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Complementary Acyclic Chromatic Preserving Sets in Graphs

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Abstract

Let G =(V, E) be a simple graph. A subset S of V(G) is called a complementary acyclic chromatic preserving set of G (c-acp set of G) if $< V - S >$ is acyclic and $\chi(< S >) = \chi(G)$. The minimum cardinality of a c-acp set in G is called the complementary acyclic chromatic preserving number of G and is denoted by c-acpn(G). A c-acp set of G of cardinality c-acpn(G) is called a c-acpn- set of G. A study of chromatic preserving sets has been made in detail in [5]. In this paper, a study of complementary acyclic chromatic preserving sets is initiated.Further chromatic complementary acyclic dominating sets are defined and studied.

Keywords: complementary acyclic chromatic preserving set,complementary acyclic chromatic preserving number, chromatic complementary acyclic dominating set.

Mathematics Subject Classification : 05C15

1 Introduction

Let G = (V,E) be a simple graph. Let P be a graph property and let t be a parameter of G. Let S be a subset of V(G). S is said to be a P-set which is t-preserving if S satisfies P and $t(< S >) = t(G)$. Depending on the nature of P, either minimum cardinality or maximum cardinality of such a set is taken as a new parameter. If P is the property that the set is complementary acyclic and t is the chromatic number, then the t-preserving P-set is called the complementary acyclic chromatic preserving set . A study of such sets is made in this paper.Further, a study of chromatic complementary acyclic dominating sets is also made.

2 Preliminaries

Definition 2.1 *A subset S of V(G) is called a complementary acyclic chromatic preserving set of G (c-acp set of G) if* $< V - S >$ *is acyclic and*
$\chi(< S >) = \chi(G)$.
The minimum cardinality of a c-acp set in G is called the complementary acyclic chromatic preserving number of G and is denoted by c-acpn(G). A c-acp set of G of cardinality c-acpn(G) is called a c-acpn- set of G.

Remark 2.2 *Since V(G) is a c-acp set , the existence of a c-acp sets is guaranteed in any graph.*

Remark 2.3 *The c-acp property is superhereditary, since any super set of a c-acp set is a c-acp set. Hence a c-acp set is minimal if and only if it is one minimal.*

3 Main Results

c-acpn for standard graphs

1. c-acpn$(K_n) = n$.

2. c-acpn$(K_{1,n}) = 2$.

3. c-acpn$(K_{m,n})= min\,\{m, n\}, \quad m, n \geq 2$.

4. c-acpn$(D_{r,s}) = 2$.

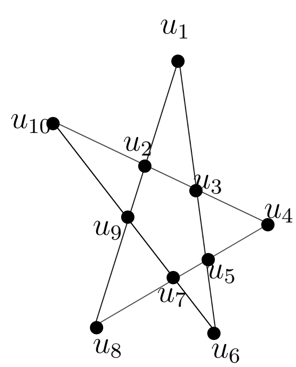
5. c-acpn$(P_n) = 2$.

6. c-acpn$(C_n) = \begin{cases} 2 & if\ n\ is\ even \\ n & if\ n\ is\ odd \end{cases}$

7. c-acpn$(nK_1) = 1$.

Observation 3.1 *A c-acpn-set of a connected graph G need not induce a connected subgraph.*

For: Consider the graph G :

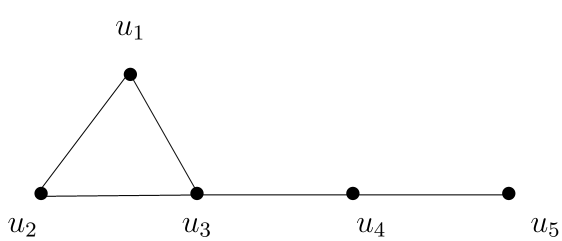


In this graph c-acpn(G) = 4 and any c-acpn-set is disconnected.$S_1 = \{u_1, u_2, u_3, u_7\}$ is a c-acpn-set which is disconnected.

Observation 3.2 *$\chi(G)$ = c-acpn(G) if and only if any c-acpn-set induces a complete subgraph.*

Proof: By hypothesis c-acpn(G) = $\chi(G)$.Let S be a c-acpn-set of G.
c-acpn(G) = $|S| = \chi(G) = \chi(< S >)$.
Therefore $< S >$ is a complete subgraph.
Conversely, let any c-acpn-set induces a complete subgraph . Let S be a c-acpn-set of G. $|S| = \chi(< S >) = \chi(G)$.
That is c-acpn(G) = $\chi(G)$.

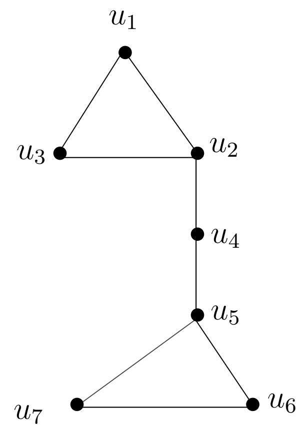
Illustration 3.3 *Consider the graph G:*



S = $\{u_1, u_2, u_3\}$, $\chi(< S >) = 3$ and S is a c-acpn-set.

Remark 3.4 *If $\chi(G) = \omega(G)$, then cpn(G) = $\chi(G)$. But c-acpn(G) may be greater than $\chi(G)$.*

For example, consider the graph G:



$\chi(G) = \omega(G) = 3$ and c-acpn(G) = 4.

Definition 3.5 *A subset S of V(G) is called a minimal c-acp set of G if S is a c-acp set of G and no subset of S is a c-acp set of G.*

Theorem 3.6 *Let S be a c-acp set of G. S is minimal if and only if for any u in S, either* $V - (S - \{u\})$ *contains a cycle or* $\chi(< S - \{u\} >) < \chi(G)$.

Proof: *Obvious.*

Definition 3.7 *The maximum cardinality of a minimal c-acp set of G is called the upper c-acp number of G and is denoted by c-acpN(G).*

Remark 3.8 *There are graphs G with c-acpn(G)< c-acpN(G).*

4 Fine c-acp Graphs

Definition 4.1 *A graph G is a fine c-acp graph if all minimal c-acp sets have the same cardinality.*

Example 4.2 *(i) Every totally disconnected graph is a fine c-acp graph.*

(ii) $K_{n,n}$ *is a fine c-acp graph.*

(iii) $K_{n_1,n_2,...,n_r}$ *wher* $n_1 = n_2 = ... = n_r = t$ *is a fine c-acp graph.*

(iv) Any acyclic graph $\neq \overline{K_n}$ *is a fine c-acp graph. All minimal c-acp sets have cardinality two.*

(v) All cycles are fine c-acp graphs.

Theorem 4.3 *Let* G_1 *and* G_2 *be graphs. Suppose* G_1 *contains a cycle,* G_2 *is acyclic and* $\chi(G_1) = \chi(G_2)$*. Then* $G_1 \cup G_2$ *is a fine c-acp graph if and only if both* G_1 *and* G_2 *are fine c-acp graphs and the cardinality of any minimal c-acp set of* G_1 *is same as the sum of cardinalities of any minimal c-acp set of* G_2 *and a minimal c-a set of* G_1*.*

Proof: *Suppose* G_1 *contains a cycle,* G_2 *is acyclic and* $\chi(G_1) = \chi(G_2)$*. Let* S_1 *be a minimal c-acp set of* G_1*. Then* S_1 *is a c-a set of* $G_1 \cup G_2$*.* $\chi(S_1) = \chi(G_1) = max\{\chi(G_1), \chi(G_2)\} = \chi(G_1 \cup G_2)$*. Therefore* S_1 *is c-acp set of* $G_1 \cup G_2$*. Since* S_1 *is a minimal c-acp set of* G_1*,* S_1 *is minimal c-acp set of* $G_1 \cup G_2$*. Any minimal c-acp set of* G_2 *is not a c-a set of* $G_1 \cup G_2$*. Let* S_2 *be a minimal c-acp set of* G_2 *and* T_2 *be a minimal c-a set of* G_1*. Then* $S_2 \cup T_2$ *is a c-a set of* $G_1 \cup G_2$*.* $\chi(S_2 \cup T_2) = max\{\chi(S_2), \chi(T_2)\}$*. Suppose* $\chi(T_2) > \chi(S_2)$*. Then* $\chi(S_2 \cup T_2) = \chi(T_2) = \chi(G_1)$ *if and only if* T_2 *is a c-a set of* G_1*. Therefore* $\chi(S_2 \cup T_2) = \chi(G_1 \cup G_2)$ *if and only if* T_2 *is a c-acp set of* G_1*. Let* S_1 *be a minimal c-acp set of* G_1 *contained in* T_2*. Then* $S_1 \subset S_2 \cup T_2$*. Therefore* $S_2 \cup T_2$ *is not a minimal c-acp set of* $G_1 \cup G_2$*. Suppose* $\chi(T_2) \leq \chi(S_2)$*. Then* $\chi(S_2 \cup T_2) = \chi(S_2) = \chi(G_2) = \chi(G_1) = \chi(G_1 \cup G_2)$*.* $S_2 \cup T_2$ *is a c-acp set of* $G_1 \cup G_2$*.* $S_2 \cup T_2$ *is a minimal c-acp set of* $G_1 \cup G_2$ *if and only if* $\chi(T_2) < \chi(G_1)$*. Conversely, let* T *be a minimal c-acp set of* $G_1 \cup G_2$*. Let* $T = T_1 \cup T_2$*, where* $T_1 \subseteq V(G_1)$ *and* $T_2 \subseteq V(G_2)$*. Since* T *is a c-a set of* $G_1 \cup G_2$*,* T_1 *is a c-a set of* G_1 *and* T_2 *is a c-a set of* G_2*.* $\chi(T) = \chi(T_1 \cup T_2) = max\{\chi(T_1), \chi(T_2)\} = \chi(G_1 \cup G_2) =$

$\chi(G_1) = \chi(G_2)$. If $\chi(T_1) > \chi(T_2)$,then $\chi(T) = max\{\chi(T_1), \chi(T_2)\} = \chi(T_1) = \chi(G_1)$.Therefore T_1 is a c-acp set of G_1 and a c-acp set of $G_1 \cup G_2$. since T is minimal, $T_2 = \Phi$.

Thus a minimal c-acp set of $G_1 \cup G_2$ is a minimal c-acp set of G_1 if $\chi(T_1) > \chi(T_2)$.Suppose $\chi(T_2) > \chi(T_1)$, then $\chi(T) = max\{\chi(T_1), \chi(T_2)\} = \chi(T_2) = \chi(G_1) = \chi(G_2)$. Therefore T_2 is a c-acp set of G_2.That is T_2 is a cp of set of $G_1 \cup G_2$ but T_2 is not a c-a set of $G_1 \cup G_2$. Also T_1 is not a cp set of G_1. Therefore $T = T_1 \cup T_2$, where T_1 is a minimal c-a set of G_1 and T_2 is a minimal c-acp set of G_2. Thus if G_1 contains a cycle and G_2 is acyclic and $\chi(G_1) = \chi(G_2)$, then $G_1 \cup G_2$ is a fine c-acp graph if and only if both G_1 and G_2 are fine c-acp graphs and the cardinality of any minimal c-acp set of G_1 is same as the sum of cardinalities of any minimal c-acp set of G_2 and a minimal c-a set of G_1.

Theorem 4.4 *Let G_1 and G_2 be graphs. Suppose G_1 contains a cycle and G_2 is acyclic and $\chi(G_1) < \chi(G_2)$. Then $G_1 \cup G_2$ is a fine c-acp graph if and only if G_2 is a fine c-acp graph and all minimal c-a sets of G_1 have equal cardinality.*

Proof: *Suppose G_1 contains a cycle and G_2 is acyclic and $\chi(G_1) < \chi(G_2)$. Any c-acp set of G_1 is a c-a set of $G_1 \cup G_2$ but it is not a cp set $G_1 \cup G_2$. Let S_2 be a minimal cp set of G_2 and S_1 be a minimal c-a set of G_1. Clearly $\chi(S_2) = \chi(G_2) > \chi(G_1) \geq \chi(S_1)$. Therefore $S_1 \cup S_2$ is a minimal c-acp set of $G_1 \cup G_2$. Suppose T is a minimal c-acp set of $G_1 \cup G_2$. Let $T = T_1 \cup T_2$, where $T_1 \subseteq V(G_1)$ and $T_2 \subseteq V(G_2)$. Since T is a c-a set of $G_1 \cup G_2$, T_1 is a c-a set of G_1 and T_2 is a c-a set of G_2. $\chi(T) = \chi(T_1 \cup T_2) = max\{\chi(T_1), \chi(T_2\} = \chi(G_1 \cup G_2) = \chi(G_2)$. If $\chi(T_1) > \chi(T_2)$, then $\chi(T_1) = \chi(G_2) > \chi(G_1)$, a contradiction, since T_1 is a subset of $V(G_1)$. Therefore $\chi(T_1) \leq \chi(T_2)$. Therefore $\chi(T) = \chi(T_2) = \chi(G_2)$. Therefore T_2 is a c-acp set of G_2 and T_1 is a c-a set of G_1.Since T is minimal,*

T_1 is a minimal c-a set of G_1 and T_2 is a minimal c-acp set of G_2. Thus $G_1 \cup G_2$ is a fine c-acp graph if and only if G_2 is a fine c-acp graph and all minimal c-a sets of G_1 have equal cardinality.
Similar argument can be given if G_2 contains a cycle and G_1 is acyclic.

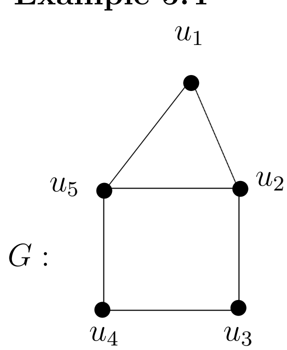
5 Chromatic Complementary Acyclic Dominating Set

Definition 5.1 *Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is called a complementary acyclic dominating set of G if D is a dominating set of G and $< V - D >$ is acyclic.*

Definition 5.2 *Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is called a chromatic complementary acyclic dominating set (chromatic c-a dominating set) of G if D is a complementary acyclic dominating set and $\chi(< D >) = \chi(G)$.*

Definition 5.3 *The minimum cardinality of a chromatic c-a dominating set of G is called the chromatic c-a domination number of G and is denoted by $\gamma^{\chi}_{c-a}(G)$.*

Example 5.4



$D = \{u_1, u_2, u_5\}$ is a chromatic c-a dominating set of G.

Here $\chi(< D >) = \chi(G) = 3$ and $\gamma^{\chi}_{c-a}(G) = 3$

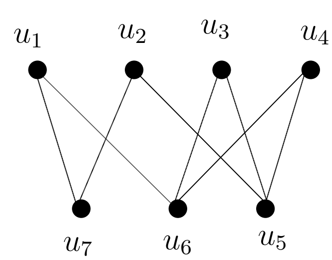
Theorem 5.5 *Let G be a graph with $\Delta(G) = 1$. Then $\gamma^{\chi}_{c-a}(G) + \Delta(G) = n$ if and only if $G = 2K_2 \cup (n-4)K_1$.*

Proof: Suppose G is a graph with $\Delta(G) = 1$ and $\gamma^{\chi}_{c-a}(G) + \Delta(G) = n$. Then each component of G is K_1 or K_2. Let t be the number of components which are K_2. Then $\gamma^{\chi}_{c-a}(G) = n - 2t + t + 1 = n - t + 1$.
Therefore $\gamma^{\chi}_{c-a}(G) + \Delta(G) = n - t + 1 + 1 = n - t + 2$ which implies t=2. Therefore $G = 2K_2 \cup (n-4)K_1$. The converse is obvious.

Theorem 5.6 *For any bipartite graph G, $\gamma_{c-a}(G) \leq \gamma^{\chi}_{c-a}(G) \leq \gamma_{c-a}(G) + 1$.*

Proof: Since G is bipartite, $\chi(G) = 2$. Let D be a γ_{c-a}-set of G. If $< D >$ contains an edge, then $\gamma^{\chi}_{c-a}(G) = |D| = \gamma_{c-a}(G)$. If $< D >$ is totally disconnected for all γ_{c-a}- sets of G, then $D \cup \{v\}$ is a γ^{χ}_{c-a}-set of G, where $v \in V - D$. Then $\gamma^{\chi}_{c-a}(G) = |D| + 1 = \gamma_{c-a}(G) + 1$. Hence $\gamma_{c-a}(G) \leq \gamma_{c-a}\chi(G) \leq \gamma_{c-a}(G) + 1$

Example 5.7



$D = \{u_2, u_6\}$ is a c-a dominating set, $\gamma_{c-a}(G) = 2$.
But $D_1 = \{u_2, u_6, u_7\}$ is a chromatic c-a dominating set of G.
Therefore $\gamma^{\chi}_{c-a}(G) = 3 = \gamma_{c-a}(G) + 1$.

Theorem 5.8 *Let G be a graph without isolates. If G has a vertex u and* $\chi(G) > \chi(G - v)$*, for all* $v \in V(G) - \{u\}$ *and* $\chi(G) = \chi(G - u)$*, then* $\gamma^{\chi}_{c-a}(G) = n - 1$.

Proof: Let D be a γ^{χ}_{c-a}-set of G. Then $\chi(< D >) = \chi(G)$.

Suppose $|D| < n-1$. Then D does not contain at least one vertex $v \neq u$. Therefore $D \subset V(G) - \{v\}$.

Therefore $\chi(< D >) \leq \chi(< V(G) - \{v\} >) < \chi(G)$, a contradiction. Therefore $|D| \geq n - 1$. Consider $S = V(G) - \{u\}$. Then S dominates u and $\chi(< S >) = \chi(G)$. Therefore S is a c-a dominating set of G and so $\gamma^{\chi}_{c-a}(G) \leq |S| = n - 1$. Therefore $\gamma^{\chi}_{c-a}(G) = n - 1$.

Theorem 5.9 *Let T be a tree. Then* $\gamma^{\chi}_{c-a}(T) = 2$ *if and only if T is either* $K_{1,n}$ *or* $D_{r,s}$

Proof: Let $\gamma^{\chi}_{c-a}(T) = 2$. Let $D = \{u, v\}$ be a γ^{χ}_{c-a}-set of T.

$\gamma^{\chi}_{c-a}(T) = \gamma_{c-a}(T)$ or $\gamma_{c-a}(T) + 1$

case (i): Let $\gamma^{\chi}_{c-a}(T) = \gamma_{c-a}(T)$

Therefore D is not independent and D is a γ^{χ}_{c-a}-set.

Since T is a tree, every point other than u and v is adjacent to exactly one of u and v. Also every point of T other than u and v is a pendent vertex.

Since u, v have private neighbour, $T = D_{r,s}$.

Case (ii): $\gamma^{\chi}_{c-a}(T) = \gamma_{c-a}(T) + 1$.

Therefore $\gamma_{c-a}(T) = 1$.

Therefore T is $K_{1,n}$.

Converse is obvious.

Theorem 5.10 *If G is a planar graph with diam(G)=2,* $\chi(G) = 3$ *and* $\gamma(G) = 2$, *then* $3 \leq \gamma^{\chi}_{c-a}(G) \leq 5$

Proof: Lower bound is trivial. Let $S = \{a, b\}$ be a γ-set of G. Since diam(G)=2, $g_0(G) = 3$ or 5, where $g_0(G)$ is the length of the smallest odd cycle of G.

Case (i): $g_0(G) = 3$. Let C be a 3-cycle xyzx. If a,b $\notin$ C, then two vertices of C are adjacent to a and one verex is adjacent to b or vice versa, for otherwise K_4 is induced, a contradiction. Let x and y be adjacent to a and z be adjacent to b. Then axya is a 3-cycle. Hence $\{a, x, y, b\}$ is a chromatic c-a dominating set of G. If a or b is in the 3-cycle, then the 3-cycle together with the remaining vertex of S is a chromatic c-a dominating set of G.

Case (ii): $g_0(G) = 5$. Let C be a 5-cycle uvwxyu. If a,b$\notin$C, then as S is a dominating set, vertices of C are adjacent to a or b and not to both, otherwise a 3-cycle is induced. Moreover no two consecutive vertices of C can be both adjacent to a or b, otherwise a 3-cycle is induced. Then S can dominate at most 4-vertices of C, a contradiction. Hence a or b $\in$ C. Let $a \in C$ and $b \notin C$. Let u=a. Then x and w are adjacent to b and hence a 3-cycle is induced, a contradiction. Therefore both a,b $\in$ C and hence $V(C)$ is a γ^{χ}_{c-a}-set of G. Case(i) and case(ii) show that the upper bound is attained.

Theorem 5.11 *For any integer* $N \geq 0$*, there exists a coneected graph* G *and a graph* G' *such that* G' *is obtained from* G *by adding exactly one vertex and* $\gamma^{\chi}_{c-a}(G) - \gamma^{\chi}_{c-a}(G') = N$*.*

Proof: Let $G = P_{3(N+2)}$, a path on 3(N+2) vertices. Then $\gamma^{\chi}_{c-a}(G) = \left\lceil \frac{3(N+2)}{3} \right\rceil + 1 = N+3$. Let G' be the graph obtained from G by adding a new vertex v and joining v to all the vertices of G. Therefore $\gamma^{\chi}_{c-a}(G') = 3$ Hence $\gamma^{\chi}_{c-a}(G) - \gamma^{\chi}_{c-a}(G') = N + 3 - 3 = N$

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