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Classification of 3-Lie bialgebras of type (L_b, C_b)

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Abstract

The 4-dimensional 3-Lie coalgebras with one-dimensional derived algebra, and 4-dimensional 3-Lie bialgebras of type (L_b, C_b) are classified. It is proved that there exist two classes 4-dimensional 3-Lie coalgebras with one-dimensional derived algebra which are (L, C_{b_1}) and (L, C_{b_2}) (Theorem 3.2), and seven classes of 4-dimensional 3-Lie bialgebras of type (L_b, C_b) (Theorem 3.3).

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1 Introduction

Lie coalgebras and Lie bialgebras (cf [1, 2, 3]) are important in Lie algebras, and it has been playing an important role in mathematics and physics. Motivated by this, authors in paper [4] studied 3-Lie coalgebras and 3-Lie bialgebras. In this paper, we give the classification of 4-dimensional 3-Lie coalgebras with one-dimensional derived algebra, and then classify 4-dimensional 3-Lie bialgebras of type (L_b, C_b). In the following, we suppose that F is a field of characteristic zero, and the zero multiplication of basis vectors in the multiplication table of 3-Lie algebras and 3-Lie coalgebras are omitted.

2 Preliminaries

A 3-Lie algebra [5] is a vector space L endowed with a linear multiplication $\mu : L^{\wedge 3} \to L$ such that for all $x, y, z, u, v \in L$,
$\mu(u, v, \mu(x, y, z)) = \mu(x, y, \mu(u, v, z)) + \mu(y, z, \mu(u, v, x)) + \mu(z, x, \mu(u, v, y))$.

3-Lie algebras can be described as follows. A 3-Lie algebra (L, μ) is a vector space L endowed with a 3-ary linear multiplication $\mu : L\otimes L\otimes L \to L$ satisfying $\mu(1-\tau) = 0$, $\mu(1\otimes 1\otimes \mu)(1-\omega_1-\omega_2-\omega_3) = 0$, where $\tau : L\otimes L\otimes L \to L\otimes L\otimes L$, $\tau(x_1 \otimes x_2 \otimes x_3) = sign(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}, \sigma \in S_3, \forall x_1, x_2, x_3 \in L$.

Define $\omega_i, 1 : L \otimes L \otimes L \otimes L \otimes L \to L \otimes L \otimes L \otimes L \otimes L,\ 1 \le i \le 3$, by

$$1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5,$$

$$\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5, \quad (1)$$

$$\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3, \quad (2)$$

$$\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4. \quad (3)$$

Let (L, μ) be a 3-Lie algebra, L^* be the dual space of L. The dual map $\mu^* : L^* \to L^* \otimes L^* \otimes L^*$ of μ, satisfies $\forall x, y, z \in L$ and $\xi, \eta, \zeta \in L^*$, $\langle \mu^*(\xi), x \otimes y \otimes z\rangle = \langle \xi, \mu(x, y, z)\rangle$, and $\langle \xi \otimes \eta \otimes \zeta, x \otimes y \otimes z\rangle = \langle \xi, x\rangle\langle \eta, y\rangle\langle \zeta, z\rangle$, where $\langle , \rangle$ is the natural nondegenerate symmetric bilinear form on $L \oplus L^*$ defined by $\langle \xi, x\rangle = \xi(x), \xi \in L^*, x \in L$. Then $Im(\mu^*) \subseteq L^* \wedge L^* \wedge L^*$, and $(1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \mu^*)\mu^* = 0$, that is, $\forall x, y, z, u, v \in L$ and $\xi, \eta, \zeta, \alpha, \beta \in L^*$,

$$\begin{aligned}
&\langle (1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \mu^*)\mu^*(\xi),\ x \otimes y \otimes z \otimes u \otimes v\rangle \\
= &\langle (1 \otimes 1 \otimes \mu^*)\mu^*(\xi),\ (1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v)\rangle \\
= &\langle \mu^*(\xi),\ (1 \otimes 1 \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v)\rangle \\
= &\langle \xi,\ \mu(1 \otimes 1 \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v)\rangle.
\end{aligned}$$

A 3-Lie coalgebra (L, Δ) [4] is a vector space L with a linear map $\Delta : L \to L\otimes L\otimes L$ satisfying $Im(\Delta) \subset L\wedge L\wedge L$, and $(1-\omega_1-\omega_2-\omega_3)(1\otimes 1\otimes \Delta)\Delta = 0$, where $\omega_1, \omega_2, \omega_3 : L^{\otimes 5} \to L^{\otimes 5}$ satisfy identities (1), (2) and (3), respectively, and 1 is the identity of $L^{\otimes 5}$.

Let L^* be the dual space of L, $e^1, \cdots, e^m$ be the dual basis of $e_1, \cdots, e_m$, that is, $\langle e^i, e_j\rangle = \delta_{ij}, 1 \le i, j \le m$. Let $\mu^* : L^* \to L^* \otimes L^* \otimes L^*$ be the dual map of μ, for every $\xi \in L^*, x, y, z \in L$, $\langle \mu^*(\xi), x \otimes y \otimes z\rangle = \langle \xi, \mu(x, y, z)\rangle$.

Then we have the following result.

Lemma 2.1 [4] *Let L be a vector space over F, and $\mu : L \otimes L \otimes L \to L$ be a 3-ary linear map. Then (L, μ) is a 3-Lie algebra if and only if (L^*, μ^*) is a 3-Lie coalgebra with $\mu^* : L^* \to L^* \otimes L^* \otimes L^*$.*

Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. If there is a linear isomorphism $\varphi : L_1 \to L_2$ satisfying $(\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e))$, for all $e \in L_1$, then (L_1, Δ_1) is isomorphic to (L_2, Δ_2), and φ is called *a 3-Lie coalgebra isomorphism*, where $(\varphi \otimes \varphi \otimes \varphi) \sum_i (a_i \otimes b_i \otimes c_i) = \sum_i \varphi(a_i) \otimes \varphi(b_i) \otimes \varphi(c_i)$.

Lemma 2.2 [4] *Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. Then $\varphi : L_1 \to L_2$ is a 3-Lie coalgebra isomorphism if and only if the dual map φ^* :*

$L_2^* \to L_1^*$ *is a* 3*-Lie algebra isomorphism from* (L_2^*, Δ_2^*) *to* (L_1^*, Δ_1^*), *where for every* $\xi \in L_2^*, v \in L_1$, $\langle \varphi^*(\xi), v\rangle = \langle \xi, \varphi(v)\rangle$.

A 3-Lie bialgebra[4] is a triple (L, μ, Δ) such that
(1) (L, μ) is a 3-Lie algebra with the multiplication $\mu : L \wedge L \wedge L \to L$,
(2) (L, Δ) is a 3-Lie coalgebra with $\Delta : L \to L \wedge L \wedge L$,
(3) Δ and μ satisfy the following identity

$$\Delta\mu(x, y, z) = ad_\mu^{(3)}(x, y)\Delta(z) + ad_\mu^{(3)}(y, z)\Delta(x) + ad_\mu^{(3)}(z, x)\Delta(y),$$

where $ad_\mu(x, y) : L \wedge L \to End(L), ad_\mu(x, y)(z) = \mu(x, y, z)$ for $x, y, z \in L$, $ad_\mu^{(3)}(x, y)$, $ad_\mu^{(3)}(z, x)$, $ad_\mu^{(3)}(y, z) : L \otimes L \otimes L \to L \otimes L \otimes L$ are 3-ary linear maps satisfying for $u, v, w \in L$

$$ad_\mu^{(3)}(x, y)(u \otimes v \otimes w) = (ad_\mu(x, y) \otimes 1 \otimes 1)(u \otimes v \otimes w)$$
$$+(1 \otimes ad_\mu(x, y) \otimes 1)(u \otimes v \otimes w) + (1 \otimes 1 \otimes ad_\mu(x, y))(u \otimes v \otimes w)$$
$$= \mu(x, y, u) \otimes v \otimes w + u \otimes \mu(x, y, v) \otimes w + u \otimes v \otimes \mu(x, y, w),$$

and similar for $ad_\mu^{(3)}(z, x)$ and $ad_\mu^{(3)}(y, z)$.

Two 3-Lie bialgebras (L_1, μ_1, Δ_1) and (L_2, μ_2, Δ_2) are called *equivalent* if there exists a linear isomorphism $f : L_1 \to L_2$ such that
(1) $f : (L_1, \mu_1) \to (L_2, \mu_2)$ is a 3-Lie algebra isomorphism, that is,
$f\mu_1(x, y, z) = \mu_2(f(x), f(y), f(z))$ for $\forall x, y, z \in L_1$;
(2) $f : (L_1, \Delta_1) \to (L_2, \Delta_2)$ is a 3-Lie coalgebra isomorphism, that is,
$\Delta_2(f(x)) = (f \otimes f \otimes f)\Delta_1(x)$ for every $x \in L_1$.

3 The classification of 3-Lie coalgebras and 3-Lie bialgebras

We first give the following lemma.

Lemma 3.1[5] *Let* (L, μ) *be a* 4*-dimensional* 3*-Lie algebra with* $\dim L^1 = 1$, *and* e_1, e_2, e_3, e_4 *be a basis of* L. *Then* L *is isomorphic to one and only one of the two* 3*-Lie algebras* $L_{b_1} = (L, \mu_{b_1})$ *and* $L_{b_2} = (L, \mu_{b_2})$, *where*

$L_{b_1} : \mu_{b_1}(e_2, e_3, e_4) = e_1$; $L_{b_2} : \mu_{b_2}(e_1, e_2, e_3) = e_1$.

Theorem 3.2 *Let* (L, Δ) *be a* 4*-dimensional* 3*-Lie coalgebra with* $dimL^1 = 1$, *and* e^1, e^2, e^3, e^4 *be a basis of* L. *Then* (L, Δ) *is isomorphic to one and only one of the two* 3*-Lie coalgebras* $C_{b_1} = (L, \Delta_{b_1})$ *and* $C_{b_2} = (L, \Delta_{b_2})$, *where*

$\Delta_{b_1}(e^1) = e^2 \wedge e^3 \wedge e^4$; $\Delta_{b_2}(e^1) = e^1 \wedge e^2 \wedge e^3$.

Proof The result follows from Lemma 2.1 and 2.2 and a direct computation, we omit the consideration process.

For a given 3-Lie algebra L, in order to find all the 3-Lie bialgebra structures on L, we should find all the 3-Lie coalgebra structures on L which are compatible with the 3-Lie algebra L. Although a permutation of a basis vectors of L gives isomorphic 3-Lie coalgebra structures, it may lead to the non-equivalent 3-Lie bialgebra structures on L. In the following, for a 3-Lie bialgebra (L, μ, Δ), if the 3-Lie algebra (L, μ) is the case (L, μ_{b_i}) in Lemma 3.1 and the 3-Lie coalgebra (L, Δ) is the case (L, Δ_{b_j}) in Theorem 3.2, then we denote $(L, \mu_{b_i}, \Delta_{b_j})$ by $(L_{b_i}, C_{b_j}, \Delta_{b_k})$ or simply, by (L_{b_i}, C_{b_j}). *The* 3-*Lie bialgebras* (L_{b_i}, C_{b_j}) *for* $1 \le i, j \le 2$ *are called the* 3-*Lie bialgebras of type* (L_b, C_b).

Theorem 3.3 The non-equivalent 4-dimensional 3-Lie bialgebras of type (L_b, C_b) are only as follows, for a basis e_1, e_2, e_3, e_4 of L,
$(L_{b_1}, C_{b_1}, \Delta_1)$: $\Delta_1(e_2) = e_1 \wedge e_3 \wedge e_4$; $(L_{b_2}, C_{b_1}, \Delta_2)$: $\Delta_1(e_2) = e_1 \wedge e_4 \wedge e_3$;
$(L_{b_1}, C_{b_2}, \Delta_3)$: $\Delta_3(e_2) = e_2 \wedge e_1 \wedge e_3$; $(L_{b_2}, C_{b_2}, \Delta_4)$: $\Delta_4(e_1) = e_1 \wedge e_2 \wedge e_4$;
$(L_{b_2}, C_{b_2}, \Delta_5)$: $\Delta_5(e_1) = e_1 \wedge e_2 \wedge e_3$; $(L_{b_2}, C_{b_2}, \Delta_3)$: $\Delta_3(e_2) = e_2 \wedge e_1 \wedge e_3$;
$(L_{b_2}, C_{b_2}, \Delta_6)$: $\Delta_6(e_2) = e_2 \wedge e_1 \wedge e_4$.

Proof From Lemma 3.1 and Theorem 3.2 we need to verify that whether the following eight 3-Lie coalgebras of type C_{b_1}, which are obtained by permuting the basis vectors e_1, e_2, e_3, e_4, are compatible with the 3-Lie algebra L_{b_1}

(1) $\Delta(e_1) = e_2 \wedge e_3 \wedge e_4, \Delta(e_2) = \Delta(e_3) = \Delta(e_4) = 0$;

(2) $\Delta(e_1) = e_2 \wedge e_4 \wedge e_3, \Delta(e_2) = \Delta(e_3) = \Delta(e_4) = 0$;

(3) $\Delta(e_2) = e_1 \wedge e_4 \wedge e_3, \Delta(e_1) = \Delta(e_3) = \Delta(e_4) = 0$;

(4) $\Delta(e_2) = e_1 \wedge e_3 \wedge e_4, \Delta(e_1) = \Delta(e_3) = \Delta(e_4) = 0$;

(5) $\Delta(e_3) = e_1 \wedge e_2 \wedge e_4, \Delta(e_1) = \Delta(e_2) = \Delta(e_4) = 0$;

(6) $\Delta(e_3) = e_1 \wedge e_4 \wedge e_2, \Delta(e_1) = \Delta(e_2) = \Delta(e_4) = 0$;

(7) $\Delta(e_4) = e_1 \wedge e_2 \wedge e_3, \Delta(e_1) = \Delta(e_2) = \Delta(e_3) = 0$;

(8) $\Delta(e_4) = e_2 \wedge e_1 \wedge e_3, \Delta(e_1) = \Delta(e_2) = \Delta(e_3) = 0$.

First we discuss the case (1). Thanks to Lemma 3.1, $\Delta\mu_{b_1}(e_1, e_2, e_3) = 0$. But
$ad^{(3)}_{\mu_{b_1}}(e_1, e_2)\Delta(e_3) + ad^{(3)}_{\mu_{b_1}}(e_3, e_1)\Delta(e_2) + ad^{(3)}_{\mu_{b_1}}(e_2, e_3)\Delta(e_1)$
$= \mu_{b_1}(e_2, e_3, e_2) \wedge e_3 \wedge e_4 + e_2 \wedge \mu_{b_1}(e_2, e_3, e_3) \wedge e_4 + e_2 \wedge e_3 \wedge \mu_{b_1}(e_2, e_3, e_4)$
$= e_2 \wedge e_3 \wedge e_1 \ne 0$, that is,
$\Delta\mu_{b_1}(e_1, e_2, e_3) \ne ad^{(3)}_{\mu_{b_1}}(e_1, e_2)\Delta(e_3) + ad^{(3)}_{\mu_{b_1}}(e_3, e_1)\Delta(e_2) + ad^{(3)}_{\mu_{b_1}}(e_2, e_3)\Delta(e_1)$.
Therefore, 3-Lie algebra L_{b_1} is not compatible with the case (1).

By the similar discussions to the case (1), the 3-Lie algebra L_{b_1} is not compatible with the case (2), and the 3-Lie algebra L_{b_2} is not compatible with the cases (1) and (2). And the 3-Lie algebras L_{b_1} and L_{b_2} are compatible with all the cases (3)-(8), respectively.

By the following 3-Lie bialgebra isomorphisms $f : L \to L$, in the case (L_{b_1}, C_{b_1}) :
$(4) \to (6), (3) \to (5) : f(e_1) = -e_1, f(e_2) = e_3, f(e_3) = e_2, f(e_4) = e_4$;
$(5) \to (8), (6) \to (7) : f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_4, f(e_4) = e_3$.

In the case (L_{b_2}, C_{b_1}):
$(4) \to (6), (3) \to (5) : f(e_1) = e_1, f(e_2) = e_3, f(e_3) = -e_2, f(e_4) = e_4$;
$(3) \to (4), (7) \to (8) : f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = e_4$,
and e_4 is in the center of the 3-Lie algebra $L_{\mu_{b_2}}$, we get the non-equivalent 3-Lie bialgebras of the types $(L_{b_1}, C_{b_1}, \Delta_1)$, $(L_{b_2}, C_{b_1}, \Delta_2)$.

Second, we verify that whether the following twenty-four isomorphic 3-Lie coalgebras of the type C_{b_2} are compatible with the 3-Lie algebra L_{b_1} and L_{b_2}, respectively,
$(1)\Delta e_1 = e_1 \wedge e_2 \wedge e_3, \Delta e_i = 0, i = 2, 3, 4$; $(2)\Delta e_1 = e_1 \wedge e_2 \wedge e_4, \Delta e_i = 0, i = 2, 3, 4$;
$(3)\Delta e_1 = e_1 \wedge e_3 \wedge e_4, \Delta e_i = 0, i = 2, 3, 4$; $(4)\Delta e_1 = e_1 \wedge e_3 \wedge e_2, \Delta e_i = 0, i = 2, 3, 4$;
$(5)\Delta e_1 = e_1 \wedge e_4 \wedge e_2, \Delta e_i = 0, i = 2, 3, 4$; $(6)\Delta e_1 = e_1 \wedge e_4 \wedge e_3, \Delta e_i = 0, i = 2, 3, 4$;
$(7)\Delta e_2 = e_2 \wedge e_1 \wedge e_3, \Delta e_i = 0, i = 1, 3, 4$; $(8)\Delta e_2 = e_2 \wedge e_1 \wedge e_4, \Delta e_i = 0, i = 1, 3, 4$;

(9)$\Delta e_2 = e_2 \wedge e_3 \wedge e_4, \Delta e_i = 0, i = 1, 3, 4$; (10)$\Delta e_2 = e_2 \wedge e_3 \wedge e_1, \Delta e_i = 0, i = 1, 3, 4$;
(11)$\Delta e_2 = e_2 \wedge e_4 \wedge e_1, \Delta e_i = 0, i = 1, 3, 4$; (12)$\Delta e_2 = e_2 \wedge e_4 \wedge e_3, \Delta e_i = 0, i = 1, 3, 4$;
(13)$\Delta e_3 = e_3 \wedge e_1 \wedge e_2, \Delta e_i = 0, i = 1, 2, 4$; (14)$\Delta e_3 = e_3 \wedge e_1 \wedge e_4, \Delta e_i = 0, i = 1, 2, 4$;
(15)$\Delta e_3 = e_3 \wedge e_2 \wedge e_4, \Delta e_i = 0, i = 1, 2, 4$; (16)$\Delta e_3 = e_3 \wedge e_2 \wedge e_1, \Delta e_i = 0, i = 1, 2, 4$;
(17)$\Delta e_3 = e_3 \wedge e_4 \wedge e_1, \Delta e_i = 0, i = 1, 2, 4$; (18)$\Delta e_3 = e_3 \wedge e_4 \wedge e_2, \Delta e_i = 0, i = 1, 2, 4$;
(19)$\Delta e_4 = e_4 \wedge e_1 \wedge e_2, \Delta e_i = 0, i = 1, 2, 3$; (20)$\Delta e_4 = e_4 \wedge e_1 \wedge e_3, \Delta e_i = 0, i = 1, 2, 3$;
(21)$\Delta e_4 = e_4 \wedge e_2 \wedge e_3, \Delta e_i = 0, i = 1, 2, 3$; (22)$\Delta e_4 = e_4 \wedge e_2 \wedge e_1, \Delta e_i = 0, i = 1, 2, 3$;
(23)$\Delta e_4 = e_4 \wedge e_3 \wedge e_1, \Delta e_i = 0, i = 1, 2, 3$; (24)$\Delta e_4 = e_4 \wedge e_3 \wedge e_2, \Delta e_i = 0, i = 1, 2, 3$.

By the similar discussions to the casae C_{b_1}, the only the cases (7), (8), (10), (11), (13), (14), (16), (17), (19), (20), (22) and (23) are compatible with L_{b_1}. And the only cases (9), (12), (15), (18), (19), (20), (21), (22), (23) and (24) are not compatible with the 3-Lie algebra L_{b_2}. By the following 3-Lie bialgebra isomorphisms $f : L \to L$, in the case (L_{b_1}, C_{b_2}) : $(8) \to (22)$: $(e_1) = -e_1,\ f(e_2) = e_4,\ f(e_3) = e_3, f(e_4) = e_2$;
$(7) \to (8), (10) \to (11) : f(e_1) = e_1, f(e_2) = -e_2, f(e_3) = e_4, f(e_4) = e_3$;
$(13) \to (14), (16) \to (17) : f(e_1) = e_1,\ f(e_2) = e_4,\ f(e_3) = -e_3,\ f(e_4) = e_2$;
$(19) \to (20), (22) \to (23) : f(e_1) = e_1,\ f(e_2) = e_3,\ f(e_3) = e_2,\ f(e_4) = -e_4$;
$(7) \to (10) : f(e_1) = -e_1,\ f(e_2) = -e_2,\ f(e_3) = e_3,\ f(e_4) = e_4$;
$(7) \to (13) : f(e_1) = -e_1,\ f(e_2) = e_3,\ f(e_3) = -e_2,\ f(e_4) = -e_4$;
$(7) \to (16) : f(e_1) = -e_1,\ f(e_2) = e_3,\ f(e_3) = e_2,\ f(e_4) = e_4$;
$(8) \to (19) : f(e_1) = -e_1,\ f(e_2) = e_4,\ f(e_3) = -e_3,\ f(e_4) = -e_2$.

(L_{b_2}, C_{b_2}) : $(1) \to (4), (2) \to (5), (3) \to (6), (7) \to (10), (8) \to (11), (13) \to (16), (14) \to (17) : f(e_1) = -e_1, f(e_2) = e_2, f(e_3) = e_3, f(e_4) = e_4$;
$(2) \to (3)$: $f(e_1) = -e_1, f(e_2) = e_3, f(e_3) = -e_2, f(e_4) = -e_4$;
$(7) \to (13), (8) \to (14) : f(e_1) = e_1, f(e_2) = -e_3, f(e_3) = e_2, f(e_4) = e_4$,
and e_4 is in the center of the 3-Lie algebra L_{b_2}, we obtain the result.

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