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Conformable Fractional Integral Equations of the Second Kind

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Abstract

In this paper, we prove the existence and uniqueness of the solution involving the conformable fractional integral equation and give error estimates of the approximations using the contraction principle.

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1 Introduction

The concepts of fractional integral assume various forms not always equivalent and also not compatible with each other and play a vital role in the theory of the most of the scientific areas such as physics and applied sciences.

Fractional integral equations are studied in various fields of physics and engineering, specifically in signal processing, control engineering, biosciences, fluid mechanics, diffusion processes and dynamic of viscoelastic material, as shown in [1, 2, 3, 4, 5, 6, 7, 8, 9].

These types of integral equations naturally appear in certain modeling and theoretical problems, for example, porous medium equations [10], and numerical analysis [11], among other applications. Moreover the theory of integral equations is rapidly developing using the tools of functional analysis, topology and fixed point theory. The theory of such integral equations is developed intensively in recent years together with the theory of differential equations of fractional order

The Banach contraction principle [12] provides the most simple and efficient tools in fixed point theory. In this paper, we considerer a fractional integral equation of the type

$$x(t) = a(t)\int_0^t b(u)x(u)d_\alpha u + f(t), \quad t \in [0,T], \tag{1}$$

where $\frac{1}{2} < \alpha < 1$ and $a, b, f : [0,T] \to \mathbb{R}$ are continuous functions.

To solve (1), we employ fixed point theory. We recall the main result for fixed points of an operator on a Banach space.

2 Basic definitions and tools

Conformable fractional integral was first introduced by Khalil et al. (2015) [13] as a generalization of n-fold integral and developed by Abdeljawad [14]; their definition is presented below.

Definition 2.1 (Fractional Integral)**.** *The (left) conformable fractional integral of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ of a function $f \in L^1_\alpha[a,b]$ is defined by*

$$I^a_\alpha f(t) = \int_a^t f(u)\, d^a_\alpha u = \int_a^t f(u)(u-a)^{\alpha-1}\, du, \tag{2}$$

if the Riemann improper integral exists.

When $a = 0$ we write I_α and $d_\alpha u$. The operator I^a_α is called conformable (left) fractional integral of order $\alpha \in (0,1)$.

Remark 2.1. *Note that the relation between the Riemann integral and the conformable fractional integral is given by*

$$I^a_\alpha f(t) = I^a_1\big(t^{\alpha-1}f(t)\big) = \int_a^t f(u)u^{\alpha-1}\, du. \tag{3}$$

Remark 2.2. *When $\alpha \to 1$ in Eq. (2), the conformable fractional integrals reduce to ordinary first order integrals.*

Definition 2.2 (Contraction mapping)**.** *Let $(X, ||\,.\,||)$ be a Banach space and let $T : X \to X$ be a self-mapping. Then T is said to be k-contraction if there exists a constant $k \in [0,1)$ such that*

$$||Tx - Ty|| \leq k||x-y||, \quad \textit{for all } x, y \in X. \tag{4}$$

We normally refer to the infimum of all k values satisfying Eq. (4) as the contraction factor of T.

The existence results will be based on the following fixed-point theorems and definitions.

Theorem 2.1 (Banach Fixed Point Theorem [12])**.** *Let $(X, ||\centerdot||)$ be a Banach space and let $T : X \to X$ be a k-contraction. Then*

(a) $Tx = x$ has exactly one solution, that is, T has exactly one fixed point $x^* \in X$.

(b) The sequence $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$, is convergent to x^*, for any arbitrary choice of initial point $x_0 \in X$. In other words, the fixed point x^* is globally attractive.

(c) The error estimate

$$||x_n - x^*|| \leq \frac{k^n}{1-k}||x_1 - x_0|| \tag{5}$$

holds for every $n \in \mathbb{N}$.

Lemma 2.2. *Let $T : X \to X$ be a contraction mapping on a Banach space and $M \subseteq X$ be a closed subset such that $f(M) \subseteq M$. Then, the unique fixed point of f is in M.*

Let $||\centerdot||_{B,s} : C[0,T] \to \mathbb{R}_+$ be the Bielecki norm,

$$||x||_{B,s} = \max_{t\in[0,T]} |x(t)|e^{-st}, \tag{6}$$

for some suitable $s > 0$, and let $||\centerdot||_C$ be the Chebyshev norm on $C[0,T]$, defined by $||x||_C = \max_{t\in[0,T]} |x(t)|$. It is easy to show that $||\centerdot||_{B,s}$ and $||\centerdot||_C$ are a norms of $C[0,T]$. Moreover, $(X, ||\centerdot||_{B,s})$ and $(X, ||\centerdot||_C)$ are Banach spaces.

3 Main Results

In this section, we discuss th existence and uniqueness of the solutions of conformable fractional integral equations.

For Eq. (1), we define the associated integral operator T by

$$Tx(t) = a(t)\int_0^t b(u)x(u)\, d_\alpha u + f(t), \quad t \in [0,T], \tag{7}$$

where $a, b, f \in X = C[0,T]$.

By this construction, $T\big(X\big) \subset X$, so $T : X \to X$ is well defined [15, 16, 17]. Observe that problem (1) has solution if the operator (7) has fixed point. We have:

Theorem 3.1. *Let $T : (X, ||\cdot||_{B,s}) \to (X, ||\cdot||_{B,s})$ be defined by Eq. (7), with the constant s chosen so that*

$$s \geq \left(\frac{2T^{2\alpha-1}||a||_C \cdot ||b||_C}{2\alpha - 1} \right)^{\frac{1}{1-\alpha}}, \quad \alpha \in \big(\frac{1}{2}, 1\big). \tag{8}$$

Choose R satisfying $R \geq \max\{-R_1, R_2\}$, where $R_1 = \min_{t\in[0,T]} f(t)$, $R_2 = \max_{t\in[0,T]} f(t)$. Then

(a) $Tx = x$ has exactly one solution $x^* \in \overline{B}_R(f) := \big\{x \in X \big| ||x - f||_{B,s} \leq R\big\}$;

(b) for any arbitrary initial point $x_0 \in \overline{B}_R(f)$, the sequence $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$, converges to solution x^*;

(c) for every $n \in \mathbb{N}$, the error estimate

$$||x_n - x^*|| \leq \frac{k^n}{1-k}||x_1 - x_0|| \tag{9}$$

holds for every $n \in \mathbb{N}$.

Proof. First, let us show that $T\big(\overline{B}_R(f)\big) \subseteq \overline{B}_R(f) \subseteq X$. Let $x \in \overline{B}_R(f)$. Since $||x - f||_{B,s} \leq R$, we have, for every $t \in [0, T]$,

$$R_1 - Re^{st} \leq f(t) - Re^{st} \leq x(t) \leq f(t) + Re^{st} \leq R_2 + Re^{st}. \tag{10}$$

Multiplying by e^{-st}, using fact that $e^{-sT} \leq e^{-st} \leq 1$, it follows that

$$-2R \leq x(t)e^{-st} \leq 2R, \tag{11}$$

so $||x||_{B,s} \leq 2R$. Now, fix $t \in [0, t]$. We have

$$\begin{aligned} |Tx(t) - f(t)| &\leq |a(t)| \int_0^t |b(u)|\,|x(u)|\,|u^{\alpha-1}|\,du && (12)\\ &\leq ||a||_C \cdot ||b||_C \cdot ||x||_{B,s} \int_0^t u^{\alpha-1}e^{su}\,du && (13) \end{aligned}$$

Making the change of variables $w = su$, $0 \leq w \leq st$, we get

$$|Tx(t) - f(t)| \leq ||a||_C \cdot ||b||_C \cdot ||x||_{B,s}\frac{1}{s^\alpha}\int_0^{st} w^{\alpha-1}e^w\,dw. \tag{14}$$

Note that, by the well-known Hölder's inequality, for $\alpha \in (\frac{1}{2}, 1)$ we have

$$\int_0^{st} w^{\alpha-1}e^w\,dw \leq \left(\int_0^{st} w^{2(\alpha-1)}\,dw\right)^{\frac{1}{2}} \left(\int_0^{st} e^{2w}\,dw\right)^{\frac{1}{2}} \leq \frac{(sT)^{2\alpha-1}}{2\alpha-1}e^{st}. \tag{15}$$

Then

$$
\begin{aligned}
|Tx(t) - f(t)| &\leq ||a||_C \cdot ||b||_C \cdot 2R \frac{T^{2\alpha-1}}{2\alpha - 1} e^{st} s^{\alpha-1} &(16)\\
&\leq Re^{st}, &(17)
\end{aligned}
$$

by Eqs. (8) and (11). Then $||Tx - f||_{B,s} \leq R$ and $T\big(\overline{B}_R(f)\big) \subseteq \overline{B}_R(f)$.

Next, for every fixed $x, y \in X$, similar computations lead to

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\leq ||a||_C \cdot ||b||_C \left| \int_0^t |x(u) - y(u)| \, |u^{\alpha-1}| \, du \right| &(18)\\
&\leq ||a||_C \cdot ||b||_C \cdot ||x - y||_{B,s} \frac{T^{2\alpha-1}}{2\alpha - 1} e^{st} s^{\alpha-1} &(19)
\end{aligned}
$$

So, again, by (8), T is a contraction with constant

$$
k = ||a||_C \cdot ||b||_C \frac{T^{2\alpha-1}}{2\alpha - 1} s^{\alpha-1} < 1. \tag{20}
$$

It easily shows that all the hypotheses of Theorem (2.1) and Lemma (2.2) are satisfied and hence the mapping has a fixed point that is a solution in closed ball $M = \overline{B}_R(f)$ of the integral equation (1). □

Remark 3.1. *The above proof gives a constructive method to find a sequence of Picard iterations that converges to the exact solution of the fractional integral equation* (1)*.*

Theorem 3.2. *Assume the conditions of Theorem* (3.1) *are satisfied. If, in addition, $a, b, f \in C^2[0, T]$, then $x^* \in C^2[0, T]$, also.*

4 Concluding Remark

There are many results devoted to the well-known integral equation that in most cases extremely difficult. In this paper, we give results for integral equation containing conformable fractional integral. We used Picard iteration for finding fixed points of a singular integral operator (Banachs fixed point theorem). The presented idea may stimulate further research in the theory of conformable fractional integrals.

Conflicts of Interest: The author declares no conict of interest.

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