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Bipolar Valued Fuzzy Translation in Semigroups

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Abstract

In this paper the notions of bipolar valued fuzzy translation and a bipolar valued fuzzy S-extension of a bipolar valued fuzzy subsemigroup (bi-ideal) in a semigroup are introduced and some of their important properties have been investigated. As an application we also introduce the notion of bipolar valued fuzzy equivalence relation.

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1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by Lofti Zadeh[8] in his classic paper in 1965. The concept of bipolar-valued fuzzy sets was introduced by Lee[4, 5]. In traditional fuzzy sets the membership degree range is $[0, 1]$. The membership degree is the degree of belonging ness of an element to a set. The membership degree 1 indicates that an element completely belongs to its corresponding set, the membership degree 0 indicates that an element does not belong to the corresponding set and the membership degree on the interval $(0, 1)$ indicate the partial membership to the corresponding set. Sometimes, membership degree also means the satisfaction degree of elements to some property corresponding to a set and its counter property. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is increased from the interval $[0, 1]$ to the interval $[-1, 1]$. In a bipolar-valued fuzzy set the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property and the membership degrees on $[-1, 0]$ indicate that elements somewhat satisfy the implicit counter-property. In 2000 Lee[4] introduced the notion of bipolar valued fuzzy sets and in 2009 Lee[6] also introduced the notion of bipolar fuzzy subalgebra and bipolar fuzzy ideal in BCK/BCI-algebras. Jun et al.[2] introduced the notion of bipolar fuzzy subalgebra and bipolar fuzzy ideal in BCH-algebras. The concept of bipolar valued fuzzy translation and bipolar valued fuzzy S-extension of a bipolar valued fuzzy subalgebra in BCK/BCI-algebra was introduced by Y.B. Jun, H.S. Kim and K.J. Lee[3]. The purpose of this paper is as stated in the abstract.

2 Preliminary Notes

In this section we discuss some elementary definitions that we use in the sequel.

Definition 2.1 [*1*] *If* $(S, *)$ *is a mathematical system such that* $\forall a, b, c \in S$, $(a*b)*c = a*(b*c)$, *then* $*$ *is called associative and* $(S, *)$ *is called a semigroup.*

Definition 2.2 [*1*] *A non-empty subset* A *of a semigroup* S *is called a subsemigroup of* S *if* $A^2 \subseteq A$.

Definition 2.3 [*1*] *A subsemigroup* A *of a semigroup* S *is called a bi-ideal of* S *if* $ASA \subseteq A$.

Definition 2.4 [3] *Let S be the universe of discourse. A bipolar valued fuzzy set φ in S is an object having the form $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ where the negative membership degree $\varphi^- : S \to [-1, 0]$ is a mapping that denotes the satisfaction degree of an element x to some implicit counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ and the positive membership degree $\varphi^+ : S \to [0, 1]$ is a mapping that denotes the satisfaction degree of an element x to the property corresponding to $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.*

Remark 1 [3] (1) If $\varphi^+(x) \neq 0$ and $\varphi^-(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

(2) If $\varphi^+(x) = 0$ and $\varphi^-(x) \neq 0$, then it is the situation that x does not satisfy the property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$ but some what satisfies the counter property of $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

(3) If $\varphi^+(x) \neq 0$ and $\varphi^-(x) \neq 0$, then the membership function of the property overlaps that of its counter property over some portion of the domain.

Remark 2 For the sake of simplicity we shall use the symbol $\varphi = (S; \varphi^-, \varphi^+)$ for the bipolar valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) : x \in S\}$.

Definition 2.5 [3] *The union of any two bipolar valued fuzzy sets $\varphi = (S; \varphi^-, \varphi^+)$ and $\psi = (S; \psi^-, \psi^+)$ is a bipolar valued fuzzy set $\varphi \cup \psi = (S; (\varphi \cup \psi)^-, (\varphi \cup \psi)^+)$ where $(\varphi \cup \psi)^- : S \to [-1, 0]$ is a mapping defined by $(\varphi \cup \psi)^-(x) = \min\{\varphi^-(x), \psi^-(x)\}\ \forall x \in S$ and $(\varphi \cup \psi)^+ : S \to [0, 1]$ is a mapping defined by $(\varphi \cup \psi)^+(x) = \max\{\varphi^+(x), \psi^+(x)\}\ \forall x \in S$.*

Definition 2.6 [3] *The intersection of any two bipolar valued fuzzy sets $\varphi = (S; \varphi^-, \varphi^+)$ and $\psi = (S; \psi^-, \psi^+)$ is a bipolar valued fuzzy set $\varphi \cap \psi = (S; (\varphi \cap \psi)^-, (\varphi \cap \psi)^+)$ where $(\varphi \cap \psi)^- : S \to [-1, 0]$ is a mapping defined by $(\varphi \cap \psi)^-(x) = \max\{\varphi^-(x), \psi^-(x)\}\ \forall x \in S$ and $(\varphi \cap \psi)^+ : S \to [0, 1]$ is a mapping defined by $(\varphi \cap \psi)^+(x) = \min\{\varphi^+(x), \psi^+(x)\}\ \forall x \in S$.*

Remark 3 [3] For any bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ in S, we denote $\bot := -1 - \inf\{\varphi^-(x) : x \in S\}$ and $\top := 1 - \sup\{\varphi^+(x) : x \in S\}$.

Definition 2.7 [3] *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy set of a semigroup S and $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$. By a bipolar valued fuzzy (α, β)-translation of $\varphi = (S; \varphi^-, \varphi^+)$ we mean a bipolar valued fuzzy set $\varphi^T_{(\alpha,\beta)} = (S; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)})$ where $\varphi^-_{(\alpha,T)} : S \to [-1, 0]$ is a mapping defined by $\varphi^-_{(\alpha,T)}(x) = \varphi^-(x) + \alpha\ \forall x \in S$ and $\varphi^+_{(\beta,T)} : S \to [0, 1]$ is a mapping defined by $\varphi^+_{(\beta,T)}(x) = \varphi^+(x) + \beta\ \forall x \in S$.*

Example 1 Let $S = \{e, a, b\}$ be a set with following caley table:

$$\begin{array}{c|ccc} & e & a & b \\ \hline e & e & e & e \\ a & e & a & e \\ b & e & e & b \end{array}$$

Then S is a semigroup. We define a bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of S as follows:

S	e	a	b
φ^-	-0.5	-0.3	-0.1
φ^+	0.6	0.5	0.5

Let $\alpha = -0.2$ and $\beta = 0.3$. Then the bipolar valued fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = (S; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)})$ of $\varphi = (S; \varphi^-, \varphi^+)$ is

S	e	a	b
$\varphi^-_{(\alpha,T)}$	-0.7	-0.5	-0.3
$\varphi^+_{(\beta,T)})$	0.9	0.8	0.8

Definition 2.8 *A non-empty bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of a semigroup S is called a bipolar valued fuzzy subsemigroup of S if* (1) $\varphi^-(xy) \leq \max\{\varphi^-(x), \varphi^-(y)\}$, (2) $\varphi^+(xy) \geq \min\{\varphi^+(x), \varphi^+(y)\}$ $\forall x, y \in S$.

Definition 2.9 *A bipolar valued fuzzy subsemigroup $\varphi = (S; \varphi^-, \varphi^+)$ of a semigroup S is called a bipolar valued fuzzy bi-ideal of S if* (1) $\varphi^-(xyz) \leq \max\{\varphi^-(x), \varphi^-(z)\}$, (2) $\varphi^+(xyz) \geq \min\{\varphi^+(x), \varphi^+(z)\}$ $\forall x, y, z \in S$.

3 Bipolar Valued Fuzzy Translation and Extension

Theorem 3.1 *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a non-empty bipolar valued fuzzy subset of S and $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$. Then bipolar valued fuzzy translation $\varphi^T_{(\alpha,\beta)} = (S; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)})$ of $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup(fuzzy bi-ideal) of S if and only if $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup(fuzzy bi-ideal) of S.*

Proof: Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy subsemigroup of S and $x, y \in S$. Then

$$\begin{aligned} \varphi^-_{(\alpha,T)}(xy) &= \varphi^-(xy) + \alpha \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-_{(\alpha,T)}(x), \varphi^-_{(\alpha,T)}(y)\} \end{aligned}$$

and

$$\begin{aligned}\varphi^{+}_{(\beta,T)}(xy) &= \varphi^{+}(xy) + \beta \\ &\geq \min\{\varphi^{+}(x), \varphi^{+}(y)\} + \beta \\ &= \min\{\varphi^{+}(x) + \beta, \varphi^{+}(y) + \beta\} \\ &= \min\{\varphi^{+}_{(\beta,T)}(x), \varphi^{+}_{(\beta,T)}(y)\}.\end{aligned}$$

Hence $\varphi^{T}_{(\alpha,\beta)} = (S; \varphi^{-}_{(\alpha,T)}, \varphi^{+}_{(\beta,T)})$ is a bipolar valued fuzzy subsemigroup of S.

Conversely, let $\varphi^{T}_{(\alpha,\beta)} = (S; \varphi^{-}_{(\alpha,T)}, \varphi^{+}_{(\beta,T)})$ be a bipolar valued fuzzy subsemigroup of S for some $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$. Then for any $x, y \in S$ we have

$$\begin{aligned}\varphi^{-}(xy) + \alpha &= \varphi^{-}_{(\alpha,T)}(xy) \\ &\leq \max\{\varphi^{-}_{(\alpha,T)}(x), \varphi^{-}_{(\alpha,T)}(y)\} \\ &= \max\{\varphi^{-}(x) + \alpha, \varphi^{-}(y) + \alpha\} \\ &= \max\{\varphi^{-}(x), \varphi^{-}(y)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^{+}(xy) + \beta &= \varphi^{+}_{(\beta,T)}(xy) \\ &\geq \min\{\varphi^{+}_{(\beta,T)}(x), \varphi^{+}_{(\beta,T)}(y)\} \\ &= \min\{\varphi^{+}(x) + \beta, \varphi^{+}(y) + \beta\} \\ &= \min\{\varphi^{+}(x), \varphi^{+}(y)\} + \beta\end{aligned}$$

which imply that $\varphi^{-}(xy) \leq \max\{\varphi^{-}(x), \varphi^{-}(y)\}$ and $\varphi^{+}(xy) \geq \min\{\varphi^{+}(x), \varphi^{+}(y)\}$ respectively for all $x, y \in S$. Hence $\varphi = (S; \varphi^{-}, \varphi^{+})$ is a bipolar valued fuzzy subsemigroup of S. Similarly we can prove the other case also.

Definition 3.2 [*3*] *Let $\varphi = (S; \varphi^{-}, \varphi^{+})$ and $\psi = (S; \psi^{-}, \psi^{+})$ be two bipolar valued fuzzy sets of S. If $\varphi^{-}(x) \geq \psi^{-}(x)$ and $\varphi^{+}(x) \leq \psi^{+}(x)$ for all $x \in S$, then we say that $\psi = (S; \psi^{-}, \psi^{+})$ is a bipolar valued fuzzy extension of $\varphi = (S; \varphi^{-}, \varphi^{+})$.*

Example 2 Let $\psi = (S; \psi^{-}, \psi^{+})$ be a bipolar valued fuzzy set of a semigroups S in Example 1 and it is defined as follows:

S	e	a	b
ψ^{-}	-0.53	-0.36	-0.21
ψ^{+}	0.64	0.71	0.73

Then $\psi = (S; \psi^{-}, \psi^{+})$ is a bipolar valued fuzzy extension of $\varphi = (S; \varphi^{-}, \varphi^{+})$.

Definition 3.3 *Let $\varphi = (S; \varphi^-, \varphi^+)$ and $\psi = (S; \psi^-, \psi^+)$ be two bipolar valued fuzzy sets of S. Then $\psi = (S; \psi^-, \psi^+)$ is called a bipolar valued fuzzy S-extension(B-extension) of $\varphi = (S; \varphi^-, \varphi^+)$ if the following hold:* (1) $\psi = (S; \psi^-, \psi^+)$ *is a bipolar valued fuzzy extension of* $\varphi = (S; \varphi^-, \varphi^+)$, (2) $\varphi = (S; \varphi^-, \varphi^+)$ *is a bipolar valued fuzzy subsemigroup(bi-ideal) of S.*

Example 3 $\psi = (S; \psi^-, \psi^+)$ in Example 2 is a bipolar valued fuzzy S-extension of $\varphi = (S; \varphi^-, \varphi^+)$ in Example 1.

Remark 4 From the definition of bipolar valued fuzzy (α, β)-translation it is clear that $\varphi^-_{(\alpha,T)}(x) \leq \varphi^-(x)$ and $\varphi^+_{(\beta,T)}(x) \geq \varphi^+(x)$ $\forall x \in S$.

Using the above remark we have the following theorem.

Theorem 3.4 *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy subsemigroup(bi-ideal) of a semigroup S and $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$. Then bipolar valued fuzzy (α, β)-translation $\varphi^T_{(\alpha,\beta)} = (S; \varphi^-_{(\alpha,T)}, \varphi^+_{(\beta,T)})$ of $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy S-extension(B-extension) of $\varphi = (S; \varphi^-, \varphi^+)$.*

Remark 5 The converse of Theorem 3.4 is not true in general. In fact $\psi = (S; \psi^-, \psi^+)$(see Example 2) is a bipolar valued fuzzy S-extension of $\varphi = (S; \varphi^-, \varphi^+)$ of S(see Example 1) but it is not a bipolar valued fuzzy (α, β)-translation of $\varphi = (S; \varphi^-, \varphi^+)$ of S.

Theorem 3.5 *The intersection of two bipolar valued fuzzy S-extensions(B-extensions) of a bipolar valued fuzzy subsemigroup(bi-ideal) $\varphi = (S; \varphi^-, \varphi^+)$ in S is a bipolar valued fuzzy S-extension(B-extension) of $\varphi = (S; \varphi^-, \varphi^+)$ in S, provided it is non-empty.*

Proof: Let $\psi = (S; \psi^-, \psi^+)$ and $\pi = (S; \pi^-, \pi^+)$ be two bipolar valued fuzzy S-extensions of $\varphi = (S; \varphi^-, \varphi^+)$ in S. Then $\varphi^-(x) \geq \psi^-(x), \varphi^-(x) \geq \pi^-(x)$ and $\varphi^+(x) \leq \psi^+(x), \varphi^+(x) \leq \pi^+(x) \forall x \in S$. Now

$$(\psi \cap \pi)^-(x) = \max\{\psi^-(x), \pi^-(x)\} \leq \varphi^-(x)$$

and

$$(\psi \cap \pi)^+(x) = \min\{\psi^+(x), \pi^+(x)\} \geq \varphi^+(x).$$

Consequently, $\psi \cap \pi = (S; (\psi \cap \pi)^-, (\psi \cap \pi)^+)$ is a bipolar valued fuzzy extension of $\varphi = (S; \varphi^-, \varphi^+)$. Since $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup of S, so $\psi \cap \pi = (S; (\psi \cap \pi)^-, (\psi \cap \pi)^+)$ is a bipolar valued fuzzy S-extension of $\varphi = (S; \varphi^-, \varphi^+)$. Similarly we can prove the other case also.

Theorem 3.6 *Union of two bipolar valued fuzzy S-extensions(B-extensions) of a bipolar valued fuzzy subsemigroup(bi-ideal) $\varphi = (S; \varphi^-, \varphi^+)$ in S is a bipolar valued fuzzy S-extension(B-extension) of $\varphi = (S; \varphi^-, \varphi^+)$ in S.*

Proof: Let $\psi = (S; \psi^-, \psi^+)$ and $\pi = (S; \pi^-, \pi^+)$ be two bipolar valued fuzzy S-extensions of $\varphi = (S; \varphi^-, \varphi^+)$ in S. Then $\varphi^-(x) \geq \psi^-(x)$, $\varphi^-(x) \geq \pi^-(x)$ and $\varphi^+(x) \leq \psi^+(x)$, $\varphi^+(x) \leq \pi^+(x)$ $\forall x \in S$. Now

$$(\psi \cup \pi)^-(x) = \min\{\psi^-(x), \pi^-(x)\} \leq \varphi^-(x)$$

and

$$(\psi \cup \pi)^+(x) = \max\{\psi^+(x), \pi^+(x)\} \geq \varphi^+(x).$$

Consequently, let $\psi \cup \pi = (S; (\psi \cup \pi)^-, (\psi \cup \pi)^+)$ is a bipolar valued fuzzy extension of $\varphi = (S; \varphi^-, \varphi^+)$. Since $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup of S, so $\psi \cup \pi = (S; (\psi \cup \pi)^-, (\psi \cup \pi)^+)$ is a bipolar valued fuzzy S-extension of $\varphi = (S; \varphi^-, \varphi^+)$. Similarly we can prove the other case also.

Definition 3.7 [3] *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy set of S. Let us consider the two sets $N_\alpha(\varphi^-; t^-) := \{x \in S : \varphi^-(x) \leq t^- - \alpha\}$ and $P_\beta(\varphi^+; t^+) := \{x \in S : \varphi^+(x) \geq t^+ - \beta\}$ where $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and $(t^-, t^+) \in [-1, \alpha] \times [\beta, 1]$.*

By routine verification we deduce the following theorem.

Theorem 3.8 *If $\varphi = (S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy subsemigroup(bi-ideal) of a semigroup S, then $N_\alpha(\varphi^-; t^-)$ and $P_\beta(\varphi^+; t^+)$ are subsemigroups(bi-ideals) of S for all $(t^-, t^+) \in Im(\varphi^-) \times Im(\varphi^+)$ with $t^- \leq \alpha$ and $t^+ \geq \beta$.*

Remark 6 In Theorem 3.8, if a bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of S is not a bipolar valued fuzzy subsemigroup(bi-ideal) of S then at least one of $N_\alpha(\varphi^-; t^-)$ and $P_\beta(\varphi^+; t^+)$ may not be a subsemigroup(bi-ideal) of S which is clear from the following example.

Example 4 Let $S = \{e, a, b\}$ be a set with following caley table:

	e	a	b
e	e	e	e
a	e	a	e
b	e	e	b

Then S is a semigroup. We define a bipolar valued fuzzy set $\varphi = (S; \varphi^-, \varphi^+)$ of S as follows:

S	e	a	b
φ^-	-0.7	-0.5	-0.3
φ^+	0.3	0.5	0.5

Then $\varphi = (S; \varphi^-, \varphi^+)$ is not a bipolar valued fuzzy subsemigroup of S since $\varphi^+(ab) = \varphi^+(e) = 0.3$ and $\min\{\varphi^+(a), \varphi^+(b)\} = 0.5$. Let $(\alpha, \beta) = (-0.15, 0.1)$ and $(t^-, t^+) = (-0.5, 0.5)$. Then $N_\alpha(\varphi^-; t^-) = \{e, a\}$ is a subsemigroup of S, but $P_\beta(\varphi^+; t^+) = \{a, b\}$ is not a subsemigroup of S.

4 Bipolar Valued Fuzzy Relation

In this section we introduce the notion of bipolar valued fuzzy equivalence relation using the notion of bipolar valued fuzzy extension.

Definition 4.1 *Let S be a non-empty set. Then a BVFR*[2] *on S is a bipolar valued fuzzy set $\varphi = \{((x, y), \varphi^-(x, y), \varphi^+(x, y)) : (x, y) \in S \times S\}$ on $S \times S$ where $\varphi^- : S \times S \to [-1, 0]$ and $\varphi^+ : S \times S \to [0, 1]$.*

Definition 4.2 *A BVFR $\varphi = \{((x, y), \varphi^-(x, y), \varphi^+(x, y)) : (x, y) \in S \times S\}$ on S is said to be reflexive if $\varphi^+(x, x) = 1$ and $\varphi^-(x, x) = -1$ $\forall x \in S$. Also φ is said to be symmetric if $\varphi^+(x, y) = \varphi^+(y, x)$ and $\varphi^-(x, y) = \varphi^-(y, x)$ $\forall x, y \in S$.*

Definition 4.3 *If $\varphi_1 = \{((x, y), \varphi_1^-(x, y), \varphi_1^+(x, y)) : (x, y) \in S \times S\}$ and $\varphi_2 = \{((x, y), \varphi_2^-(x, y), \varphi_2^+(x, y)) : (x, y) \in S \times S\}$ be two BVFRs on S then composition of φ_1 and φ_2 denoted by $\varphi_1 \circ \varphi_2$ is defined by $\varphi_1 \circ \varphi_2 = \{((x, y), (\varphi_1 \circ \varphi_2)^-(x, y), (\varphi_1 \circ \varphi_2)^+(x, y)) : (x, y) \in S \times S\}$, where $(\varphi_1 \circ \varphi_2)^-(x, y) = \inf_{z \in S}\{\max\{\varphi_1^-(x, z), \varphi_2^-(z, y)\}\}$ and $(\varphi_1 \circ \varphi_2)^+(x, y) = \sup_{z \in S}\{\min\{\varphi_1^+(x, z), \varphi_2^+(z, y)\}\}$.*

Definition 4.4 *A BVFR φ on S is called transitive if $\varphi = (S \times S; \varphi^-, \varphi^+)$ is a bipolar valued fuzzy extension of $\varphi \circ \varphi = (S \times S; (\varphi \circ \varphi)^-, (\varphi \circ \varphi)^+)$.*

Definition 4.5 *A BVFR φ on a semigroup S is called a BVFER if φ is reflexive, symmetric and transitive.*

Definition 4.6 *Let φ be a BVFR on S. Then the inverse relation of φ denoted by φ^{-1} is $\varphi^{-1} = \{((x, y), \varphi^{-1^-}(x, y), \varphi^{-1^+}(x, y)) : (x, y) \in S \times S\}$ where $\varphi^{-1^-}(x, y) = \varphi^-(y, x)$ and $\varphi^{-1^+}(x, y) = \varphi^+(y, x)$ $\forall x, y \in S$.*

Proposition 4.7 *Let φ and ψ be two BVFR on S. Then*

[2]BVFR and BVFER respectively stands for bipolar valued fuzzy relation and bipolar valued fuzzy equivalence relation

(1) $(\varphi^{-1})^{-1} = \varphi$,

(2) $(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}$.

Proof: (1) By routine verification we can prove it.
(2) Let $x, y \in S$. Then

$$\begin{aligned}
(\varphi \circ \psi)^{-1^-}(x,y) &= (\varphi \circ \psi)^-(y,x) \\
&= \inf_{z\in S}\{\max\{\varphi^-(y,z), \psi^-(z,x)\}\} \\
&= \inf_{z\in S}\{\max\{\varphi^{-1^-}(z,y), \psi^{-1^-}(x,z)\}\} \\
&= \inf_{z\in S}\{\max\{\psi^{-1^-}(x,z), \varphi^{-1^-}(z,y)\}\} \\
&= (\psi^{-1} \circ \varphi^{-1})^-(x,y).
\end{aligned}$$

Similarly we can prove that $(\varphi \circ \psi)^{-1^+}(x,y) = (\psi^{-1} \circ \varphi^{-1})^+(x,y)$. Hence $(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}$.

Definition 4.8 *The transitive closure of a BVFR φ on a set S is φ^∞ defined by*

$$\varphi^\infty = \varphi \cup \varphi^2 \cup \varphi^3 \cup \ldots.$$

We can prove the following lemma by routine verification.

Lemma 4.9 *The transitive closure φ^∞ of a BVFR φ on S is the smallest transitive relation on S containing φ.*

Definition 4.10 *Let $\varphi = (S; \varphi^-, \varphi^+)$ be a bipolar valued fuzzy set on a set S. Let us consider the two sets $N(\varphi; \alpha) := \{x \in S : \varphi^-(x) \leq \alpha\}$ and $P(\varphi; \beta) := \{x \in S : \varphi^+(x) \geq \beta\}$ where $(\alpha, \beta) \in [-1, 0] \times [0, 1]$. Then the set $C(\varphi; (\alpha, \beta)) := N(\varphi; \alpha) \cap P(\varphi; \beta)$ is called the bipolar (α, β)-cut set of φ in S.*

Theorem 4.11 *Let $\varphi = (S \times S; \varphi^-, \varphi^+)$ be a BVFR on S. Then φ is a BVFER on S if and only if each $C(\varphi; (\alpha, \beta))$ is an equivalence relation on S, with $(\alpha, \beta) \in [-1, 0] \times [0, 1]$.*

Proof: Let φ be a BVFER on S. Let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$. Then by definition, $C(\varphi; (\alpha, \beta)) = \{(x, y) \in S \times S : \varphi^-(x, y) \leq \alpha, \varphi^+(x, y) \geq \beta\}$. Since φ is a BVFER, so $\varphi^-(x, x) = -1 \leq \alpha$ and $\varphi^+(x, x) = 1 \geq \beta$ $\forall x \in S$. So $(x, x) \in C(\varphi; (\alpha, \beta))$ $\forall x \in S$. Hence $C(\varphi; (\alpha, \beta))$ is reflexive.

Next let $(x, y) \in C(\varphi; (\alpha, \beta))$ for $x, y \in S$. Then $\varphi^-(x, y) \leq \alpha$ and $\varphi^+(x, y) \geq$

β. So by symmetry of φ, $\varphi^-(y,x) = \varphi^-(x,y) \leq \alpha$ and $\varphi^+(y,x) = \varphi^+(x,y) \geq \beta$ and hence $(y,x) \in C(\varphi;(\alpha,\beta))$. Therefore $C(\varphi;(\alpha,\beta))$ is symmetric.

Finally let $(x,y) \in C(\varphi;(\alpha,\beta))$ and $(y,z) \in C(\varphi;(\alpha,\beta))$ for $x,y,z \in S$. Then $\varphi^-(x,y) \leq \alpha$, $\varphi^+(x,y) \geq \beta$ and $\varphi^-(y,z) \leq \alpha$, $\varphi^+(y,z) \geq \beta$. So $\max\{\varphi^-(x,y), \varphi^-(y,z)\} \leq \alpha$ and $\min\{\varphi^+(x,y), \varphi^+(y,z)\} \geq \beta$ whence $\inf\limits_{y\in S}\{\max \{\varphi^-(x,y), \varphi^-(y,z)\}\} \leq \alpha$ and $\sup\limits_{y\in S}\{\min\{\varphi^+(x,y), \varphi^+(y,z)\}\} \geq \beta$. So $(\varphi \circ \varphi)^-(x,z) \leq \alpha$ and $(\varphi \circ \varphi)^+(x,z) \geq \beta$. But φ is a BVF transitive relation, so $\varphi^-(x,z) \leq (\varphi \circ \varphi)^-(x,z) \leq \alpha$ and $\varphi^+(x,z) \geq (\varphi \circ \varphi)^+(x,z) \geq \beta$. Hence $(x,z) \in C(\varphi;(\alpha,\beta))$ which shows that $C(\varphi;(\alpha,\beta))$ is transitive.

Conversely, suppose that each $C(\varphi;(\alpha,\beta))$ is an equivalence relation on S with $(\alpha,\beta) \in [-1,0] \times [0,1]$. Then for $\alpha = -1$ and $\beta = 1$, by reflexivity, $(x,x) \in C(\varphi;(-1,1))$ $\forall x \in S$ which implies $\varphi^-(x,x) \leq -1$, $\varphi^+(x,x) \geq 1$ and consequently $\varphi^-(x,x) = -1$, $\varphi^+(x,x) = 1$ $\forall x \in S$. Hence φ is reflexive.

Next for $x,y \in S$, let $\varphi^-(x,y) = \alpha$ and $\varphi^+(x,y) = \beta$. Then $(x,y) \in C(\varphi;(\alpha,\beta))$. So by symmetry, $(y,x) \in C(\varphi;(\alpha,\beta))$. Therefore $\varphi^-(y,x) \leq \alpha = \varphi^-(x,y)$ and $\varphi^+(y,x) \geq \beta = \varphi^+(x,y)$. Similarly we get $\varphi^-(x,y) \leq \varphi^-(y,x)$ and $\varphi^+(x,y) \geq \varphi^+(y,x)$. Hence $\varphi^-(x,y) = \varphi^-(y,x)$ and $\varphi^+(x,y) = \varphi^+(y,x)$. So the BVFR φ is symmetric.

Finally let $x,y,z \in S$ and $\max\{\varphi^-(x,z), \varphi^-(z,y)\} = \alpha$ and $\min\{\varphi^+(x,z), \varphi^+(z,y)\} = \beta$. Then $\varphi^-(x,z) \leq \alpha$, $\varphi^-(z,y) \leq \alpha$ and $\varphi^+(x,z) \geq \beta$, $\varphi^+(z,y) \geq \beta$. So $(x,z) \in C(\varphi;(\alpha,\beta))$ and $(z,y) \in C(\varphi;(\alpha,\beta))$. So by transitivity of $C(\varphi;(\alpha,\beta))$, we get $(x,y) \in C(\varphi;(\alpha,\beta))$. Therefore $\varphi^-(x,y) \leq \alpha$ and $\varphi^+(x,y) \geq \beta$. Then $\varphi^-(x,y) \leq \max\{\varphi^-(x,z), \varphi^-(z,y)\}$ and $\varphi^+(x,y) \geq \min\{\varphi^+(x,z), \varphi^+(z,y)\}$ $\forall z \in S$. Therefore $\varphi^-(x,y) \leq \inf\limits_{z\in S}\{\max\{\varphi^-(x,z), \varphi^-(z,y)\}\}$ and $\varphi^+(x,y) \geq \sup\limits_{z\in S}\{\min\{\varphi^+(x,z), \varphi^+(z,y)\}\}$. So $\varphi^- \subseteq (\varphi \circ \varphi)^-$ and $(\varphi \circ \varphi)^+ \subseteq \varphi^+$. Hence $\varphi \circ \varphi \subseteq \varphi$ which shows that the BVFR φ is transitive. Consequently φ is a BVFER.

Definition 4.12 *Let $\varphi = (S \times S; \varphi^-, \varphi^+)$ be a BVFER on a set S and $a \in S$. Then the bipolar valued fuzzy set defined by $a\varphi = (S; a\varphi^-, a\varphi^+)$, where $(a\varphi^-)(x) = \varphi^-(a,x)$, $(a\varphi^+)(x) = \varphi^+(a,x)$ $\forall x \in S$, is called a bipolar valued fuzzy class of a with respect to φ.*

Theorem 4.13 *Let $\varphi = (S \times S; \varphi^-, \varphi^+)$ be a BVFER on a set S and $a \in S$. Then for $(\alpha,\beta) \in [-1,0] \times [0,1]$, $C(a\varphi;(\alpha,\beta)) = [a]$, the equivalence class of a with respect to the equivalence relation $C(\varphi;(\alpha,\beta))$ on S.*

Proof: We have

$$\begin{aligned}
[a] &= \{x \in S : (a, x) \in C(\varphi; (\alpha, \beta))\} \\
&= \{x \in S : \varphi^-(a, x) \leq \alpha, \varphi^+(a, x) \geq \beta\} \\
&= \{x \in S : (a\varphi^-)(x) \leq \alpha, (a\varphi^+)(x) \geq \beta\} \\
&= C(a\varphi; (\alpha, \beta)).
\end{aligned}$$

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