and*

$$\mu(x, y) \in W, \forall x, y \in W.$$

(5) A subvector space $W \subseteq V$ *is a Hom ideal of* (V, μ, α) *if* $\alpha(W) \subseteq W$ *and*

$$\mu(x, y) \in W, \forall x \in W, y \in V.$$

Remark 2.3 *Since the multiplication is commutative, one may write the identity* (3) *as*

$$\mu(\mu(y, \mu(x, x)), \alpha^2(x)) = \mu(\mu(y, \alpha(x)), \alpha(\mu(x, x))). \tag{4}$$

When the twisting map α is the identity map, we recover the classical notion of Jordan algebra.

Definition 2.4 *Let* (V, μ, α) *and* $(V^{'}, \mu^{'}, \beta)$ *be two Hom-Jordan algebras. A linear map* $\phi : V \to V^{'}$ *is said to be a morphism of Hom-Jordan algebra if*

$$\phi(\mu(x, y)) = \mu^{'}(\phi(x), \phi(y)), \forall x, y \in V, \tag{5}$$

$$\phi \circ \alpha = \beta \circ \phi. \tag{6}$$

Denote by $\mathfrak{G}_\phi \subset V \oplus V^{'}$ *is the graph of a linear map* $\phi : V \to V'$.

Definition 2.5 [4] *A Hom-associative algebra is a triple* (V, m, α) *consisting of a linear space* V*, a bilinear map* $m : V \times V \to V$ *and a homomophism* $\alpha : V \to V$*, satisfying*

$$m(\alpha(x), m(y, z)) = m(m(x, y), \alpha(z)). \tag{7}$$

Definition 2.6 [5] *Let* V_1, V_2 *be two rings, a linear map* $f : V_1 \to V_2$ *is called an anti-homomorphism if the linear map* f *is satisfying for any* $a, b \in V$

$$f(a + b) = f(a) + f(b),$$

$$f(ab) = f(b)f(a).$$

If the anti-homomorphism f *is a bijection, we called the linear map* f *is an anti-isomorphism. When we have* $V_1 = V_2$*, we called the linear map* f *is an anti-automorphism.*

3 Main Results

Example 3.1 [2] *Let (V, m, α) be a Hom-associative algebra. Then the Hom-algebra (V, μ, α), where the multiplication μ is defined for $x, y \in V$ by*

$$\mu(x, y) = \frac{1}{2}(m(x, y) + m(y, x)),$$

is a Hom-Jordan algebra, which is denoted V^+. The Hom-algebra $(V, [\cdot, \cdot], \alpha)$, where the bracket $[\cdot, \cdot]$ is defined for $x, y \in V$ by

$$[x, y] = m(x, y) - m(y, x),$$

is a Hom-Lie algebra, which is denoted V^-.

Example 3.2 *Let (V, m, α) be a Hom-Jordan algebra, we define the subspace W of $End(V)$ where $W = \{\omega \in End(V) | \omega\alpha = \alpha\omega\}$, $\sigma : W \to W$ is a map satisfying $\sigma(\omega) = \alpha\omega$.*

(1)The (W, ν, σ), where the multiplication $\nu : W \to W$ is defined for $\omega_1, \omega_2 \in W$ by

$$\nu(\omega_1, \omega_2) = \omega_1\omega_2 + \omega_2\omega_1,$$

is a Hom-Jordan algebra.

(2)The (W, ν', σ), where the multiplication $\nu' : W \to W$ is defined for $\omega_1, \omega_2 \in W$ by

$$\nu'(\omega_1, \omega_2) = \omega_1\omega_2 - \omega_2\omega_1,$$

is a Hom-Lie algebra over $\mathbb{F}$.

Proof. (1)For any $\omega_1, \omega_2 \in W$, we have

$$\begin{aligned}
&\nu(\omega_1, \omega_2) = \omega_1\omega_2 + \omega_2\omega_1 = \omega_2\omega_1 + \omega_1\omega_2 = \nu(\omega_2, \omega_1),\\
&\nu(\sigma^2(\omega_1), \nu(\omega_2, \nu(\omega_1, \omega_1)))\\
=\ &\nu(\alpha^2\omega_1, \nu(\omega_2, 2\omega_1^2))\\
=\ &\nu(\alpha^2\omega_1, 2\omega_2\omega_1^2 + 2\omega_1^2\omega_2)\\
=\ &2\alpha^2\omega_1\omega_2\omega_1^2 + 2\alpha^2\omega_1^3\omega_2 + 2\omega_2\omega_1^2\alpha^2\omega_1 + 2\omega_1^2\omega_2\alpha^2\omega_1\\
=\ &2\alpha^2\omega_1\omega_2\omega_1^2 + 2\alpha^2\omega_1^3\omega_2 + 2\alpha^2\omega_2\omega_1^3 + 2\alpha^2\omega_1^2\omega_2\omega_1,\\
&\nu(\nu(\sigma(\omega_1), \omega_2), \sigma(\nu(\omega_1, \omega_1)))\\
=\ &\nu(\nu(\alpha\omega_1, \omega_2), \sigma(2\omega_1^2))\\
=\ &\nu(\alpha\omega_1\omega_2 + \omega_2\alpha\omega_1, 2\alpha\omega_1^2)\\
=\ &2\alpha\omega_1\omega_2\alpha\omega_1^2 + 2\omega_2\alpha\omega_1\alpha\omega_1^2 + 2\alpha\omega_1^2\alpha\omega_1\omega_2 + 2\alpha\omega_1^2\omega_2\alpha\omega_1\\
=\ &2\alpha^2\omega_1\omega_2\omega_1^2 + 2\alpha^2\omega_2\omega_1^3 + 2\alpha^2\omega_1^3\omega_2 + 2\alpha^2\omega_1^2\omega_2\omega_1.
\end{aligned}$$

We find that

$$\nu(\sigma^2(\omega_1), \nu(\omega_2, \nu(\omega_1, \omega_1))) = \nu(\nu(\sigma(\omega_1), \omega_2), \sigma(\nu(\omega_1, \omega_1))).$$

Therefore, (W, ν, σ) is a Hom-Jordan algebra.

(2)For any $\omega_1, \omega_2, \omega_3 \in W, k_1, k_2 \in \mathbb{F}$, we have

$$\begin{aligned}
& \nu'(\omega_1, \omega_1) = \omega_1\omega_1 - \omega_1\omega_1 = 0, \\
& \nu'(k_1\omega_1 + k_2\omega_2, \omega_3) \\
= & (k_1\omega_1 + k_2\omega_2)\omega_3 - \omega_3(k_1\omega_1 + k_2\omega_2) \\
= & k_1(\omega_1\omega_3 - \omega_3\omega_1) + k_2(\omega_2\omega_3 - \omega_3\omega_2) \\
= & k_1\nu'(\omega_1, \omega_3) + k_2\nu'(\omega_2, \omega_3). \\
& \nu'(\sigma(\omega_1), \nu'(\omega_2, \omega_3)) + \nu'(\sigma(\omega_2), \nu'(\omega_3, \omega_1)) + \nu'(\sigma(\omega_3), \nu'(\omega_1, \omega_2)) \\
= & \alpha\omega_1\omega_2\omega_3 - \alpha\omega_1\omega_3\omega_2 - \alpha\omega_2\omega_3\omega_1 + \alpha\omega_3\omega_2\omega_1 \\
& +\alpha\omega_2\omega_3\omega_1 - \alpha\omega_2\omega_1\omega_3 - \alpha\omega_3\omega_1\omega_2 + \alpha\omega_1\omega_3\omega_2 \\
& +\alpha\omega_3\omega_1\omega_2 - \alpha\omega_3\omega_2\omega_1 - \alpha\omega_1\omega_2\omega_3 + \alpha\omega_2\omega_1\omega_3 \\
= & 0.
\end{aligned}$$

Therefore, (W, ν', σ) is a Hom-Lie algebra. □

Example 3.3 *Let (V, m, α) be a Hom-associative algebra over a field $\mathbb{F}$, ι is an anti-automorphism of V and $\iota^2 = id_V$, then the characteristic subspace $E_1(\iota) = \{x|\iota(x) = x\}$ of V is a Hom subalgebra of the Hom-Jordan algebra (V, μ, α), which is structured by Ex 3.1.*

Proof. First, $E_1(\iota)$ is a subspace of V. For any $x, y \in E_1(\iota)$, we have

$$\begin{aligned}
\iota(\mu(x, y)) &= \iota(\frac{1}{2}(m(x, y) + m(y, x))) \\
&= \frac{1}{2}(\iota(m(x, y)) + \iota(m(y, x))) \\
&= \frac{1}{2}(m(\iota(y), \iota(x)) + m(\iota(x), \iota(y))) \\
&= \frac{1}{2}(m(y, x) + m(x, y)) \\
&= \mu(x, y).
\end{aligned}$$

For any $x \in E_1(\iota)$, since ι is a morphism of Hom-Jordan algebra, we have

$$\iota \circ \alpha = \alpha \circ \iota,$$

then

$$\iota(\alpha(x)) = \alpha(\iota(x)) = \alpha(x) \in E_1(\iota).$$

Therefore, $E_1(\iota)$ is a Hom subalgebra of V^+. □

Theorem 3.4 *Given two Hom-Jordan algebras* (V,μ,α) *and* $(V^{'},\mu^{'},\beta)$*, there is a Hom-Jordan algebra* $(V\oplus V^{'},\mu^{''},\alpha+\beta)$*, where the multiplication* $\mu^{''}:(V\oplus V^{'})\times(V\oplus V^{'})\to V\oplus V^{'}$ *is given by*

$$\mu^{''}(x+x^{'},y+y^{'})=\mu(x,y)+\mu^{'}(x^{'}+y^{'}),\forall x\in V,y\in V,x^{'}\in V^{'},y^{'}\in V^{'},$$

and the linear map $(\alpha+\beta):V\oplus V^{'}\to V\oplus V^{'}$ *is given by*

$$(\alpha+\beta)(x+x^{'})=\alpha(x)+\beta(x^{'}),\forall x\in V,x^{'}\in V^{'}.$$

Proof. First, for any $x,y\in V$, $x^{'},y^{'}\in V^{'}$, we have

$$\mu^{''}(x+x^{'},y+y^{'})=\mu(x,y)+\mu^{'}(x^{'}+y^{'}),$$

$$\mu^{''}(y+y^{'},x+x^{'})=\mu(y,x)+\mu^{'}(y^{'}+x^{'})=\mu(x,y)+\mu^{'}(x^{'}+y^{'}).$$

Since (V,μ,α) and $(V^{'},\mu^{'},\beta)$ are Hom-Jordan algebras, we have

$$\mu(\alpha^2(x),\mu(y,\mu(x,x)))=\mu(\mu(\alpha(x),y),\alpha(\mu(x,x))),$$

$$\mu^{'}(\beta^2(x^{'}),\mu^{'}(y^{'},\mu^{'}(x^{'},x^{'})))=\mu^{'}(\mu^{'}(\beta(x^{'}),y^{'}),\beta(\mu^{'}(x^{'},x^{'}))).$$

By a direct computation, for any $x+x^{'},y+y^{'}\in V\oplus V^{'}$ we have

$$\begin{aligned}
& \mu^{''}((\alpha+\beta)^2(x+x^{'}),\mu^{''}(y+y^{'},\mu^{''}(x+x^{'},x+x^{'})))\\
= & \mu^{''}((\alpha+\beta)(\alpha(x)+\beta(x^{'})),\mu^{''}(y+y^{'},\mu(x,x)+\mu^{'}(x^{'},x^{'})))\\
= & \mu^{''}(\alpha^2(x)+\beta^2(x^{'}),\mu(y,\mu(x,x))+\mu^{'}(y^{'},\mu^{'}(x^{'},x^{'})))\\
= & \mu(\alpha^2(x),\mu(y,\mu(x,x)))+\mu^{'}(\beta^2(x^{'}),\mu^{'}(y^{'},\mu^{'}(x^{'},x^{'})))\\
= & \mu(\mu(\alpha(x),y),\alpha(\mu(x,x)))+\mu^{'}(\mu^{'}(\beta(x^{'}),y^{'}),\beta(\mu^{'}(x^{'},x^{'})))\\
= & \mu^{''}(\mu(\alpha(x),y)+\mu^{'}(\beta(x^{'}),y^{'}),\alpha(\mu(x,x))+\beta(\mu^{'}(x^{'},x^{'})))\\
= & \mu^{''}(\mu^{''}(\alpha(x)+\beta(x^{'}),y+y^{'}),(\alpha+\beta)(\mu(x,x)+\mu^{'}(x^{'},x^{'})))\\
= & \mu^{''}(\mu^{''}((\alpha+\beta)(x+x^{'}),y+y^{'}),(\alpha+\beta)(\mu^{''}(x+x^{'},x+x^{'}))).
\end{aligned}$$

Thus $(V\oplus V^{'},\mu^{''},\alpha+\beta)$ is a Hom-Jordan algebra. □

Theorem 3.5 *A map* $\phi:(V,\mu,\alpha)\to(V^{'},\mu^{'},\beta)$ *is a morphism of Hom-Jordan algebras if and only if the graph* $\mathfrak{G}_\phi\subset V\oplus V^{'}$ *is a Hom subalgebra of* $(V\oplus V^{'},\mu^{''},\alpha+\beta)$.

Proof. Let $\phi:(V,\mu,\alpha)\to(V,\mu^{'},\beta)$ be a morphism of Hom-Jordan algebras, then for any $x,y\in V$, we have

$$\mu^{''}(x+\phi(x),y+\phi(y))=\mu(x,y)+\mu^{'}(\phi(x),\phi(y))=\mu(x,y)+\phi(\mu(x,y))\in\mathfrak{G}_\phi.$$

Thus the graph $\mathfrak{G}_\phi$ is closed under the multiplication $\mu^{''}$. By (6), we have

$$(\alpha+\beta)(x+\phi(x))=\alpha(x)+\beta\circ\phi(x)=\alpha(x)+\phi\circ\alpha(x),$$

which implies that $(\alpha+\beta)(\mathfrak{G}_\phi) \subset \mathfrak{G}_\phi$. Thus $\mathfrak{G}_\phi$ is a Hom subalgebra of $(V \oplus V^{'}, \mu^{''}, \alpha+\beta)$.

Conversely, if the graph $\mathfrak{G}_\phi \subset V \oplus V^{'}$ is a Hom subalgebra of $(V \oplus V^{'}, \mu^{''}, \alpha+\beta)$, then we have

$$\mu^{''}(x+\phi(x), y+\phi(y)) = \mu(x,y) + \mu^{'}(\phi(x), \phi(y)) \in \mathfrak{G}_\phi,$$

which implies that

$$\phi(\mu(x,y)) = \mu^{'}(\phi(x), \phi(y)).$$

Furthermore, $(\alpha+\beta)(\mathfrak{G}_\phi) \subset \mathfrak{G}_\phi$ yields that

$$(\alpha+\beta)(x+\phi(x)) = \alpha(x) + \beta \circ \phi(x) \in \mathfrak{G}_\phi,$$

which is equivalent to the condition $\beta \circ \phi(x) = \phi \circ \alpha(x)$, i.e. $\beta \circ \phi = \phi \circ \alpha$. Therefore, ϕ is a morphism of Hom-Jordan algebras. □

ACKNOWLEDGEMENTS. The author thanks Professor Liangyun Chen for his helpful comments and suggestions. The author also give her special thanks to referees for many helpful suggestions.

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Received: July, 2016