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Composition Operators From Zygmund Spaces Into $Q_{k,\omega}(p,q)$ Spaces

Jianren LONG

School of Mathematics and Computer Science, Guizhou Normal University, Guiyang, Guizhou, 550001, P.R.China

Deiqiang SHANG

College of Science, Guizhou Minzu University, Guiyang, Guizhou, 550025, P.R.China

Abstract

The purpose of this paper is to study composition operators on Zygmund spaces and $Q_{k,\omega}(p,q)$ spaces. Moreover, we study the boundedness and compactness of the composition operator from Zygmund spaces into $Q_{k,\omega}(p,q)$ and $Q_{k,\omega,0}(p,q)$ spaces.

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1 Introduction

Let $\mathbf{D} = \{z : |z| < 1\}$ be the open unit disc in the complex plane $\mathbf{C}$, and let $H(\mathbf{D})$ be the class of all analytic functions on $\mathbf{D}$. An f in $H(\mathbf{D})$ is said to belong to the Zygmund space, denoted by $\mathbf{Z}$, if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbf{D}$ and $h > 0$. By Theorem 5.3 in [4], we see that $f \in \mathbf{Z}$ if and only if

$$\|f\|_{\mathbf{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbf{D}}(1 - |z|^2)|f''(z)| < \infty.$$

It is easy to check that $\mathbf{Z}$ is a Banach space under the above norm. Let $\mathbf{Z}_0$ denote the subspace of $\mathbf{Z}$ consisting of those $f \in \mathbf{Z}$ for which

$$\lim_{|z|\to 1}(1 - |z|^2)|f''(z)| = 0.$$

The space $\mathbf{Z}_0$ is called the little Zygmund space. Throughout this paper, the closed unit ball in $\mathbf{Z}$ and $\mathbf{Z}_0$ will be denoted by $\mathbf{B_Z}$ and $\mathbf{B_{Z_0}}$, respectively.

For $a \in \mathbf{D}$, the Mӧbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \ for \ z \in \mathbf{D}.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a^{'}(z)|.$$

Note that $\varphi_a(\varphi_a(z)) = z$ and $\varphi_a^{-1}(z) = \varphi_a(z)$. Denote by

$$g(z,a) = \log|\frac{1-\bar{a}z}{a-z}| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of $\mathbf{D}$ with logarithmic singularity at $a \in \mathbf{D}$.

The α-Bloch space B^α $(0 < \alpha < \infty)$ is, by definition, the set of all function f in $H(\mathbf{D})$ such that

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in D}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

Under the above norm, B^α is a Banach space (see, [25]). When α=1, $B^1 = B$ is the well-known Bloch space. Let B_0^α denote the subspace of B^α, for f

$$B_0^\alpha = \{f : (1-|z|^2)^\alpha|f'(z)| \to 0 \ \ as \ \ |z| \to 1, f \in B^\alpha\}.$$

This space is called the little α-Bloch space.

Now, given a reasonable function $\omega : (0,1] \to [0,\infty)$, the weighted Bloch space B_ω (see, [5]) is defined as the set of all analytic functions f on $\mathbf{D}$ satisfying

$$(1-|z|^2)|f'(z)| \le C\omega(1-|z|), \ z \in \mathbf{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1$, B_ω reduces to the classical Bloch space B. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.

In [14], the authors introduce the following definitions.

Definition 1.1 *For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbf{D}$ is said to belong to the $\alpha-$weighted Bloch space B_ω^α if*

$$\|f\|_{B_\omega^\alpha} = \sup_{z\in\mathbf{D}}\frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| < \infty.$$

In the special case where $\omega \equiv 1$, B^{α}_{ω} reduces to the classical $\alpha-$Bloch space B^{α}.

Definition 1.2 *For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbf{D}$ is said to belong to the little weighted Bloch space $B^{\alpha}_{\omega,0}$ if*

$$\|f\|_{B^{\alpha}_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}|f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$. Now we introduce a particular class of Möbius-invariant function spaces, the so-called $Q_{k,\omega}(p,q)$ spaces, has attracted a lot of attention in recent years. It defined as follows(see [13]).

Definition 1.3 *For a nondecreasing function $K : [0,\infty) \to [0,\infty)$, $0 < p < \infty, -2 < q < \infty$ and for a given reasonable function $\omega : (0,1] \to (0,\infty)$, an analytic function f on $\mathbf{D}$ is said to belong to the $Q_{k,\omega}(p,q)$ if*

$$\|f\|_{k,\omega,p,q} = \{\sup_{z\in\mathbf{D}} \int_D |f'(z)|^p(1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)\}^{\frac{1}{p}} < \infty,$$

where dA denotes the normalized Lebesgue area measure on $\mathbf{D}$. When $p > 1$, $Q_{k,\omega}(p,q)$ is a Banach space under the norm $|f(0)| + \|f\|_{k,\omega,p,q}$.

Remark 1. It should be remarked that our $Q_{k,\omega}(p,q)$ classes are more general than many classes of analytic functions. If $\omega \equiv 1$, we obtain $Q_k(p,q)$ type spaces(see, [19-20]). If $p = q = 2$, and $\omega(t) = t$, we obtain Q_k spaces as studied in [6] and others. If $p = q = 2$, $\omega(t) = t$ and $K(t) = t^p$, we obtain Q_p spaces as studied in [21] and others. If $\omega \equiv 1$ and $K(t) = t^s$, then $Q_{k,\omega}(p,q) = F(p,q,s)$ classes (see, [1, 24]).

We say that $f \in Q_{k,\omega,0}(p,q)$ if

$$\lim_{|a|\to 1^-} \int_D |f'(z)|^p(1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z) = 0.$$

In [13], the author collected the following immediate relations of $Q_{k,\omega}(p,q)$ and $Q_{k,\omega,0}(p,q)$

(i) $Q_{k,\omega}(p,q) \subset B_{\omega}^{\frac{q+2}{p}}$ and $Q_{k,\omega,0}(p,q) \subset B_{\omega,0}^{\frac{q+2}{p}}$.

(ii) $Q_{k,\omega}(p,q) = B_{\omega}^{\frac{q+2}{p}}$ (or $Q_{k,\omega,0}(p,q) = B_{\omega,0}^{\frac{q+2}{p}}$), iff

$$\int_0^1 K(\log\frac{1}{r})\frac{r}{(1-r^2)^2}dr < \infty.$$

Let φ be an analytic self-map of $\mathbf{D}$. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)),\ f \in H(\mathbf{D}).$$

It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. For a study of the composition operators, see [2] and [15]. The composition operator from Bloch spaces to Q_k and $Q_{k,0}$ was investigated in [18, 23]. Some characterizations of the boundedness and compactness of the composition operator, as well as Volterra type operator, on Zygmund space can be found in [2, 9-11, 17]. The purpose of this paper is to study the boundedness and compactness of the operator C_φ from the Zygmund spaces and little Zygmund space into $Q_{k,\omega}(p,q)$ and $Q_{k,\omega,0}(p,q)$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $\frac{B}{C} \leq A \leq CB$.

2 Main results and proofs

In this section, we state and prove our main results. In order to formulate our main results, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma can be proved in a standard way (see, e.g., Theorem 3.11 in [3]).

Lemma 2.1 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then $C_\varphi : \mathbf{Z} \to Q_{k,\omega}(p,q)$ is compact if and only if $C_\varphi : \mathbf{Z} \to Q_{k,\omega}(p,q)$ is bounded and for every bounded sequence (f_n) in $\mathbf{Z}$ which converges to 0 uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$, we have $\lim_{n\to\infty} \|C_\varphi f_n\|_{K,\omega,p,q} = 0$.*

By using the methods of [18] (see also [8]), we can obtain the following lemma. Since the proof is similar, we omit the details.

Lemma 2.2 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. If $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega}(p,q)$ is compact, then for any $\varepsilon > 0$, there exists a δ, $0 < \delta < 1$, such that for all f in $\mathbf{Z}(\mathbf{Z}_0)$,*

$$\sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |f'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \varepsilon \tag{1}$$

holds whenever $\delta < r < 1$.

By modifying the proof of Theorem 3.1 of [12] (or see [16]), we can prove the following lemma, we omit the details.

Lemma 2.3 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then $C_\varphi : \mathbf{Z} \to Q_{k,\omega,0}(p,q)$ is compact if and only if $C_\varphi : \mathbf{Z} \to Q_{k,\omega,0}(p,q)$ is bounded and*

$$\lim_{|a|\to 1} \sup_{\|f\|_{\mathbf{Z}}\le 1} \int_{\mathbf{D}} |(C_\varphi f)'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) = 0. \tag{2}$$

Lemma 2.4 [9] *Suppose that $f \in \mathbf{Z}_0$, then*

$$\lim_{|z|\to 1} \frac{|f'(z)|}{\ln \frac{e}{1-|z|^2}} = 0. \tag{3}$$

Lemma 2.5 [26] *Suppose that (n_k) is an increasing sequence of positive integers satisfying $\frac{n_{k+1}}{n_k} \ge \lambda > 1$ for all $k \in \mathbf{N}$. Let $0 < p < \infty$. Then there are two positive constants C_1 and C_2, depending only on p and λ such that*

$$C_1(\Sigma_{k=1}^{\infty}|a_k|^2)^{\frac{1}{2}} \le (\frac{1}{2\pi}\int_0^{2\pi} |\Sigma_{k=1}^{\infty} a_k e^{in_k\theta}|^p d\theta)^{\frac{1}{p}} \le C_2(\Sigma_{k=1}^{\infty}|a_k|^2)^{\frac{1}{2}}.$$

Now we are in a position to state and prove our main results in this paper.

Theorem 2.6 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then the following statements hold.*

(I) If

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi'(z)|^p (\ln \frac{e}{1-|\varphi(z)|^2})^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty, \tag{4}$$

then $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega}(p,q)$ is bounded.

(II) If $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega}(p,q)$ is bounded, then

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi'(z)|^p (\ln \frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty. \tag{5}$$

Proof. (I). Let $f \in \mathbf{Z}$. Then by the following result (see, [9]):

$$|f'(z)| \le C\|f\|_Z \ln \frac{e}{1-|z|^2}, \tag{6}$$

we have

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|(C_\varphi f)'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$=\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le C\|f\|_{\mathbf{Z}}^p\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|\varphi'(z)|^p(\ln\frac{e}{1-|\varphi(z)|^2})^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\infty.$$

In addition, by the well-known fact that $\|f\|_\infty\le C\|f\|_{\mathbf{Z}}$, we obtain

$$|f(\varphi(0))|\le C\|f\|_{\mathbf{Z}}.$$

Therefore, $C_\varphi:\mathbf{Z}\to Q_{k,\omega}(p,q)$ is bounded, and hence $C_\varphi:\mathbf{Z}_0\to Q_{k,\omega}(p,q)$ is bounded.

(II). First, we suppose that $C_\varphi:\mathbf{Z}\to Q_{k,\omega}(p,q)$ is bounded. Let $g(z)=z\in\mathbf{Z}$. By the boundedness of $C_\varphi:\mathbf{Z}\to Q_{k,\omega}(p,q)$, we have that $\varphi=C_\varphi g\in Q_{k,\omega}(p,q)$. Hence, we have

$$\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|\le\frac{1}{\sqrt{e}}}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le(\ln\frac{e}{e-1})^{\frac{p}{2}}\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|\le\frac{1}{\sqrt{e}}}|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le(\ln\frac{e}{e-1})^{\frac{p}{2}}\sup_{\mathbf{D}}\int_{a\in\mathbf{D}}|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\infty. \tag{7}$$

For $z\in\mathbf{D}$, such that $|z|=r\ge\frac{1}{\sqrt{e}}$. Let

$$f(z)=\sum_{k=0}^\infty\frac{1}{2^k+1}z^{2^k+1}.$$

Then by the fact that $p(z)=\sum_{k=0}^\infty z^{2^k}$ belongs to Bloch space (see, [22, Theorem 1]) and the relationship of Bloch function and Zygmund function, we see that $f\in\mathbf{Z}$. Let

$$h_\theta(z)=f(e^{i\theta}z)=\sum_{k=0}^\infty\frac{1}{2^k+1}(e^{i\theta}z)^{2^k+1}.$$

Then $h_\theta\in\mathbf{Z}$ and $\|h_\theta\|_{\mathbf{Z}}=\|f\|_{\mathbf{Z}}$. We have

$$\infty>\|C_\varphi\|^p\|h_\theta\|_{\mathbf{Z}}^p\ge\|C_\varphi h_\theta\|_{K,\omega,p,q}^p$$

$$\geq \sup_{a\in \mathbf{D}}\int_{\mathbf{D}}|(C_\varphi h_\theta)'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\geq \sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}|h_\theta'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\geq \sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}|\sum_{k=0}^{\infty}e^{i(2^k+1)\theta}\varphi^{2^k}(z)|^p|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z). \quad (8)$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^p\|h_\theta\|_{\mathbf{Z}}^p d\theta=\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^p\|f\|_{\mathbf{Z}}^p d\theta=\|C_\varphi\|^p\|f\|_{\mathbf{Z}}^p=\|C_\varphi\|^p\|h_\theta\|_{\mathbf{Z}}^p,$$

by (8), Lemma 2.5 and Fubini's theorem we have

$$\infty>\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^p\|h_\theta\|_{\mathbf{Z}}^p d\theta$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi}\sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}|\sum_{k=0}^{\infty}e^{i(2^k+1)\theta}\varphi^{2^k}(z)|^p|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)d\theta$$

$$=\sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}\{\frac{1}{2\pi}\int_0^{2\pi}|\sum_{k=0}^{\infty}e^{i(2^k+1)\theta}\varphi^{2^k}(z)|^p d\theta\}|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\geq \sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}(\sum_{k=0}^{\infty}|\varphi(z)|^{2^{k+1}})^{\frac{p}{2}}|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z).$$

For any $r\in(0,1)$, from [7] we have

$$\ln\frac{1}{1-r^2}\leq\sum_{k=0}^{\infty}r^{2^{k+1}}, \quad (9)$$

since the number of terms in the sum from 2^k to $2^{k+1}-1$ is 2^k. Therefore,

$$\infty>\frac{1}{2\pi}\int_0^{2\pi}\|C_\varphi\|^p\|h_\theta\|_{\mathbf{Z}}^p d\theta$$

$$\geq \sup_{a\in \mathbf{D}}\int_{|\varphi(z)|>\frac{1}{\sqrt{e}}}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z), \quad (10)$$

which together with (7) implies that (5) holds.

Now suppose that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$ is bounded. Take the function $f(z)$ given by the above. Set

$$f_r(z) = f(rz) = \sum_{k=0}^{\infty} \frac{1}{2^k+1}(rz)^{2^k+1},\ r \in (0,1).$$

Then $f_r \in \mathbf{Z}_0$. Then, as argued the same with the case of $C_\varphi : \mathbf{Z} \to Q_{k,\omega}(p,q)$ and $r \to 1$, we get the desired result. The proof of the theorem 2.6 is completed.

Theorem 2.7 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then the following statements hold.*

(I) If $\varphi \in Q_{k,\omega}(p,q)$ and

$$\lim_{r\to 1}\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>r} |\varphi'(z)|^p(\ln\frac{e}{1-|\varphi(z)|^2})^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z) = 0, \quad (11)$$

then $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega}(p,q)$ is compact.

(II) If $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega}(p,q)$ is compact, then $\varphi \in Q_{k,\omega}(p,q)$ and

$$\lim_{r\to 1}\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>r} |\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z) = 0. \quad (12)$$

Proof. (I). Assume that $\varphi \in Q_{k,\omega}(p,q)$ and (11) holds. Let $(f_k)_{k\in N}$ be a bounded sequence in $\mathbf{Z}$ which converges to 0 uniformly on compact subsets of $\mathbf{D}$. We need to show that $(C_\varphi f_k)$ converges to 0 in $Q_{k,\omega}(p,q)$ norm. By (11), for any given $\varepsilon > 0$, there is an $r \in (0,1)$, such that

$$\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>r} |\varphi'(z)|^p(\ln\frac{e}{1-|\varphi(z)|^2})^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z) < \varepsilon.$$

Therefore, by (6), we have

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}} |(C_\varphi f_k)'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$= \sup_{a\in\mathbf{D}}\{\int_{|\varphi(z)|>r} + \int_{|\varphi(z)|\le r}\}|f_k'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le C\|f_k\|_{\mathbf{Z}}^p\varepsilon + \sup_{|\nu|\le r}|f_k'(\nu)|^p\sup_{a\in\mathbf{D}}\int_{\mathbf{D}} |\varphi'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z). \quad (13)$$

From the assumption, we see that (f_k') also converges to 0 uniformly on compact subsets of $\mathbf{D}$ by Cauchy's estimates. It follows that $\|C_\varphi f_k\|_{K,\omega,p,q} \to 0$ since $|f_k(\varphi(0))| \to 0$ and $\sup_{|\nu|\le r} |f_k'(\nu)| \to 0$ as $k \to \infty$. By Lemma 2.1, $C_\varphi : \mathbf{Z} \to Q_{k,\omega}(p,q)$ is compact, and hence $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$ is also compact.

(II). We only need to prove the case of $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$. Assume that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$ is compact. By taking $g(z) = z \in \mathbf{Z}_0$ we get $\varphi \in Q_{k,\omega}(p,q)$. Now we choose the function $f(z)$ given in the proof of Theorem 2.6. Then $f \in \mathbf{Z}$. Choose a sequence (λ_j) in $\mathbf{D}$ which converges to 1 as $j \to \infty$, and let $f_j(z) = f(\lambda_j z)$ for $j \in \mathbf{N}$. Then, $f_j \in \mathbf{Z}_0$ for all $j \in \mathbf{N}$ and $\|f_j\|_{\mathbf{Z}} \le C$. Let $f_{j,\theta}(z) = f_j(e^{i\theta} z)$. Then $f_{j,\theta} \in \mathbf{Z}_0$. Replace f by $f_{j,\theta}$ in (1) and then integrate both side with respect to θ. By Fubini's theorem and Lemma 2.5, we obtain

$$\varepsilon > \sup_{a\in\mathbf{D}} \frac{1}{2\pi} \int_{|\varphi(z)|>r} \Big(\int_0^{2\pi} |f_j'(e^{i\theta}\varphi(z))|^p d\theta\Big) |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$= \sup_{a\in\mathbf{D}} \frac{1}{2\pi} \int_{|\varphi(z)|>r} \int_0^{2\pi} |\sum_{k=0}^{\infty} (\lambda_j \varphi(z) e^{i\theta})^{2^k}|^p d\theta |\lambda_j|^p |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$= \sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} \Big(\sum_{k=0}^{\infty} |\lambda_j\varphi(z)|^{2^{k+1}}\Big)^{\frac{p}{2}} |\lambda_j|^p |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z). \quad (14)$$

From the proof of Theorem 2.6, for $\frac{1}{\sqrt{e}} < r < 1$ and for sufficiently large j, (14) gives

$$\sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>r} |\lambda_j|^p |\varphi'(z)|^p \Big(\ln \frac{1}{1-|\lambda_j\varphi(z)|^2}\Big)^{\frac{p}{2}} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \varepsilon.$$

By Fatou's Lemma, we get (12). The proof of the theorem 2.7 is completed.

Theorem 2.8 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then the following statements hold.*

(I) If $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega,0}(p,q)$ is bounded, then $\varphi \in Q_{k,\omega,0}(p,q)$ and

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi'(z)|^p \Big(\ln \frac{1}{1-|\varphi(z)|^2}\Big)^{\frac{p}{2}} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty. \quad (15)$$

(II) If $\varphi \in Q_{k,\omega,0}(p,q)$ and

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |\varphi'(z)|^p \Big(\ln \frac{e}{1-|\varphi(z)|^2}\Big)^{p} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty, \quad (16)$$

then $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega,0}(p,q)$ is bounded.

Proof. (I). Assume that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega,0}(p,q)$ is bounded. Then it is obvious that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$ is bounded. By Theorem 2.6, (15) holds. Taking $g(z) = z$ and using the boundedness of $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega,0}(p,q)$, we obtain $\varphi \in Q_{k,\omega,0}(p,q)$.

(II). Suppose that $\varphi \in Q_{k,\omega,0}(p,q)$ and (16) holds. From Theorem 2.6, we see that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega}(p,q)$ is bounded. In order to prove that $C_\varphi : \mathbf{Z}_0 \to Q_{k,\omega,0}(p,q)$ is bounded, it suffices to prove that $C_\varphi f \in Q_{k,\omega,0}(p,q)$, for any $f \in \mathbf{Z}_0$. Let $f \in \mathbf{Z}_0$. By Lemma 2.4, for every given $\varepsilon > 0$, we can choose $\rho \in (0,1)$ such that $|f'(\nu)| < \varepsilon \ln \frac{e}{1-|\nu|^2}$ for all $\nu \in \mathbf{D} - \rho\bar{\mathbf{D}}$. Then by (6), we have

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |(C_\varphi f)'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$= \lim_{|a|\to 1} \{\int_{|\varphi(z)|>\rho} + \int_{|\varphi(z)|\le\rho}\} |f'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$\le \varepsilon^p \sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>\rho} |\varphi'(z)|^p (\ln \frac{e}{1-|\varphi(z)|^2})^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$+C\|f\|^p_{\mathbf{Z}} (\ln \frac{e}{1-\rho^2})^p \lim_{|a|\to 1} \int_{|\varphi(z)|\le\rho} |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$\le \varepsilon^p \sup_{a\in\mathbf{D}} \int_{|\varphi(z)|>\rho} |\varphi'(z)|^p (\ln \frac{e}{1-|\varphi(z)|^2})^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)$$

$$+C\|f\|^p_{\mathbf{Z}} (\ln \frac{e}{1-\rho^2})^p \lim_{|a|\to 1} \int_{\mathbf{D}} |\varphi'(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z),$$

which together with the assumed conditions imply the desired result. The proof of Theorem 2.8 is completed.

Theorem 2.9 *Let $0 < p < \infty$, $-2 < q < \infty$, $\omega : (0,1] \to (0,\infty)$. Let K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ ia an analytic self-map of $\mathbf{D}$. Then the following statements hold.*

(I) If

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |\varphi'(z)|^p (\ln \frac{e}{1-|\varphi(z)|^2})^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) = 0, \qquad (17)$$

then $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega,0}(p,q)$ is compact.

(II) If $C_\varphi : \mathbf{Z}(\mathbf{Z}_0) \to Q_{k,\omega,0}(p,q)$ is compact, then

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |\varphi'(z)|^p (\ln \frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}} (1-|z|^2)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) = 0. \qquad (18)$$

Proof. (I). Assume that (17) holds. Set

$$h_{\varphi,\omega,K}(a)=\int_{\mathbf{D}}|\varphi'(z)|^p(\ln\frac{e}{1-|\varphi(z)|^2})^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z).$$

From the assumption, we have that for every given $\varepsilon>0$, there is a $s\in(0,1)$ such that for $|a|>s$, $h_{\varphi,\omega,K}(a)<\varepsilon$. Similarly to the proof of Lemma 2.3 of [16], we see that $h_{\varphi,\omega,K}(a)$ is continuous on $|a|\le s$. Therefore, $h_{\varphi,\omega,K}(a)$ is bounded on $\mathbf{D}$. From Theorem 2.6, we see that $C_\varphi:\mathbf{Z}\to Q_{k,\omega}(p,q)$ is bounded.

For any $f\in\mathbf{Z}$, by (6), we have

$$\int_{\mathbf{D}}|(C_\varphi f)'(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le C\|f\|_{\mathbf{Z}}^p\int_{\mathbf{D}}|\varphi'(z)|^p(\ln\frac{e}{1-|\varphi(z)|^2})^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z),\qquad(19)$$

which together with (17) imply $C_\varphi:\mathbf{Z}\to Q_{k,\omega,0}(p,q)$ is bounded. Fix $f\in\mathbf{B_Z}$. The right-hand side (19) tend to 0, as $|a|\to1$ by (17). From Lemma 2.3, we see that $C_\varphi:\mathbf{Z}\to Q_{k,\omega,0}(p,q)$ is compact, and hence $C_\varphi:\mathbf{Z}_0\to Q_{k,\omega,0}(p,q)$ is compact.

(II). We only need to prove the case of $C_\varphi:\mathbf{Z}_0\to Q_{k,\omega,0}(p,q)$ is compact. From the assumption, we see that $C_\varphi:\mathbf{Z}_0\to Q_{k,\omega,0}(p,q)$ is bounded and $C_\varphi:\mathbf{Z}_0\to Q_{k,\omega}(p,q)$ is compact. From Theorem 2.7 and 2.8, we have $\varphi\in Q_{k,\omega,0}(p,q)$ and

$$\lim_{|a|\to1}\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>r}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)=0.\quad(20)$$

Hence, for any given $\varepsilon>0$, there exists a $s\in(0,1)$ such that

$$\sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>s}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\varepsilon.\qquad(21)$$

Therefore, by (21) and the fact that $\varphi\in Q_{k,\omega,0}(p,q)$, we have

$$\lim_{|a|\to1}\int_{\mathbf{D}}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\le\lim_{|a|\to1}\int_{|\varphi(z)|>s}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$+\lim_{|a|\to1}\int_{|\varphi(z)|\le s}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$\leq \sup_{a\in\mathbf{D}}\int_{|\varphi(z)|>s}|\varphi'(z)|^p(\ln\frac{1}{1-|\varphi(z)|^2})^{\frac{p}{2}}(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)$$

$$+(\ln\frac{1}{1-s^2})^{\frac{p}{2}}\lim_{|a|\to 1}\int_{\mathbf{D}}|\varphi(z)|^p(1-|z|^2)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\varepsilon.$$

By the arbitrary of ε, we get the desired result. The proof of Theorem 2.9 is completed.

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