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# Blow-up and global existence for the periodic two-component $\mu$-Hunter-Saxton system

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**Abstract**

The two-component $\mu$-Hunter-Saxton system is considered in the spatially periodic setting. Firstly, a wave-breaking criterion is derived by employing the localization analysis of the transport equation theory. Using this criterion, then we prove the global existence of strong solutions for the system.

**Mathematics Subject Classification:** 35D05, 35G25, 35L05, 35Q35

**Keywords:**Two-component $\mu$-Hunter-Saxton system; Wave-breaking; Global existence.

## 1 Introduction

In this article, we will consider the periodic two-component $\mu$-Hunter-Saxton system derived by Zuo [11]

$$\begin{cases} \mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx} + \rho\rho_x - \gamma_1 u_{xxx}, & t > 0, x \in \mathbb{R}, \\ \rho_t = (u\rho)_x + 2\gamma_2\rho_x, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t > 0, x \in \mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t > 0, x \in \mathbb{R}, \end{cases} \tag{1}$$

where $u(t,x)$ and $\rho(t,x)$ are time-dependent functions on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, $\mu(u) = \int_{\mathbb{S}} u dx$ denotes its mean and $\gamma_i \in \mathbb{R}$, $i = 1, 2$. It is shown in [11] that system (1) is an Euler equation with bi-Hamilton structure

$$\Gamma_1 = \begin{pmatrix} \partial_x A & 0 \\ 0 & \partial_x \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} A(u)\partial_x + \partial_x A(u) - \gamma_1\partial_x^3 & \rho\partial_x \\ \partial_x\rho & 2\gamma_2\partial_x \end{pmatrix},$$

where $A(u) = \mu(u) - u_{xx}$, and also be viewed as a bi-variational equation. Moreover, for $\gamma_i = 0$, $i = 1, 2$, system (1) has a Lax pair given by

$$\psi_{xx} = \lambda(A(u) - \lambda^2\rho^2)\psi, \quad \psi_t = (u - \frac{1}{2\lambda})\psi_x - \frac{1}{2}u_x\psi,$$

where $\lambda$ is a spectral parameter (see [11]).

In fact, system (1) is a generalization of the generalized Hunter-Saxton equation [5, 6]

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \tag{2}$$

which describes the geodesic flow on $D^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product $\langle u, v\rangle = \mu(u)\mu(v) + \int_{\mathbb{S}} u_xv_xdx$, and models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic nematic field and self-interaction. Here, the solution $u(t, x)$ denotes the director field of a nematic liquid crystal. It was observed in [5, 6] that the $\mu$-Hunter-Saxton equation is formally integrable, has bi-Hamiltonian structure and infinite hierarchy of conservation laws. Further, the development of singularities in finite time and geometric descriptions of the system from nonstretching invariant curve flows in centro-equiaffine geometries, pseudo-spherical surfaces and affine surfaces are described by Fu et al [3] .

Recently, Liu and Yin [7, 8] investigated the Cauchy problem for system (1). In [7], the local well-posedness and several precise blow-up criteria for the system were obtained. Under the conditions $\mu_0 = 0$ and $\mu_0 \neq 0$, the sufficient conditions of blow-up solutions were presented. The global existence for strong solution for system (1) in the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s = 2$ is also given [7], and in [8], existence of global weak solution is established for the periodic two-component $\mu$-Hunter-Saxton system. The objective of the present paper is to focus mainly on wave-breaking criterion and several sufficient conditions of blow-up solutions.

Motivated by the works in [4, 9], in the present paper, the localization analysis in the transport equation theory is employed to derive a new wave-breaking criterion of solutions for the system (1) in the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$. It implies that the wave-breaking criterion is determined only by the slope of the component $u$ of solution definitely. Further, by using the wave-breaking criterion, we also present a sufficient condition for the existence of global strong solutions. Motivated by the work in [2]. These results obtained in this paper are new and different from those in Liu and Yin's work [7].

The rest of this paper is organized as follows. Section 2 states several properties for the periodic two-component $\mu$-Hunter-Saxton system and gives several lemmas. In Section 3, we present a wave-breaking criterion in the Soblev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$. An improved result of the global existence of solutions for system (1) is given in Section 4.

# 2 Preliminary

**Lemma 2.1** ([7]) *Given $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, then there exists a maximal $T = T(\| z_0 \|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}) > 0$ and an unique solution $z = (u, \rho)$ to system (1) such that*

$$z = z(\cdot, z_0) \in C([0,T); H^s(\mathbb{S})) \bigcap C^1([0,T); H^{s-1}(\mathbb{S})).$$

**Lemma 2.2** ([1]) *For every $f(x) \in H^1(a,b)$ periodic and with zero average, i.e.such that $\int_a^b f(x)dx = 0$, it holds that*

$$\int_a^b f^2(x)dx \leq (\frac{b-a}{2\pi})^2 \int_a^b \mid f'(x) \mid^2 dx,$$

*and equality holds if and only if*

$$f(x) = A\cos(\frac{2\pi x}{b-a}) + B\sin(\frac{2\pi x}{b-a}).$$

Integrating the first equation of system (1) over the circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and noting the periodicity of $u$, we have $\mu(u_t) = 0$. Making use of system (1), we have that $\int_{\mathbb{S}}(u_x^2 + \rho^2)dx$ is conserved in time (see [7]). In what follows we denote

$$\mu_0 = \mu(u_0) = \mu(u) = \int_{\mathbb{S}} u(t,x)dx \tag{3}$$

and

$$\mu_1 = \left(\int_{\mathbb{S}} u_x^2(t,x) + \rho^2(t,x)dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}} u_x^2(0,x) + \rho^2(0,x)dx\right)^{\frac{1}{2}}. \tag{4}$$

Then $\mu_0$ and $\mu_1$ are constants and independent of time $t$.

Notice that $\int_{\mathbb{S}}(u(t,x) - \mu_0)dx = \mu_0 - \mu_0 = 0$. From Lemma 2.5, we get

$$\begin{aligned}\max_{x\in\mathbb{S}}[u(t,x) - \mu_0]^2 \quad &\leq \frac{1}{12}\int_{\mathbb{S}} u_x^2(t,x)dx \leq \frac{1}{12}\int_{\mathbb{S}} u_x^2(t,x) + \rho^2(t,x)dx \\ &= \frac{1}{12}\int_{\mathbb{S}} u_x^2(0,x) + \rho^2(0,x)dx = \frac{1}{12}\mu_1^2, \end{aligned} \tag{5}$$

which implies that the amplitude of wave remains bounded in any time. Namely, we have

$$\| u(t,\cdot) \|_{L^\infty(\mathbb{S})} - |\mu_0| \leq \| u(t,\cdot) - \mu_0 \|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6}\mu_1, \tag{6}$$

which results in

$$\| u(t,\cdot) \|_{L^\infty(\mathbb{S})} \leq |\mu_0| + \frac{\sqrt{3}}{6}\mu_1. \tag{7}$$

In fact, the initial-value problem (1) can be recast in the following

$$\begin{cases} u_t - (u+\gamma_1)u_x = A^{-1}\partial_x(2\mu_0 u + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), & t>0, x\in\mathbb{R}, \\ \rho_t - (u+2\gamma_2)\rho_x = \rho u_x, & t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \\ u(t,x+1) = u(t,x), & t>0, x\in\mathbb{R}, \\ \rho(t,x+1) = \rho(t,x), & t>0, x\in\mathbb{R}, \end{cases} \tag{8}$$

where $A = \mu - \partial_x^2$ is an isomorphism between $H^s$ and $H^{s-2}$ with the inverse $\nu = A^{-1}\omega$ given explicitly by

$$\begin{aligned} \nu(x) = \quad & (\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12})\mu(\omega) + (x-\frac{1}{2})\int_0^1\int_0^y \omega(s)dsdy \\ & - \int_0^x\int_0^y \omega(s)dsdy + \int_0^1\int_0^y\int_0^s \omega(r)drdsdy. \end{aligned} \tag{9}$$

Commuting $A^{-1}$ and $\partial_x$, we get

$$A^{-1}\partial_x\omega(x) = (x-\frac{1}{2})\int_0^1 \omega(x) - \int_0^x \omega(y)dy + \int_0^1\int_0^x \omega(y)dydx \tag{10}$$

and

$$A^{-1}\partial_x^2\omega(x) = -\omega(x) + \int_0^1 \omega(x)dx. \tag{11}$$

Note that if $f\in L^2(\mathbb{S})$, we have $A^{-1}f = (\mu - \partial_x^2)^{-1}f = g*f$, where we denotes by $*$ convolution and $g$ is the Green's function of the operator $A^{-1}$, given by

$$g(x) = \frac{1}{2}(x-\frac{1}{2})^2 + \frac{23}{24}, \tag{12}$$

and the derivative of $g$ can be assigned

$$g_x(x) = \begin{cases} 0, & x=0, \\ x-\frac{1}{2}, & x\in(0,1). \end{cases} \tag{13}$$

Now, consider the initial value problem for the Lagrangian flow map:

$$\begin{cases} \eta_t = u(t,-\eta) + 2\gamma_2, & t\in[0,T), \\ \eta(0,x) = x, & x\in\mathbb{R}, \end{cases} \tag{14}$$

where $u$ denotes the first component of the solution $z=(u,\rho)$ to system (1). Applying classical results from ordinary differential equations, one can obtain the result.

**Lemma 2.3** ([7]) *Let $u\in C([0,T);H^s(\mathbb{R}))\bigcap C^1([0,T);H^{s-1}(\mathbb{R}))$, $s\geq 2$. Then Eq.(14) has an unique solution $\eta\in C^1([0,T)\times\mathbb{R};\mathbb{R})$. Moreover, the map $\eta(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$\eta_x(t,x)=\exp(-\int_0^t u_x(s,-\eta(s,x))ds)>0,\quad (t,x)\in[0,T)\times\mathbb{R}. \tag{15}$$

**Lemma 2.4** ([7]) *Let $z_0=(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geq 2$ and let $T>0$ be the maximal existence time of the corresponding solution $z=(u,\rho)$ to system (1). Then it has*

$$\rho(t,-\eta(t,x))\eta_x(t,x)=\rho_0(-x),\quad (t,x)\in[0,T)\times\mathbb{R}. \tag{16}$$

# 3 Wave-breaking criterion

**Theorem 3.1** *Let $z_0=(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geq 2$, and $z=(u,\rho)$ be the corresponding solution to (1). Assume that $T>0$ is the maximal existence time. Then*

$$T<\infty\Rightarrow\int_0^T\parallel u_x\parallel_{L^\infty(\mathbb{S})}d\tau=\infty. \tag{17}$$

**Proof.** The proof is similar with that of Theorem 1 in [12] , hence, we omit the proof of Theorem 3.1.

# 4 Global existence

In this section, using the above criterion of wave breaking, we provide a sufficient condition for the global solution of system (1).

**Theorem 4.1** *Let $z_0=(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s\geq 2$ and let $T$ be the maximal time of existence. If $\gamma_1=2\gamma_2$ and $\rho_0(-x)\neq 0$, then the solution $z=(u,\rho)$ of system (1) with initial value $z_0=(u_0,\rho_0)$ is global.*

**Proof.** Applying a simple density argument, we only need to consider the case $s=3$. From Lemma 2.3, we know that $\eta(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$. Setting $M(t)=u_x(t,-\eta(t,x))$ and $\psi(t)=\rho(t,-\eta(t,x))$

and applying the assumption of theorem and (14), system (8) becomes the following ordinary differential equations

$$\begin{aligned}&M'(t) = \frac{1}{2}M^2 - \frac{1}{2}\psi^2(t) + f(t, -\eta(t.x)), \quad \text{a.e.} \quad t \in [t_0, T),\\ &\psi'(t) = \psi M, \quad \text{a.e.} \quad t \in [t_0, T),\end{aligned} \tag{18}$$

where $f = -2\mu_0 u + 2\mu_0^2 + \frac{1}{2}\mu_1^2$. For every $x \in \mathbb{S}$, we know from Lemma 2.4 that $\psi(0)$ and $\psi(t)$ are of the same sign. Define the Lyapunov function

$$\omega(t) = \psi(0)\psi(t) + \frac{\psi(0)}{\psi(t)}(1 + M^2(t)), \quad (t, x) \in [0, T) \times R, \tag{19}$$

which is a positive function of $t \in [0, T)$. From (18), it yields

$$\begin{aligned}\omega'(t) &= \psi(0)\psi'(t) - \frac{\psi(0)}{\psi^2(t)}\psi'(t)(1 + M^2(t)) + \frac{2\psi(0)}{\psi(t)}MM'\\ &= \frac{2\psi(0)}{\psi(t)}M(f(t, -\eta(t, x)) - \frac{1}{2})\\ &\le \frac{\psi(0)}{\psi(t)}(1 + M^2)(|f(t, -\eta(t, x))| + \frac{1}{2})\\ &\le (4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})\omega(t), \quad (t, x) \in [0, T) \times R.\end{aligned} \tag{20}$$

Using the Gronwall's inequality, we have

$$\begin{aligned}\omega(t) \quad &\le \omega(0)e^{(4\mu_0^2+\frac{1}{2}\mu_1^2+\frac{\sqrt{3}}{3}|\mu_0|\mu_1+\frac{1}{2})t}\\ &= C_3e^{C_2t}, \quad t \in [0, T),\end{aligned} \tag{21}$$

where $\omega(0) = \rho^2(-x) + 1 + u_{0,x}^2(-x) \le 1+ \parallel \rho_0 \parallel_{L^\infty}^2 + \parallel u_{0,x} \parallel_{L^\infty}^2 = C_3$ and $C_2 = 4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2}$.

Since $\psi(t)$ and $\psi(0)$ are of the same sign. The definition of $\omega(t)$ implies $|\psi(0)||M(t)| \le \omega(t)$ and $\psi(0)\psi(t) \le \omega(t)$. From (21), we obtain

$$\begin{aligned}|u_x(t, -\eta(t, x))| \quad &= |M(t)| \le \frac{1}{|\psi(0)|}\omega(t)\\ &\le \frac{1}{|\rho_0(-x)|}C_3e^{C_2t}, \quad t \in [0, T)\end{aligned} \tag{22}$$

and

$$\begin{aligned}|\rho(t, -\eta(t, x))| \quad &= |\psi(t)| \le \frac{1}{|\psi(0)|}\omega(t)\\ &\le \frac{1}{|\rho_0(-x)|}C_3e^{C_2t}, \quad t \in [0, T).\end{aligned} \tag{23}$$

Now, we assume on the contrary that $T < \infty$ and the solution blows up in finite time. It follows from Theorem 3.1 that

$$\int_0^T \| u_x(t,x) \|_{L^\infty} \, dt = \infty. \tag{24}$$

From (22), we have

$$|u_x(t,x)| \quad \le \frac{1}{|\rho_0(-x)|} C_3 e^{C_2 t} < \infty, \quad (t,x) \in [0,T) \times R, \tag{25}$$

which leads to a contradiction. Thus, $T = +\infty$, and the solution $z = (u, \rho)$ is global. This completes the proof of Theorem 4.1.

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