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Completeness criterion for a system of powers with degenerate coefficients in weighted spaces

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Abstract

System of powers with degenerate coefficients is considered. Completeness criterion for this system in the weighted Lebesgue spaces is found.

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1 Introduction

Let us consider the following system of powers

$$\left\{A^{+}(t)\,\omega^{+}(t)\,\varphi^{n}(t)\,;\;A^{-}(t)\,\omega^{-}(t)\,\bar{\varphi}^{n}(t)\right\}_{n\geq 0}, \tag{1}$$

where $A^{\pm}(t) \equiv \left|A^{\pm}(t)\right| e^{ia^{\pm}(t)}$ and $\varphi(t)$ are complex-valued functions on $[a, b]$ with the degenerate coefficients $\omega^{\pm}(\cdot)$:

$$\omega^{\pm}(t) \equiv \prod_{k=1}^{\tilde{r}^{\pm}} \left|t - \tilde{t}_k^{\pm}\right|^{\tilde{\beta}_k^{\pm}}, \left\{\tilde{t}_k^{\pm}\right\}_1^{\tilde{r}^{\pm}} \subset [a, b).$$

Basis properties (completeness, minimality, basicity) of the systems like (1) have been previously studied by many mathematicians (see, e.g., [1-4;11-13]). These systems are the natural generalizations of the classical exponential systems. For $\varphi(t) \equiv e^{it}$, the basicity (including completeness and minimality) of the system (1) in $L_p(-\pi, \pi)$ was studied in [4]. In case of no degeneration, the necessary and sufficient condition for the completeness and minimality of the system (1) in $L_p(a, b)$ was found in [3].

In this work, we present the completeness criterion for the system (1) in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(a, b)$, $1 < p < +\infty$, with the weight function $\rho : [a, b] \to (0, +\infty)$. Let us note that in unweighted case this question previously were studied in [14].

2 Assumptions and necessary information

We make the following assumptions

1) $[A^+(t)]^{\pm 1}; [A^-(t)]^{\pm 1}; [\varphi'(t)]^{\pm 1} \in L_\infty$;

2) $\Gamma = \varphi\{[a, b]\}$ is a simple closed $(\varphi(a) = \varphi(b))$ rectifiable Jordan curve. Γ is either a Radon curve (i.e. the angle $\theta_0(\varphi(t))$ between the tangent line to Γ at the point $\varphi = \varphi(t)$ and the real axis is a function of bounded variation on $[a, b]$), or a piecewise Lyapunov curve. Γ has a finite number of corner points and no cusps. Denote by φ_k the points of discontinuity of the function $\arg\varphi'(t)$ on $[a, b]$, $k = \overline{1, r}$.

For definiteness, we will assume that when the point $\varphi = \varphi(t)$ moves across the curve Γ as t increases, the internal domain $D \equiv int\Gamma$ stays on the left side.

We define the function $\arg\varphi'(t)$ as follows. At the every initial point φ_k we define the branch $\arg\varphi'(\varphi_k + 0)$, where $(\varphi_k, \varphi_{k+1})$ is the interval of continuity of the function $\arg\varphi'(t)$. And, at the endpoint φ_{k+1} the value $\arg\varphi'(\varphi_{k+1} - 0)$ is obtained from the chosen branch $\arg\varphi'(\varphi_k + 0)$ by continuously changing $\arg\varphi'(t)$. Without loss of generality, we define the values $\arg\varphi'(\varphi_k + 0)$ at the points φ_k by the following conditions

$$0 \le \arg\varphi'(a + 0) < 2\pi;$$

$$|\arg\varphi'(\varphi_k + 0) - \arg(\varphi_k - 0)| < \pi.$$

We will need weighted Sobolev classes and the Riemann problem in them.

Let $E_1(D)$ be a usual Smirnov class and $\nu(\tau)$, $\tau \in \Gamma$, be some weight function. Denote

$$E_{p,\nu}(D) \equiv \left\{ f \in E_1(D) : \int_\Gamma \left|f^+(\tau)\right|^p \nu(\tau)\, |d\tau| < +\infty \right\},$$

where $f^+(\tau)$'s are non-tangential boundary values of the function $f(z)$ on Γ.

Consider the following conjugation problem

$$F_1^+(\tau) + G(\tau)\overline{F_2^+(\tau)} = g(\tau)\ ,\ \tau \in \Gamma, \tag{2}$$

where $g(\tau) \in L_{p,\nu}(\Gamma)$ is the right-hand side, $G(\tau)$ is the coefficient of the problem, and $L_{p,\nu}(\Gamma)$ is a weighted Lebesgue class equipped with the norm

$$\|f\|_{p,\nu} = \left(\int_\Gamma |f(\tau)|^p \nu(\tau)\,|d\tau|\right)^{\frac{1}{p}}.$$

We seek a pair of analytic functions in D :

$$(F_1(z)\ ;\ F_2(z))\ :\ F_i \in E_{p,\nu}(D)\ ,\ i = 1, 2\,,$$

whose non-tangential boundary values satisfy the equality (2) almost everywhere on Γ.

It should be noted that the Riemann problem in the Smirnov classes $E_p(D)$ has been studied in detail by I.I. Danilyuk [5]. Generally, in case of no weight the problem (2) can be reduced to the Riemann problem by means of a conformal mapping. But this is not necessary for accomplishing our goals in this paper. We will take a different way. Namely, we will use a lemma that can be proved similar to [6].

Lemma 2.1 *Let $\rho : [a,b] \to (0,+\infty)$ be some weight function, the functions $A^{\pm}(t)$, $\varphi(t)$ and curve Γ satisfy the conditions 1), 2). If $\omega^{\pm} \in L_{p,\rho}$, $p \in (1,+\infty)$, then the system* (1) *is complete in $L_{p,\rho}$ only when the homogeneous conjugation problem*

$$F_1^+(\tau) - G(\tau)\overline{F_1^+(\tau)} = 0\,,\ \tau \in \Gamma, \tag{3}$$

has only the trivial solution in the classes $E_{q,\rho^{\pm}}(D)$: $F_1 \in E_{q,\rho^{\pm}}(D)$; $F_2 \in E_{q,\rho^-}(D)\,,\ \frac{1}{p} + \frac{1}{q} = 1$, where $\rho^{\pm}(\varphi(t)) \equiv |\omega^{\pm}(t)|^{-q}\rho^{1-q}(t)$, and the coefficient $G(\tau)$ is defined as

$$G(\varphi(t)) = \frac{A^+(t)\,\omega^+(t)\,\bar{\varphi}'(t)}{A^-(t)\,\omega^-(t)\,\varphi'(t)}, t \in (a,b)\,.$$

3 Main Results

We will study the trivial solvability of the conjugation problem (3) in the classes $E_{q,\rho^{\pm}}(D)$.

Denote by $z = \omega(\xi)\ ,\ \omega'(0) > 0\,,\ \omega(-\pi) = \varphi(a)$, the function that performs the univalent conformal mapping of the unit circle $\{\xi : |\xi| < 1\}$ onto the domain. Introduce the following analytic functions in the unit circle

$$\Phi_i(\xi) \equiv F_i[\omega(\xi)]\,\omega'(\xi)\ ,\ \ i = 1, 2.$$

As is known, F_i belongs to the class $E_1(D)$ only when Φ_i belongs to H_1, where H_1 is a usual Hardy class. Let $F_i \in E_{p,\nu}(D)$, i.e. $F_i \in E_1(D)$ and $F_i^+ \in L_{p,\nu}(\Gamma)$. Obviously, $\Phi_i \in H_1$. We have

$$\int_\Gamma \left|F_i^+(\tau)\right|^p \nu(\tau)\, |d\tau| = \int_{|\xi|=1} \left|F_i^+[\omega(\xi)]\right|^p \nu[\omega(\xi)]\, |\omega'(\xi)|\, |d\xi| =$$

$$= \int_{|\xi|=1} \left|\Phi_i^+(\xi)\right|^p \frac{\nu[\omega(\xi)]}{|\omega'(\xi)|^{p-1}}\, |d\xi| = \int_{|\xi|=1} \left|\Phi_i^+(\xi)\right|^p \mu(\xi)\, |d\xi| < +\infty. \qquad (4)$$

It follows that $\Phi_i \in H_{p,\mu}$, where

$$\mu(\xi) \equiv \frac{\nu[\omega(\xi)]}{|\omega'(\xi)|^{p-1}} ,$$

and the weighted class $H_{p,\mu}$ is defined by the norm

$$H_{p,\mu} \equiv \left\{ \Phi \in H_1 : \int_{|\xi|=1} \left|\Phi^+(\xi)\right|^p \mu(\xi)\, |d\xi| < +\infty \right\}.$$

Thus, if $F_i \in E_{p,\nu}(D)$, then $\Phi_i \in H_{p,\mu}$. It follows immediately from (4) that the contrary is also true, i.e. if $\Phi_i \in H_{p,\mu}$, then $F_i \in E_{p,\nu}(D)$. Consequently, $F_i \in E_{p,\nu}(D)$ only when $\Phi_i \in H_{p,\mu}$. Based on this conclusion, from (3) we obtain

$$\Phi_1^+(\xi) - D(\xi)\, \overline{\Phi_2^+(\xi)} = 0\,,\ |\xi| = 1, \qquad (5)$$

where $D(\xi) = G(\omega(\xi))$.

So we get the validity of the following lemma.

Lemma 3.1 *The homogeneous problem* (3) *is trivially solvable in the class* $E_{q,\rho^+}(D) \times E_{q,\rho^-}(D)$ *(i.e.* $F_1 \in E_{q,\rho^+}(D)$, $F_2 \in E_{q,\rho^-}(D)$*) only when the problem* (5) *is trivially solvable in the class* $H_{q,\mu^+} \times H_{q,\mu^-}$ *(i.e.* $\Phi_1 \in H_{q,\mu^+}$*;* $\Phi_2 \in H_{q,\mu^-}$*), where* $\mu^\pm(\xi) = \frac{\rho^\pm[\omega(\xi)]}{|\omega'(\xi)|^{q-1}}$.

Now we will proceed with the solving of the problem (5). Denote by $\xi = \omega_{-1}(z)$ the inverse function of $z = \omega(\xi)$ that performs a univalent conformal mapping of the domain D onto the unit circle. Let $\xi_k = \omega_{-1}[\varphi_k]$, $k = \overline{1, r}$; where φ_k is a corner point of the curve $\Gamma \backslash \{\varphi(a)\}$. It is known that $\omega'(\xi)$ is discontinuous at the points ξ_k, and the following relations are true in the neighborhood of these points (see, e.g., [8]):

$$|\omega'(\xi)| \sim |\xi - \xi_k|^{\nu_k - 1} ,\ \xi \to \xi_k;$$

$$\left(|\varphi(\xi)| \sim |\psi(\xi)| \Leftrightarrow 0 < \delta \le \frac{|\varphi(\xi)|}{|\psi(\xi)|} \le \delta^{-1} < +\infty\,,\ \delta > 0 \right),$$

where $\nu_k\pi$ are the internal angles of the curve Γat the points $\varphi(\varphi_k)$. Therefore

$$|\omega'(\xi)| \sim \prod_{k=1}^{r} |\xi - \xi_k|^{\nu_k - 1} \ , \ |\xi| = 1.$$

Let

$$A_1^+(t) \equiv A^+(t)\,\overline{\varphi'(t)};\, A_1^-(t) \equiv A^-(t)\,\varphi'(t)\,;$$

$$\tilde{A}(\xi) \equiv \xi^{-1} A_1^+\left[\varphi_{-1}(\omega(\xi))\right] \frac{\omega^+\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}};$$

$$\tilde{B}(\xi) \equiv \xi^{-1} A_1^-\left[\varphi_{-1}(\omega(\xi))\right] \frac{\omega^-\left[\varphi_{-1}(\omega(\xi))\right]}{|\omega'(\xi)|^{-\frac{1}{p}}} \frac{\overline{\omega'(\xi)}}{\omega'(\xi)},$$

where $\varphi_{-1} : \Gamma \backslash \{\varphi(a)\} \to (a,b)$ is the inverse function of $\varphi = \varphi(t)$. Consider the system

$$\left\{\tilde{A}\left(e^{ix}\right)\, e^{inx};\, \tilde{B}\left(e^{ix}\right)\, e^{-inx}\right\}_{n\geq 0}. \tag{6}$$

Absolutely similar to Lemma 2.1, we can prove the validity of the following one.

Lemma 3.2 *System* (6) *is complete in* $L_p(-\pi,\pi)$ *only when the homogeneous conjugation problem* (5) *has only the trivial solution in the classes* $H_{q,\mu^+} \times H_{q,\mu^-}$.

In fact, assuming the existence of the function $f \in L_q(-\pi,\pi)$ that annihilates the system (6), we have

$$\int_{-\pi}^{\pi} \tilde{A}\left(e^{ix}\right)\, e^{inx}\overline{f(x)}\, dx = 0,$$

$$\int_{-\pi}^{\pi} \tilde{B}\left(e^{ix}\right)\, e^{-inx}\overline{f(x)}\, dx = 0\,,\ \forall n \geq 0.$$

From the first equality above we obtain

$$\int_{-\pi}^{\pi} \tilde{A}\left(e^{ix}\right)\, e^{-ix}\overline{f(x)}\, e^{inx} de^{ix} = \int_{|\xi|=1} \tilde{A}(\xi)\ \overline{\xi f(\arg\xi)}\xi^n d\xi =$$

$$= \int_{|\xi|=1} f_1(\xi)\, \xi^n d\xi = 0, \tag{7}$$

where

$$f_1(\xi) = \tilde{A}(\xi)\, \bar{\xi}\, \overline{f(\arg\xi)}.$$

It follows from our assumptions that $f_1 \in L_1(\gamma)$, where $\gamma \equiv \{\xi : |\xi| = 1\}$. It is known (see, e.g., [7]) that the equalities (7) are equivalent to the existence of $\Phi_1 \in H_1 : \Phi_1^+(\xi) = f_1(\xi)$ a.e. on γ. Thus, we have

$$\Phi_1^+(\xi) = \tilde{A}(\xi)\,\bar{\xi}\,\overline{f(\arg\xi)} = A_1^+[\varphi_{-1}(\omega(\xi))]\frac{\omega^-[\varphi_{-1}(\omega(\xi))]}{|\omega'(\xi)|^{-\frac{1}{p}}}\overline{f(\arg\xi)}.$$

From the last relation we immediately obtain that
$\frac{\Phi_1^+(\xi)}{\omega^+[\varphi_{-1}(\omega(\xi))]\,|\omega'(\xi)|^{-\frac{1}{p}}} \in L_q(\gamma)$, i.e. $\Phi_1 \in H_{q,\mu^+}$.
In a similar way, we can establish that
$\exists \Phi_2 \in H_{q,\mu^-} :\ \Phi_2^+(\xi) = \overline{\tilde{B}(\xi)}\,\bar{\xi}\,f(\arg\xi)$ a.e. on γ.
Consequently

$$\overline{f(\arg\xi)} = \frac{\overline{\Phi_2^+(\xi)}}{\tilde{B}(\xi)\;\xi} = \frac{\overline{\Phi_2^+(\xi)}}{A_1^-[\varphi_{-1}(\omega(\xi))]\frac{\omega^-[\varphi_{-1}(\omega(\xi))]}{|\omega'(\xi)|^{-\frac{1}{p}}}\frac{\overline{\omega'(\xi)}}{\omega'(\xi)}} \equiv g(\xi).$$

From the above relations we obtain

$$\frac{\Phi_1^+(\xi)}{A^+[\varphi_{-1}(\omega(\xi))]\frac{\omega^+[\varphi_{-1}(\omega(\xi))]}{|\omega'(\xi)|^{-\frac{1}{p}}}} = g(\xi),$$

i.e.

$$\Phi_1^+(\xi) - \frac{A_1^+[\varphi_{-1}(\omega(\xi))]\,\omega^+[\varphi_{-1}(\omega(\xi))]}{A_1^-[\varphi_{-1}(\omega(\xi))]\,\omega^-[\varphi_{-1}(\omega(\xi))]}\frac{\omega'(\xi)}{\overline{\omega'(\xi)}}\overline{\Phi_2^+(\xi)} = 0;$$

$\Phi_1^+(\xi) - D(\xi)\,\overline{\Phi_2^+(\xi)} = 0$ a.e. on γ.

Thus, we get the relation (5). The contrary can be proved in a similar way as Lemma 2.1.

So the following theorem is true.

Theorem 3.3 *Let the conditions 1), 2) be satisfied and $\omega^\pm \in L_{p,\rho}$, $p \in (1,+\infty)$. The system* (1) *is complete in $L_{p,\rho}(a,b)$ only when the system* (6) *is complete in $L_p(-\pi,\pi)$.*

In the sequel, we will use the results of [9;10]. First, we make some additional assumptions.

We assume that the weight $\rho(\cdot)$ has the power form

$$\rho(t) = \prod_{k=1}^{l} |t-\tau_k|^{\alpha_k},$$

where $\{\tau_k\}_1^l \subset [a,b)$ and $\{\alpha_k\}_1^l \subset R$. Let's represent the product $\omega^{\pm}\rho^{\frac{1}{p}}$ in the following form

$$\omega^{\pm}(t)\,\rho^{\frac{1}{p}}(t) = \prod_{k=1}^{r^{\pm}} \left|t - t_k^{\pm}\right|^{\beta_k^{\pm}}.$$

3) $\alpha^{\pm}(t)$'s are piecewise Hölder on $[-\pi,\pi]$; $\{\tau_i\}_1^n$'s are the points of discontinuity of the function $\theta(t) \equiv \alpha^-(t) - \alpha^+(t)$, and $\alpha_i = \omega_{-1}[\varphi(\tau_i)]$.

$$4) -\frac{1}{p} < \beta_k^{\pm} < \frac{1}{q}\,,\ k = \overline{1, r^{\pm}}.$$

Let $\xi_k = \omega_{-1}[\varphi(\varphi_k)]$, $k = \overline{1,r}$; $\xi_k^{\pm} = \omega_{-1}\left[\varphi\left(t_k^{\pm}\right)\right]$, $k = \overline{1,r^{\pm}}$. Assume

$$\{\sigma_k\}_1^m \equiv \{\alpha_i\}_1^n \bigcup \{\xi_k\}_1^r \bigcup \left\{\xi_k^+\right\}_1^{r^+} \bigcup \left\{\xi_k^-\right\}_1^{r^-} \ :\ \sigma_1 < \sigma_2 < ... < \sigma_m.$$

To apply the results of [9; 10], we need to represent the functions $\tilde{A}(\xi)$ and $\tilde{B}(\xi)$ in the forms they have been considered in the above-mentioned works. Let $t = \varphi_{-1}[\omega(\xi)]$, $|\xi| = 1$. We have

$$\left|t - t_k^{\pm}\right| = \left|\varphi_{-1}[\omega(\xi)] - \varphi_{-1}\left[\omega\left(\xi_k^{\pm}\right)\right]\right|.$$

It follows from the condition 1) that

$$\left|\varphi_{-1}[\omega(\xi)] - \varphi_{-1}\left[\omega\left(\xi_k^{\pm}\right)\right]\right| \sim \left|\omega(\xi) - \omega\left(\xi_k^{\pm}\right)\right|.$$

Moreover (see, e.g., [8, p. 25]), the following relation is true

$$\left|\omega(\xi) - \omega\left(\xi_k^{\pm}\right)\right| \sim \left|\xi - \xi_k^{\pm}\right|^{\nu_k^{\pm}},$$

where $\nu_k^{\pm}\pi$ is the internal angle of the curve Γat the point $\omega\left(\xi_k^{\pm}\right)$. In particular, if $\omega\left(\xi_k^{\pm}\right)$ is the point of smoothness of Γ, then $\nu_k^{\pm} = 1$. Thus

$$\left|t - t_k^{\pm}\right| \sim \left|\xi - \xi_k^{\pm}\right|^{\nu_k^{\pm}},$$

$$\prod_{k=1}^{r^{\pm}} \left|t - t_k^{\pm}\right|^{\beta_k^{\pm}} \sim \prod_{k=1}^{r^{\pm}} \left|\xi - \xi_k^{\pm}\right|^{\beta_k^{\pm}\nu_k^{\pm}} \equiv \tilde{\omega}^{\pm}(\xi).$$

Let

$$\tilde{A}^+(\xi) \equiv \xi\, A_1^+\varphi_{-1}[\omega(\xi)]\,;\tilde{A}^-(\xi) \equiv \frac{\overline{\omega'(\xi)}}{\omega'(\xi)} A_1^-\left[\varphi_{-1}(\omega(\xi))\right]\,,\ |\xi| = 1.$$

Denote

$$\nu^{\pm}(\xi) \equiv \tilde{\omega}^{\pm}(\xi)\ \left|\omega'(\xi)\right|^{\frac{1}{p}}, |\xi| = 1.$$

Consider the system

$$\left\{\tilde{A}^{+}\left(e^{ix}\right)\,\nu^{+}\left(e^{ix}\right)e^{inx}\,;\;\tilde{A}^{-}\left(e^{ix}\right)\,\nu^{-}\left(e^{ix}\right)e^{-i(n+1)x}\right\}_{n\geq 0}. \tag{8}$$

To make it easier, we introduce some notations. Let $\tilde{\theta}\left(\arg\xi\right)\equiv\arg\tilde{A}^{-}\left(\xi\right)-\arg\tilde{A}^{+}\left(\xi\right)$. Then

$$\tilde{\theta}\left(\arg\xi\right)=-2\arg\omega'\left(\xi\right)+\alpha^{-}\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]+$$
$$+\arg\,\varphi'\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]-\left[\arg\xi+\alpha^{+}\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]\right]-$$
$$-\arg\,\varphi'\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]=\alpha^{-}\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]-\alpha^{+}\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right]+$$
$$+\arg\omega'\left(\xi\right)+\arg\varphi'\left[\varphi_{-1}\left(\omega\left(\xi\right)\right)\right].$$

According to the results of [5], the function $\arg\omega'\left(\xi\right)$ can be represented in the following form

$$\arg\omega'\left(e^{i\sigma}\right)=\theta\left(s\left(\theta\right)\right)-\sigma-\frac{\pi}{2},\ -\pi<\sigma\leq\pi,$$

where $\theta\left(s\left(\theta\right)\right)$ is the angle between the tangent line to Γ at the point $\omega\left(e^{i\sigma}\right)$ and the real axis; $s\left(\sigma\right)$ is the arc distance between the points $\varphi=\varphi\left(a\right)$ and $\omega\left(e^{i\sigma}\right)$, $-\pi<\sigma\leq\pi$, in the positive direction. Therefore, the points of discontinuity for the function $\arg\omega'\left(\xi\right)$ are $\left\{\tau_k\right\}_1^r$'s. It is not difficult to see that the system (6) is complete in $L_p\left(-\pi,\pi\right)$ only when the system (8) is complete in $L_p\left(-\pi,\pi\right)$. Let

$$\left\{\sigma_k^{\pm}\right\}_1^{m^{\pm}}\equiv\left\{\alpha_k\right\}_1^{n}\bigcup\left\{\xi_k^{\pm}\right\}_1^{r^{\pm}}.$$

We need the following set function

$$\chi\left(A\right)\equiv\begin{cases}1\,,\ A\neq\emptyset,\\ 0\,,\ A=\emptyset.\end{cases}$$

Let $\Omega_k\left(\Omega_k^{\pm}\right)$ be a set with one element σ_k $\left(\sigma_k^{\pm}\right)$, i.e. $\Omega_k\equiv\left\{\sigma_k\right\}$ $\left(\Omega_k^{\pm}\equiv\left\{\sigma_k^{\pm}\right\}\right)$. $\left\{\sigma_k^{\pm}\right\}$'s are the points of degeneration of the functions $\nu^{\pm}\left(\xi\right)$, respectively. The orders of degeneracy at these points are defined by the following relations

$$\alpha_k^{\pm}\equiv\sum_{i=1}^{r^{\pm}}\beta_i^{\pm}\,\nu_i^{\pm}\chi\left(\Omega_k^{\pm}\bigcap\left\{\xi_i^{\pm}\right\}\right)+\sum_{i=1}^{r}\frac{\nu_i-1}{p}\chi\left(\Omega_k^{\pm}\bigcap\left\{\varphi_i\right\}\right),$$

where $\left\{\xi\right\}$ is a set with one element ξ.

It is absolutely clear that the points of discontinuity of the function on $\gamma\backslash\left\{-1\right\}$ are $\left\{\sigma_k\right\}_1^m$'s. Denote by $\left\{h_k\right\}_1^m$ the jumps at these points

$$h_k=\tilde{\theta}\left(\arg\sigma_k+0\right)-\tilde{\theta}\left(\arg\sigma_k-0\right)\,,\ k=\overline{1,m}.$$

Introduce the following correspondences

$$\sigma_k^{\pm} \to \alpha_k^{\pm}\,;\; \sigma_k \to \frac{h_k}{2\pi}.$$

Define

$$\lambda_i^{\pm} = \begin{cases} \frac{\alpha_k^{\pm}}{2}\,, & if \;\; \{\sigma_i\} \bigcap \Omega^{\pm} = \sigma_k^{\pm}, \\ 0, & if \;\; \{\sigma_i\} \bigcap \Omega^{\pm} = \emptyset\,, \end{cases}$$

$$\lambda_i = \begin{cases} -\frac{h_k}{2\pi}\,, & if \;\; \{\sigma_i\} \bigcap \Omega = \sigma_k, \\ 0, & if \;\; \{\sigma_i\} \bigcap \Omega = \emptyset\,, \end{cases}$$

$$\omega_i = -\left(\lambda_i^+ + \lambda_i^- + \lambda_i\right)\,,\; i = \overline{1,m}.$$

Following [9; 10], we define the integers n_i, $i = \overline{1,m}$ by the inequalities

$$\left.\begin{array}{l} -\frac{1}{q} < \omega_i + n_{i-1} - n_i \leq \frac{1}{p}, \\ n_0 = 0\,,\; i = \overline{1,m}\,. \end{array}\right\} \tag{9}$$

Let

$$\omega = \tilde{\theta}\,(-\pi + 0) - \tilde{\theta}\,(\pi - 0) + 2n_m\pi. \tag{10}$$

The following theorem is true.

Theorem 3.4 *Let the functions $A^{\pm}(t)$, $\omega^{\pm}(t)$ satisfy the conditions 1)-4), and ω be defined by* (9), (10). *The system* (1) *is complete in $L_{p,\rho}(a,b)$, $1 < p < +\infty$, only when $\omega \leq \frac{2\pi}{p}$.*

In fact, according to the results of [9; 10], if all the conditions of Theorem 3.4 are satisfied, then the validity of the inequality $\omega \leq \frac{2\pi}{p}$ is a necessary and sufficient condition for the completeness of the system (8) in $L_p(-\pi,\pi)$. The rest follows from Lemma 3.2.

In particular, if we consider $\varphi(t) \equiv e^{it}$, $t \in [-\pi,\pi]$, then it is clear that $\nu_k^{\pm} = 1$, $\forall k$. In this case we obtain the known results of [9;10].

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