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Asymptotic behavior of solutions of mixed problem for linear thermo-elastic systems with microtemperatures

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Abstract

In this paper we study the mixed problem with dissipative boundary for linear thermo-elastic system with microtemperatures. We investigate the correctness of the mixed problem and establish the exponential decrease in the energy norm of the solutions.

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1 Introduction

Thermo-elastic systems describe the elastic and thermal behavior of elastic heat conductive media, particularly the reciprocal actions between elastic stresses and temperature differences [1–5]. In recent years, the existence, uniqueness and asymptotic behavior of solutions of the system of thermoelasticity has been analyzed intensively [6, 7, 12, 14, 15] and the references cited therein.

Eringen [16] introduced a class of micromorphic solids and called them microstretch solids. Microstretch solids of modeling porous media filled with gas or viscid fluids and composite materials with chopped elastic fibers. The material points is that of these materials can stretch and contract independently of their translations and rotations. The existence of solutions the mixed problem and the Cauchy problem for different system of this type studied in the works [6–14]. The Cauchy problem for a semilinear thermo-elastic system with microtemperatures in one space variable are considered in the follows works [8, 10, 13–15].

2 Statement problem and main result

In the domain $[0;\infty) \times [0;1]$ we consider the following thermo-elastic system with microtemperatures:

$$\begin{cases} u_{tt} - \mu^2 u_{xx} + b\theta_x = 0 \\ \varphi_{tt} - \alpha^2 \varphi_{xx} + \omega w_x = 0 \\ \theta_t - k\theta_{xx} + \beta u_{xt} + g w_x = 0 \\ w_t - \gamma w_{xx} + h\varphi_{xt} + m\theta_x = 0 \end{cases}, \tag{1}$$

where u, φ, θ and w represent the displacement vector, microstretch, absolute temperature difference $\theta = T_\alpha - T_0$ and microtemperature, respectively; $\mu, b, \alpha, \omega, k, \beta, g, \gamma, h$ and m are smooth function of $(t, x) \in [0, \infty) \times [0, 1]$ with μ, α, k and γ being positive.

For system (1) we investigate the mixed problem with boundary conditions

$$u(t, 0) = 0, \;\; \varphi(t, 0) = 0, \tag{2}$$

$$\begin{cases} u_t(t, 1) + u_x(t, 1) = 0 \\ \varphi_t(t, 1) + \varphi_x(t, 1) = 0 \end{cases}, \tag{3}$$

$$\theta(t, 0) = \theta(t, 1) = 0, \qquad w(t, 0) = w(t, 1) = 0 \tag{4}$$

and the initial conditions

$$\begin{cases} u(0, x) = u_0(x), & u_t(0, x) = u_1(x) \\ \varphi(0, x) = \varphi_0(x), & \varphi_t(0, x) = \varphi_1(x) \\ \theta(0, x) = \theta_0(x), & w(0, x) = w_0(x) \end{cases}, \;\; x \in [0, 1]. \tag{5}$$

The main purpose of this paper is to establishing the behavior of solutions of the problems (1)-(5) when $\mu, b, \alpha, \omega, k, \beta, g, \gamma, h$ and m some constants and

$$\mu > 0, \quad \alpha > 0, \quad k > 0, \quad \omega > 0. \tag{6}$$

Let there exist numbers $\lambda_i, i = 0, 1, 2$ such that

$$\begin{cases} \lambda_i > 0, i = 0, 1, 2, \\ m = \lambda_0 g, \\ \lambda_0 \beta = \lambda_1 b, \\ \lambda_2 \omega = h \end{cases}. \tag{7}$$

We introduce the following notations

$$L_2 = L_2(0,1), \quad {}_0W_2^1 = \{u : u \in W_2^1, u(0) = 0\},$$

$$\overset{0}{W}{}_2^1 = \overset{0}{W}{}_2^1(0,1) = \{u : u \in W_2^1(0,1), u(0) = u(1) = 0\}, W_2^2 = W_2^2(0,1).$$

In the space

$$H ={}_0 W_2^1 \times L_2 \times_0 W_2^1 \times L_2 \times L_2 \times L_2$$

we define the scalar product as follows:

$$\langle w, z\rangle_H = \lambda_1\mu^2 \int_0^1 \nu_{1x}\bar{z}_{1x}dx + \lambda_1 \int_0^1 \nu_{2x}\bar{z}_{2x}dx+$$

$$+\lambda_2\alpha^2 \int_0^1 \nu_{3x}\bar{z}_{3x}dx + \lambda_2 \int_0^1 \nu_{4x}\bar{z}_{4x}dx + \lambda_0 \int_0^1 \nu_{5x}\bar{z}_{5x}dx + \int_0^1 \nu_{6x}\bar{z}_{6x}dx,$$

where $w = (\nu_1, \dots, \nu_6)$, $z = (z_1, \dots, z_6) \in H$.

We denote by H_0 as following space

$$H_0 = \left\{ w : w = (\nu_1, \dots, \nu_6) \in \left[W_2^2 \cap_0 W_2^1 \times_0 W_2^1\right]^2 \times \left[W_2^2 \cap \overset{0}{W}{}_2^1\right]^2, \right.$$

$$\left. \nu_{1x}(1) + \nu_2(1) = 0, \quad \nu_{3x}(1) + \nu_4(1) = 0 \right\}$$

and by $E(t)$ the energy function

$$E\left(t\right) = \int_0^1 \left[|u_t|^2 + |\varphi_t|^2 + |u_x|^2 + |\varphi_x|^2 + |\theta|^2 + |w|^2\right] dx.$$

In this paper is obtained the following main result:

Theorem 2.1 *Suppose that conditions* (6),(7) *are fulfilled. Then there exist numbers $M \geq 1$ and $d > 0$ such that for any $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H$ the inequality*

$$E\left(t\right) \leq Me^{-dt}E\left(0\right)$$

is true, where

$$E\left(0\right) = \int_0^1 \left[|u_{0_x}|^2 + |\varphi_{0_x}|^2 + |u_1|^2 + |\varphi_1|^2 + |\theta_0|^2 + |w_0|^2\right] dx.$$

3 Existence of solutions of the problem (1)-(5)

In space H we define a linear operator A where

$$D(A) = H_0,$$
$$Aw = \big(\nu_2, \mu^2\nu_{1xx} - b\nu_{5x}, \quad \alpha^2\nu_{3xx} - \omega\nu_{6x}, \;\; k\nu_{5xx} - \beta\nu_{2x} -$$
$$-g\nu_{6x}, \gamma\,\nu_{6xx} - h\nu_{4x} - m\nu_{5x}\big),$$
$$w = (\nu_1, ..., \nu_6) \in D(A).$$

Lemma 3.1 *A is dissipative operator in H.*

Proof. Let $w = (\nu_1, ..., \nu_6) \in D(A)$. Then

$$\langle Aw, w\rangle_H = \lambda_1\mu^2 \int_0^1 \nu_{2x}\cdot\bar{\nu}_{1x}dx + \lambda_1 \int_0^1 \big(\mu^2\nu_{1xx} - b\nu_{5x}\big)\bar{\nu}_{2x}dx +$$
$$+\lambda_2\alpha^2 \int_0^1 \nu_{4x}\cdot\bar{\nu}_{3x}dx + \lambda_2 \int_0^1 \big(\alpha^2\nu_{3xx} - \omega\nu_{6x}\big)\cdot\bar{\nu}_4 dx +$$
$$+\lambda_0 \int_0^1 (k\nu_{5xx} - \beta\nu_{2x} - g\nu_{6x})\,\bar{\nu}_5 dx + \int_0^1 (\gamma\nu_{6xx} - h\nu_{4x} - m\nu_{5x})\cdot\bar{\nu}_6 dx.$$

Integrating by parts the obtained equality we have:

$$Re\,\langle Aw, w\rangle_H = -\lambda_0 k \int_0^1 |\nu_{5x}|^2\,dx -$$
$$-\gamma \int_0^1 |\nu_{6x}|^2\,dx - \lambda_1\mu^2\,|\nu_2(1)|^2 - \lambda_2\alpha^2\,|\nu_4(1)|^2. \tag{8}$$

Hence taking into account (6) that

$$Re\,\langle Aw, w\rangle_H \le 0.$$

In this way A is dissipative operator.

Lemma 3.2 *A is invertible operator in H.*

Proof. Let $h = (h_1, ..., h_6) \in H$. We consider the equation

$$Aw = h, \qquad w \in D(A). \tag{9}$$

The equation (9) is equivalent to the boundary value problem

$$\begin{cases} \nu_2 = h_1 \\ \mu^2\nu_{1xx} - b\nu_{5x} = h_2 \\ \nu_4 = h_3 \\ \alpha^2\nu_{3xx} - \omega\nu_{6x} = h_4 \\ k\nu_{5xx} - \beta\nu_{2x} - g\nu_{6x} = h_5 \\ \gamma\,\nu_{6xx} - h\nu_{4x} - m\nu_{5x} = h_6 \end{cases} \tag{10}$$

with the boundary conditions:

$$\begin{cases} \nu_1(0) = 0, \ \nu_2(0) = 0, \ \nu_3(0) = 0, \ \nu_4(0) = 0, \\ \nu_5(0) = \nu_5(1) = 0, \qquad \nu_6(0) = \nu_6(1) = 0 \end{cases}, \tag{11}$$

$$\begin{cases} \nu_{1x}(1) + \nu_2(1) = 0 \\ \nu_{3x}(1) + \nu_4(1) = 0 \end{cases}. \tag{12}$$

Substituting $\nu_2 = h_1$ and $\nu_4 = h_3$ in other equation system (10) and boundary conditions (11),(12) we obtain

$$\begin{cases} \mu^2 \nu_{1xx} - b\nu_{5x} = h_2 \\ \alpha^2 \nu_{3xx} - \omega\nu_{6x} = h_4 \\ k\nu_{5xx} - g\nu_{6x} = h_5 + \beta h_{1x} \\ \gamma\nu_{6xx} - m\nu_{5x} = h_6 + hh_{3x} \end{cases}, \tag{13}$$

$$\begin{cases} \nu_1(0) = 0, \ \nu_3(0) = 0, \\ \nu_5(0) = \nu_5(1) = 0, \ \nu_6(0) = \nu_6(1) = 0\,, \\ \nu_{1x}(1) = -h_1(1)\,, \qquad \nu_{3x}(1) = -h_3(1) \end{cases}. \tag{14}$$

First solve the system

$$\begin{cases} k\nu_{5xx} - g\nu_{6x} = h_5 - \beta h_{1x} \\ \gamma\nu_{6xx} - m\nu_{5x} = h_6 - hh_{3x} \end{cases} \tag{15}$$

with boundary conditions

$$\begin{cases} \nu_5(0) = \nu_5(1) = 0, \\ \nu_6(0) = \nu_6(1) = 0. \end{cases} \tag{16}$$

The problem (15),(16) has a unique solution (ν_5, ν_6), where $\nu_5 \in W_2^2 \cap \overset{0}{W}{}_2^1$, $\nu_6 \in W_2^2 \cap \overset{0}{W}{}_2^1$. Substituting ν_5 and ν_6 in the first two equation of the system (13) we obtain the boundary value problem for the system with respect to the functions ν_1 and ν_3 with inhomogeneous boundary conditions. The obtained problem is also solved by the standard method.

From Lemma 3.1 and Lemma 3.2 it follows that A maximally dissipative operator, therefore A generates a strongly continuous semigroup $U(t) = e^{tA}$. In this case $w(t) = e^{tA}w_0$ is a strong solution of the problem (1)-(5), if $w_0 \in H_0$.

If $w_0 \in H$, then $w(t) = e^{tA}w_0$ is a weak solution of the problem (1)-(5). In this instance if $w_0 \in H$, $w_{0n} \in H_0$ and $w_{0n} \to w_0$ in H at $n \to \infty$, then $\lim\limits_{n\to\infty} e^{tA}w_{0n} = e^{tA}w_0$ in $C([0,T];\ H)$ [17]. In this situation boundary conditions are understood in the following sense:
-for any $\nu, z \in {}_0W_2^1$ take place the equalities

$$\frac{d}{dt}\langle u(t,1), \nu(1)\rangle - \langle u(t,1), \nu_x(1)\rangle = 0, t \in [0,T]\,, \tag{17}$$

$$\frac{d}{dt}\left\langle \varphi(t,1), z(1)\right\rangle - \left\langle \varphi(t,1), z_x(1)\right\rangle = 0, t \in [0,T]\,. \qquad (18)$$

Thus the following theorems are valid:

Theorem 3.1 *Let the conditions* (6),(7) *are fulfilled. Then for any* $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H_0$ *problem* (1)-(5) *has a unique solution* (u, φ, θ, w), *where* $u, \varphi \in C\left([0,\infty); W_2^2 \cap_0 W_2^1\right) \cap C^1\left([0,\infty);_0 W_2^1\right) \cap C^2\left([0,\infty); L_2\right)$, $\theta, w \in C\left([0,\infty); W_2^2 \cap \overset{0}{W}{}_2^1\right) \cap C^1\left([0,\infty); L_2\right)$ *and the boundary conditions* (3),(4) *are satisfied.*

Theorem 3.2 *Let the conditions* (6),(7) *are fulfilled. Then for any* $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H$ *problem* (1)-(5) *has a unique "weak" solution* (u, φ, θ, w), *where* $u, \varphi \in C\left([0,\infty);_0 W_2^1\right) \cap C^1\left([0,\infty); L_2\right)$, $\theta, w \in C\left([0,\infty); L_2\right)$ *and the boundary conditions* (3) *are satisfied in the meaning of* (17)-(18).

4 Proof of Theorem 2.1

For proof of Theorem 2.1 we'll use the following lemma, which is proved in the paper V. Komornik [18].

Lemma 4.1 *Let* $Z(t) : [0,\infty) \to [0,\infty)$ *be a non-increasing function and assume that there exists constant* $c > 0$ *such that*

$$\int_s^\infty Z(t)\,dt \le MZ(s), \quad s \ge 0.$$

Then

$$Z(t) \le Z(0)\exp\left(1 - \frac{t}{c}\right), \quad t \ge 0.$$

We define the functional

$$E_0(t) = \frac{\lambda_1}{2}\int_0^1 u_t^2 dx + \frac{\mu^2\lambda_1}{2}\int_0^1 u_x^2 dx +$$

$$+\frac{\lambda_2}{2}\int_0^1 \varphi_t^2 dx + \frac{\alpha^2\lambda_2}{2}\int_0^1 \varphi_x^2 dx + \frac{\lambda_0}{2}\int_0^1 \theta^2 dx + \frac{1}{2}\int_0^1 w^2 dx.$$

It is easy to see that the following lemma is true:

Lemma 4.2 *Let the condition* (6),(7) *are satisfied, then exists* $c_1 > 0$ *and* $c_2 > 0$ *such that* $c_1 E(t) \le E_0(t) \le c_2 E(t)$.

Assume that $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H_0$, then $E_0(t)$ is differentiable. Taking into account (1) we obtain that

$$E_0'(t) = -k\lambda_0 \int_0^1 \theta_x^2 dx -$$

$$-\gamma \int_0^1 w_x^2 dx + \lambda_1 \mu^2 u_x(1,t) \cdot u_t(1,t) + \lambda_2 \alpha^2 \varphi_x(1,t) \cdot \varphi_t(1,t). \tag{19}$$

By using (3) from (19) we have

$$E_0'(t) = -k\lambda_0 \int_0^1 \theta_x^2 dx -$$

$$-\gamma \int_0^1 w_x^2 dx - \lambda_1 \mu^2 |u_t(t,1)|^2 - \lambda_2 \alpha^2 \ |\varphi_t(t,1)|^2 \leq 0. \tag{20}$$

Consequently,

$$E_0(t) \leq E_0(s), \ t \geq s. \tag{21}$$

From the inequality (21) we directly obtain the following:

Lemma 4.3 *The following estimates are true:*

1. $\int_0^1 \theta_x^2(t,x) dx \leq -\frac{1}{k} E_0'(t), \ 0 \leq t \leq T;$
2. $\int_s^T \int_0^1 \theta_x^2(t,x) dx dt \leq \frac{1}{k} E_0(s), \ 0 \leq s \leq T;$
3. $\int_0^1 w_x^2(t,x) dx \leq \frac{1}{\gamma} E_0'(t), \ 0 \leq t \leq T;$
4. $\int_s^T \int_0^1 w_x^2(t,x) dx dt \leq \frac{1}{\gamma} E_0(s), \ 0 \leq s \leq T;$
5. $\int_s^T |u_t(t,1)|^2 dt \leq \frac{1}{\lambda_1 \mu^2} E_0(s), \ 0 \leq s \leq T;$
6. $\int_s^T |\varphi_t(t,1)|^2 dt \leq \frac{1}{\lambda_1 \alpha^2} E_0(s), \ 0 \leq s \leq T.$

Multiplying the first equation of system (1) by u and integrating over the domain $[s,T] \times [0,1]$ we get:

$$0 = \int_s^T \int_0^1 u \left(u_{tt} - \mu^2 u_{xx} + b\theta_x \right) dx dt =$$

$$= \int_0^1 u\, u_t \Big|_{t=s}^{T} dx - \int_s^T \int_0^1 |u_t|^2 \, dxdt + \mu^2 \int_s^T \int_0^1 |u_x|^2 \, dxdt -$$

$$-\mu^2 \int_s^T u(t,1)\, u_x(t,1)\, dt + \int_s^T \int_0^1 bu\theta_x dxdt. \tag{22}$$

Similarly, multiplying the second equation of system (1) by φ and integrating over the domain $[s,T] \times [0,1]$ we have:

$$0 = \int_s^T \int_0^1 \varphi \left(\varphi_{tt} - \alpha^2 \varphi_{xx} + \omega w_x \right) dxdt =$$

$$= \int_0^1 \varphi \cdot \varphi_t \Big|_{t=s}^{T} dx - \int_s^T \int_0^1 |\varphi_t|^2 \, dxdt + \alpha^2 \int_s^T \int_0^1 |\varphi_x|^2 \, dxdt -$$

$$-\alpha^2 \int_s^T \varphi(t,1)\, \varphi_x(t,1)\, dt + \int_s^T \int_0^1 \omega \varphi w_x dxdt. \tag{23}$$

From (22) and (23) follows that

$$\lambda_1 \int_s^T \int_0^1 |u_t|^2 \, dxdt + \lambda_2 \int_s^T \int_0^1 |\varphi_t|^2 \, dxdt -$$

$$-\lambda_1 \mu^2 \int_s^T \int_0^1 |u_x|^2 \, dxdt - \lambda_2 \alpha^2 \int_s^T \int_0^1 |\varphi_x|^2 \, dxdt =$$

$$= \lambda_1 \int_0^1 u \cdot u_t|_{t=s}^{T} dx - \lambda_1 \mu^2 \int_s^T u(t,1)\, u_x(t,1)\, dt +$$

$$+\lambda_2 \int_0^1 \varphi \cdot \varphi_t|_{t=s}^{T} dx - \lambda_2 \alpha^2 \int_s^T \varphi(t,1)\, \varphi_x(t,1)\, dt +$$

$$+\lambda_1 b \int_s^T \int_0^1 u\theta_x dxdt + \lambda_2 \omega \int_s^T \int_0^1 \varphi w_x dxdt.$$

Hence we obtain

$$2 \int_s^T \varepsilon(t)\, dt = \lambda_1 \int_0^1 u \cdot u_t|_{t=s}^{T} dx + \lambda_2 \int_0^1 \varphi \cdot \varphi_t|_{t=s}^{T} dx +$$

$$+ \int_0^T \int_0^1 \left(\lambda_0 \theta^2 + b\lambda_1 u\theta_x \right) dxdt + \int_0^T \int_0^1 \left(w^2 + \lambda_2 \omega w_x \varphi \right) dxdt +$$

$$+2\lambda_1 \mu^2 \int_s^T \int_0^1 |u_x|^2 \, dxdt + 2\lambda_2 \alpha^2 \int_s^T \int_0^1 \varphi_x^2 dxdt -$$

$$-\lambda_1 \mu^2 \int_s^T u(t,1)\, u_x(t,1)\, dt - \lambda_2 \alpha^2 \int_s^T \varphi(t,1)\, \varphi_x(t,1)\, dt. \tag{24}$$

Using Holder inequality, embedding theorem [19] and Lemma 4.3 we get that

$$-\lambda_1\mu^2\int_s^T u\,(t,1)\,u_x\,(t,1)\,dt\le$$

$$\le\varepsilon\lambda_1\mu^2C_0\int_s^T\int_0^1 u_x^2\,(t,x)\,dxdt+\frac{1}{\varepsilon}\int_s^T u_x^2\,(t,1)\,dt\le$$

$$\le\varepsilon\lambda_1\mu^2C_0\int_s^T\int_0^1 u_x^2\,(t,x)\,dxdt+C\,(\varepsilon)\,E\,(s)\,. \tag{25}$$

Similarly, we have

$$-\lambda_2\alpha^2\int_s^T \varphi\,(t,1)\,\varphi_x\,(t,1)\,dt\le$$

$$\le\varepsilon\lambda_2\alpha^2C_0\int_s^T\int_0^1 \varphi_x^2\,(t,x)\,dxdt+C\,(\varepsilon)\,E\,(s)\,. \tag{26}$$

Multiplying both sides of the first equation of system (1) by $2xu_x$ and integrating over the domain $[s,T]\times[0,1]$ we obtain

$$0=\int_s^T\int_0^1 2xu_{tt}\cdot u_xdxdt-$$

$$-2\mu^2\int_s^T\int_0^1 xu_xu_{xx}dxdt+2b\int_s^T\int_0^1 xu_x\cdot\theta_xdxdt. \tag{27}$$

Integrating (27) by parts and taking into account the boundary conditions (2), (3) we get:

$$\mu^2\int_s^T\int_0^1 u_x^2dxdt+\int_s^T\int_0^1 u_t^2dxdt-$$

$$-\int_0^1 xu_x^2\big|_{t=s}^T\,dx+2\int_0^1 xu_xu_t\big|_{t=s}^T\,dx-$$

$$-\mu^2\int_s^T u_x^2\,(t,1)\,dt+2b\int_s^T\int_0^1 xu_x\cdot\theta_xdxdt=0. \tag{28}$$

Similarly, we have the following identity

$$\alpha^2\int_s^T\int_0^1 \varphi_x^2dxdt+\int_s^T\int_0^1 \varphi_t^2dxdt-$$

$$-\int_0^1 x\varphi_x^2\big|_{t=s}^T\,dx+2\int_0^1 x\varphi_x\varphi_t\big|_{t=s}^T\,dx-$$

$$-\alpha^2\int_s^T \varphi_x^2\,(t,1)\,dt+2\omega\int_s^T\int_0^1 x\varphi_xw_xdxdt=0. \tag{29}$$

Using Holder's and Young's inequalities, Lemma 4.3, and inequality (21) from (28) we obtain that

$$\left(\mu^2-\varepsilon\right)\int_s^T\int_0^1 u_x^2dxdt\le C_\varepsilon E_0\left(s\right).$$

Similarly, from (29) we get

$$\left(\alpha^2-\varepsilon\right)\int_s^T\int_0^1 \varphi_x^2dxdt\le\ C_\varepsilon E_0\left(s\right). \tag{30}$$

Further, using Poincare inequality [20] and Lemma 4.3, we have that

$$\int_s^T\int_0^1 w^2dxdt\le C_0\int_s^T\int_0^1 w_x^2dxdt\le CE\left(s\right).$$

Using Holder's and Young's inequalities we also obtain

$$b\lambda_1\int_s^T\int_0^1 u\theta_xdxdt\le\varepsilon\int_s^T\int_0^1 u_x^2dxdt+\frac{C_0b^2\lambda_1^2}{\varepsilon}\int_s^T\int_0^1\theta_x^2dxdt\le$$

$$\le\varepsilon\int_s^T\int_0^1 u_x^2dxdt+C_\varepsilon E_0\left(s\right). \tag{31}$$

Similarly, we get that

$$\omega\lambda_2\int_s^T\int_0^1 w_x\varphi dxdt\le\varepsilon\int_s^T\int_0^1\varphi_x^2dxdt+C_\varepsilon E_0\left(s\right). \tag{32}$$

Choosing $\varepsilon>0$ small enough from (24)-(32), we have that

$$\int_s^T E_0\left(t\right)dt\le\ CE_0\left(s\right). \tag{33}$$

Let $(u_0,u_1,\varphi_0,\varphi_1,\theta_0,w_0)\in H$. Then there exist $(u_{0n},u_{1n},\varphi_{0n},\varphi_{1n},\theta_{0n},w_{0n})\in H_0$ such that $(u_{0n},u_{1n},\varphi_{0n},\varphi_{1n},\theta_{0n},w_{0n})\to(u_0,u_1,\varphi_0,\varphi_1,\theta_0,w_0)$ in H.

Hence, passing to the limit we obtain that inequality (21) is also valid for initial data $(u_0,u_1,\varphi_0,\varphi_1,\theta_0,w_0)\in H$, but inequality (33) is valid for weak solution too.

Thus, by Lemma 4.1 we get

$$E\left(t\right)\le Me^{-dt}E(0)\,,\ t\ge 0, \tag{34}$$

where $M>0$ and $d>0$ not depends on $t>0$.

Note that the statement of the Theorem 2.1 follows from (34) and Lemma 3.2.

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