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Passage to the limit in $\int f_n \mathrm{d}\mu_n$

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Abstract

Let $(X, \mathcal{X})$ be a measurable space, $\mu_1, \mu_2 \ldots; \mu$ be signed measures on $\mathcal{X}$ and $f_1, f_2 \ldots; f$ be $\mathcal{X}$-measurable functions on X. Several sets of sufficient conditions for $\int f_n \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ and $\int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu_n \to 0$ are found. Two statements do not contain topological assumptions and are generalizations of the dominated convergence theorem; others concern topological spaces. Furthermore, a theorem about passage to the limit in $\int \mathrm{d}\nu_n(s) \int f_n(s,x)\psi_n(s, \mathrm{d}x)$ is proved and applied to evolution equations for measures.

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Introduction

Let $(X, \mathcal{X})$ be a measurable space, $\mu_1, \mu_2 \ldots; \mu$ be signed measures on $\mathcal{X}$ and $f_1, f_2 \ldots; f$ be $\mathcal{X}$-measurable functions on X. The main goal of the article is to find sufficient conditions for $\int f_n \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ or $\int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu_n \to 0$. Two general results of this sort are proved in Section 2. They do not contain topological assumptions and generalize the dominated convergence theorem. Both contain the condition

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} |\mu_n|\{x : |f_n(x) - f(x)| > \varepsilon\} = 0$$

which is not easily verifiable unless all the μ_n's coincide (in which case this condition means that the sequence (f_n) converges to f in measure $|\mu|$). So in the remaining part of the article we search coarser but more efficiently verifiable conditions.

The Baire and the Borel σ-algebras in a topological space X will be denoted $\mathcal{B}_0(X)$ and $\mathcal{B}(X)$, respectively. It is well known that $\mathcal{B}_0(X) \subset \mathcal{B}(X)$ and the equality is attained if X is metrizable (or, more generally, completely normal). Recall that a topological space X is called *Polish* if it is separable and there exists a metric ρ in X inducing the original topology and such that the metric space (X, ρ) is complete.

In our approach, some of the conditions justifying passage to the limit under the sign of integral are of topological nature. In particular, beginning from a certain place, X is a topological space and $\mathcal{X} = \mathcal{B}_0(X)$. This opens the gate to the key condition of weak convergence of a sequence of signed measures. This condition enters all final results in Sections 3 – 6. The other conditions are of two sorts: measure-theoretical (they concern both (f_n) and (μ_n) and are cognate to uniform integrability); topological (concerning only the functions). In Section 3, we consider functions on a Polish space and proceed from the recently discovered by Bogachev generalization, for signed measures, of Prokhorov's theorem. In Section 4, we deal with functions on a topological space which is not assumed metrizable (but may be subject to some other assumptions, e.g. first-countability). In this setting, the Prokhorov – Bogachev theorem is no more applicable, so one would not expect such nice sets of conditions as in Polish spaces. Nonetheless, they are the same as in the previous section, plus the extra demand that the pre-limit functions are continuous, – but the theorems are proved quite differently. The rationale is based on two fundamental facts discovered by Alexandroff [1] and suitably for our purposes modified by Bogachev [2].

In Section 5, we prove that, under rather general assumptions, weak convergence of sequences (μ_n) and (ν_n) implies weak convergence of $(\mu_n \otimes \nu_n)$. These statements turn out simple consequences of the results of the previous sections.

In Section 6, we derive, relying on the results of Sections 3 – 5, sufficient conditions for passage to the limit in $\int \mathrm{d}\nu_n(s) \int f_n(s,x)\psi_n(s, \mathrm{d}x)$ and give an idea how such results can be used for studying evolution equations for measures.

1 Preliminaries

Let us recall some definitions. A sequence (μ_n) of signed measures on a σ-algebra $\mathcal{X} \subset 2^X$ is called *uniformly bounded in variation* if $\sup\limits_n |\mu_n|(X) < \infty$. Let X be a topological space. They say that a sequence (μ_n) of signed measures on a σ-algebra $\mathcal{X} \subset 2^X$ containing $\mathcal{B}_0(X)$ *weakly converges* to a signed measure μ if the relation

$$\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu \tag{1}$$

holds for every bounded continuous function f on X. A sequence (μ_n) of signed measures on the Borel σ-algebra of a topological space X is called *uniformly tight* if for any $\varepsilon > 0$ there exists a compact set $K \in \mathcal{B}(X)$ such that $\sup_n |\mu_n|(X \setminus K) < \varepsilon$ (so it is clear what is a tight signed measure on $\mathcal{B}(X)$). A sequence in an arbitrary set endowed with convergence (topological, ordinal or defined descriptively, as just above) is called *relatively compact* if each its subsequence contains, in turn, a convergent subsequence.

Proposition 1.1. *Every weakly convergent sequence of signed measures is uniformly bounded in variation.*

This is a particular case of Proposition 8.1.7 in [2] which is, in turn, an easy consequence of the Banach – Steinhaus theorem.

We shall use the following generalization of Prokhorov's theorem.

Theorem 1.2 ([2, Theorem 8.6.2])**.** *In order that a sequence of signed measures on the Borel σ-algebra in a Polish space be relatively compact w.r.t. the weak convergence it is necessary and sufficient that it be uniformly bounded in variation and uniformly tight.*

From now on we abridge the term "signed measure" to "measure". If $(X, \mathcal{X})$ is a measurable space (it will be implied tacitly that $\mathcal{X} = \mathcal{B}_0(X)$ in case X is a topological space), then the collection of all measures on $\mathcal{X}$ will be denoted $\mathbb{M}(X)$.

Recall that a set $F \subset X$ is called *functionally closed* if there exist a real number c and a continuous function $f : X \to \mathbb{R}$ such that $F = \{x : f(x) \geq c\}$ (a formally different but obviously equivalent definition is given in [2, Sec. 6.3]).

Proposition 1.3. *Let X be a topological space and (μ_n) be a sequence in $\mathbb{M}(X)$ such that for any $h \in \mathrm{C_b}(X)$ the sequence $\left(\int h \mathrm{d}\mu_n\right)$ converges. Let, further, (Z_n) be a sequence of pairwise disjoint functionally closed subsets in X such that for any $J \subset \mathbb{N}$ the set $\bigcup_{n \in J} Z_n$ is functionally closed. Then*

$$\lim_{n \to \infty} \sup_k |\mu_k|(Z_n) = 0. \tag{2}$$

This is Proposition 8.1.10 [2] minus the assumption, not used in the proof, that the limit of $\left(\int h \mathrm{d}\mu_n\right)$ has the form $\int h \mathrm{d}\mu$. That statement is, in turn, a slight modification of Theorem 19.2 [1]. The latter concerns more general than signed measures entities called in [1] charges and does not contain the above-mentioned assumption (which in [1] need not be an assumption, because it is the conclusion of Theorem 19.3).

Proposition 1.4. *Let X be a topological space and (μ_n) be a sequence in $\mathbb{M}(X)$ such that equality (2) is valid for every sequence (Z_n) satisfying the assumptions of Proposition 1.3. Then*

$$\lim_{n\to\infty}\sup_k |\mu_k|(F_n) = 0 \tag{3}$$

for every sequence (F_n) of functionally closed subsets in X such that $F_n \searrow \emptyset$.

This is Proposition 8.1.12 [2] and a slight modification of Theorem 19.1 [1].

Corollary 1.5. *Let X be a topological space and (μ_n) be a sequence in $\mathbb{M}(X)$ such that for any $h \in \mathrm{C_b}(X)$ the sequence $\left(\int h\mathrm{d}\mu_n\right)$ converges. Then equality (3) holds for every sequence (F_n) of functionally closed subsets in X such that $F_n \searrow \emptyset$.*

For arbitrary $d \in \mathbb{N}$, $b \in \mathbb{C}^d$ and $N > 0$ we denote $b^{[N]} = \frac{Nb}{N\vee|b|}$. For a function $g : X \to \mathbb{C}^d$ we write $g^{[N]}(x)$ instead of $g(x)^{[N]}$. All the functions under consideration are meant $\mathbb{C}^d$-valued.

Lemma 1.6. *Let X be a topological space, $\mathcal{X}$ be a σ-algebra such that $\mathcal{B}_0(X) \subset \mathcal{X} \subset 2^X$ and (μ_n) be a sequence of measures on $\mathcal{X}$ weakly converging to a measure μ. Let, further, (f_n) and f be a sequence of $\mathcal{X}$-measurable functions on X and a continuous function on X such that*

$$\int |f_n|\,\mathrm{d}|\mu_n| < \infty, \tag{4}$$

$$\lim_{N\to\infty}\overline{\lim_{n\to\infty}}\left|\int \left(f_n - f_n^{[N]}\right)\mathrm{d}\mu_n\right| = 0, \tag{5}$$

$$\int |f|\;\mathrm{d}|\mu| < \infty \tag{6}$$

and the equality

$$\lim_{n\to\infty}\left|\int \left(f_n^{[N]} - f^{[N]}\right)\mathrm{d}\mu_n\right| = 0 \tag{7}$$

holds for all $N > 0$. Then

$$\int f_n\,\mathrm{d}\mu_n \to \int f\mathrm{d}\mu. \tag{8}$$

Proof. From (6) we have by the dominated convergence theorem $\int(|f| - N)_+\mathrm{d}|\mu| \to 0$ as $N \to \infty$, which together with the evident inequality $\left|b^{[N]} - b\right| \le (|b| - N)_+$ yields

$$\lim_{N\to\infty}\int \left|f^{[N]} - f\right|\mathrm{d}|\mu| = 0. \tag{9}$$

Writing

$$\int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu = \int \left(f_n - f_n^{[N]}\right) \mathrm{d}\mu_n + \int \left(f_n^{[N]} - f^{[N]}\right) \mathrm{d}\mu_n + \int f^{[N]} \mathrm{d}\mu_n - \int f^{[N]} \mathrm{d}\mu + \int \left(f^{[N]} - f\right) \mathrm{d}\mu,$$

we get from (7), continuity of f and weak convergence of (μ_n) to μ

$$\varlimsup_{n\to\infty} \left| \int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu \right| \le \varlimsup_{n\to\infty} \left| \int \left(f_n - f_n^{[N]}\right) \mathrm{d}\mu_n \right| + \int \left| f^{[N]} - f \right| \mathrm{d}|\mu|,$$

hereupon (8) emerges from (5) and (9). □

A pair (X, cnvr), where X is a nonvoid set and cnvr is a mapping of X into $2^{X^{\mathbb{N}}}$ will be called a *convergence space* if for each $x \in X$ the set $\mathrm{cnvr}[x]$ possesses the properties: (i) $(x, x \ldots) \in \mathrm{cnvr}[x]$; (ii) if $\mathrm{cnvr}[x]$ contains some sequence, then it contains all its subsequences; (iii) if $(y_k) \in \mathrm{cnvr}[x]$ and

$$x_n = y_k \quad \text{as} \quad n_{k-1} < n \le n_k,$$

where $n_0 = 0$ and (n_k) is a strictly increasing sequence of natural numbers, then $(x_n) \in \mathrm{cnvr}[x]$. The last relation will be otherwise written as $x_n \to x$ (which does not exclude that $x_n \to y \ne x$) and read as "(x_n) converges to x", herein x will be called a (not certainly the) limit of the sequence (x_n).

A set in a convergence space will be called *sequentially compact* (respectively: *sequentially precompact*) if every sequence of its members has a subsequence converging to some point of this set (respectively: of this space).

A sequence (f_n) of functions on a convergence space X will be called *asymptotically sequentially equicontinuous* (briefly: a.s.e.c.) *at a point* x (respectively: *on* X, but "on X" will be suppressed) if the relation

$$f_n(x_n) - f_n(x) \to 0 \tag{10}$$

holds for every converging to x sequence (x_n) (respectively: for any $x \in X$ and $(x_n) \in \mathrm{cnvr}[x]$). We will say that a sequence (f_n) of mappings of a convergence space X into a convergence space Y *converges to* f *uniformly at a point* x (respectively: *firmly converges to* f) if the relation

$$f_n(x_n) \to f(x) \tag{11}$$

holds for every converging to x sequence (x_n) (respectively: for any $x \in X$ and $(x_n) \in \mathrm{cnvr}[x]$). We will also say that a sequence (f_n) of functions on a convergence space X *locally uniformly converges* to a function f if

$$\lim_{n\to\infty} \sup_{x\in K} |f_n(x) - f(x)| = 0$$

for every sequentially precompact set $K \subset X$.

We shall regard any topological space as a convergence one, identifying tacitly $\mathrm{cnvr}[x]$ with the set of sequences topologically converging to x.

The family of all neighborhoods of a point x in a topological space will be denoted $\tau(x)$. It is a directed set w.r.t. the succession relation $\supset$. The limit of a net $q : \tau(x) \to \mathbb{R}$ will be denoted $\lim\limits_{U\in\tau(x)} q(U)$.

The next three statements are obvious.

Lemma 1.7. *Let X be a convergence space, x be a point in X, f be a function on X and (f_n) be an a.s.e.c. at x sequence of functions on X such that*

$$f_n(x) \to f(x). \tag{12}$$

Then (f_n) converges to f uniformly at this point.

Lemma 1.8. *Let X be a convergence space and (f_n) be an a.s.e.c. sequence of functions on X pointwise converging to a sequentially continuous function f. Then it converges to f firmly and locally uniformly.*

Lemma 1.9. *Let X be a topological space and (f_n) be a sequence in $(\mathbb{C}^d)^X$ satisfying the condition*

$$\lim_{U\in\tau(x)} \varlimsup_{n\to\infty} \sup_{x'\in U} |f_n(x') - f_n(x)| = 0 \tag{13}$$

at a point $x \in X$. Then it is a.s.e.c. at x. If, furthermore, (f_n) pointwise converges to f in some neighborhood of x, then f is continuous at this point.

Writing

$$f_n(x_n) - f(x_n) = f_n(x_n) - f_n(x) + f_n(x) - f(x) + f(x) - f(x_n),$$

we deduce from the last three lemmas

Corollary 1.10. *Let X be a topological space and (f_n) be a sequence in $(\mathbb{C}^d)^X$ satisfying, for all $x \in X$, conditions* (12) *and* (13)*. Then f is continuous and (f_n) converges to f firmly and locally uniformly.*

2 Theorems without topological assumptions

Denote

$$H_n^\varepsilon = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}. \tag{14}$$

The indicator of a set $\{x : h(x) > N\}$ will be written as $I\{h > N\}$.

Theorem 2.1. *Let $(X, \mathcal{X})$ be a measurable space, (μ_n) be a uniformly bounded in variation sequence in $\mathbb{M}(X)$ and (f_n) be a sequence of $\mathcal{X}$-measurable functions on X. Suppose that*

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \int |f_n| I\{|f_n| > N\} \, \mathrm{d}|\mu_n| = 0, \tag{15}$$

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \int |f| I\{|f| > N\} \, \mathrm{d}|\mu_n| = 0 \tag{16}$$

and for all $\varepsilon > 0$

$$\lim_{n\to\infty} |\mu_n|(H_n^\varepsilon) = 0, \tag{17}$$

where H_n^ε is defined by (14). *Then $\int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu_n \to 0$.*

Proof. Condition (15) implies, obviously, that inequality (4) holds for eventually all n. Evidently, for any $b \in \mathbb{C}^d$ and $N > 0$ $\quad \left|b - b^{[N]}\right| \le |b| I\{|b| > N\}$. So conditions (15) and (16) yield

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \int \left|f_n - f_n^{[N]}\right| \mathrm{d}|\mu_n| = 0, \tag{18}$$

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \int \left|f - f^{[N]}\right| \mathrm{d}|\mu_n| = 0. \tag{19}$$

Obviously,

$$\int_{X \setminus H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le \varepsilon |\mu_n|(X), \qquad \int_{H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le 2N |\mu_n|(H_n^\varepsilon).$$

So condition (17) implies that for any positive N and ε

$$\overline{\lim_{n\to\infty}} \int \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le \varepsilon \sup_n |\mu_n|(X). \tag{20}$$

Hence and from uniform boundedness of (μ_n) in variation we obtain equality (7) which together with (18), (19) and the identity $\int f_n \mathrm{d}\mu_n - \int f \mathrm{d}\mu_n = \int \left(f_n - f_n^{[N]}\right) \mathrm{d}\mu_n + \int \left(f_n^{[N]} - f^{[N]}\right) \mathrm{d}\mu_n + \int \left(f^{[N]} - f\right) \mathrm{d}\mu_n$ proves the theorem. □

Corollary 2.2. *Let $(X, \mathcal{X})$ be a measurable space, (μ_n) be a uniformly bounded in variation sequence in $\mathbb{M}(X)$, μ be a measure on $\mathcal{X}$, (f_n) be a uniformly bounded sequence of $\mathcal{X}$-measurable functions on X and f be an $\mathcal{X}$-measurable function on X. Suppose that conditions* (15), (16), (6), (31) *and, for any $\varepsilon > 0$,* (17) *(with H_n^ε defined by* (14)*) are fulfilled. Then relation* (8) *holds.*

In case $\mu_n = \mu$ condition (17) becomes none other than the demand of convergence of (f_n) to f in measure $|\mu|$, condition (16) becomes a consequence of (15) and (17) and so does (6). Condition (15) in this case will be, obviously, fulfilled if there exists a function $F \in \mathrm{L}_1(\mathrm{X}, \mathcal{X}, |\mu|)$ such that $|f_n| \le F$ for all n. Thus both Theorem 2.1 and Corollary 2.2 generalize Lebesgue's dominated convergence theorem.

3 Theorems for functions on a Polish space

Theorem 3.1. *Let X be a Polish space and (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ. Let, further, (f_n) and f be a sequence of Borel functions on X and a continuous function on X such that: conditions* (4) – (6) *are fulfilled; for any compactum $K \subset X$ and positive number ε*

$$\lim_{n\to\infty} |\mu_n|(H_n^\varepsilon \cap K) = 0, \tag{21}$$

where H_n^ε is defined by (14)*. Then relation* (8) *holds.*

Proof. Obviously,

$$\left| \int \left(f_n^{[N]} - f^{[N]}\right) \mathrm{d}\mu_n \right| \le \varepsilon |\mu_n|(X \setminus H_n^\varepsilon) + \int_{H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n|.$$

The evident inclusion $H_n^\varepsilon \subset H_n^\varepsilon \cap K \cup X \setminus K$ and the definition of H_n^ε yield

$$\int_{H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le 2N|\mu_n|(H_n^\varepsilon \cap K) + 2N|\mu_n|(X \setminus K).$$

So condition (21) implies that for any $N > 0$, $\varepsilon > 0$ and compactum K

$$\varlimsup_{n\to\infty} \left| \int \left(f_n^{[N]} - f^{[N]}\right) \mathrm{d}\mu_n \right| \le \varepsilon \sup_n |\mu_n|(X) + 2N \sup_n |\mu_n|(X \setminus K).$$

Now, equality (7) follows from the properties of (μ_n) asserted by Theorem 1.2 in the necessity part. Thus all the conditions of Lemma 1.6 are fulfilled and therefore its conclusion is valid. ☐

Proposition 3.2. *Let X be a Polish space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a uniformly bounded sequence of Borel functions on X locally uniformly converging to a continuous function f. Then relation* (8) *holds and*

$$\int |f| \; \mathrm{d}|\mu| \le \varliminf_{n\to\infty} \int |f_n| \; \mathrm{d}|\mu_n|. \tag{22}$$

Proof. Obviously, for any $A \subset H_n^\varepsilon$

$$|\mu_n|(A) \le \varepsilon^{-1} \int_A |f_n - f| \, \mathrm{d}|\mu_n|.$$

Herein

$$\int_{H_n^\varepsilon \cap K} |f_n - f| \, \mathrm{d}|\mu_n| \le \sup_{x \in K} |f_n(x) - f(x)| \cdot |\mu_n|(X).$$

Hence we conclude with account of Proposition 1.1 that local uniform convergence of (f_n) to f implies (21) for all compacta K. Now, the first statement follows from Theorem 3.1, once one has noted that $f_n^{[N]} = f_n$ as $N > \sup_n \|f_n\|_\infty$.

Let us define the signed measures $\varkappa_n$ and $\varkappa$ by $\varkappa_n(A) = \int_A f_n \mathrm{d}\mu_n$, $\varkappa(A) = \int_A f \mathrm{d}\mu$. Then for any $h \in \mathrm{C_b}(X)$ $\int h \mathrm{d}\varkappa_n = \int h f_n \mathrm{d}\mu_n$ (and the same without n). So Theorem 3.1 asserts that the sequence $(\varkappa_n)$ weakly converges to $\varkappa$. Consequently, $|\varkappa|(X) \le \varliminf_{n\to\infty} |\varkappa_n|(X)$. Herein by construction $|\varkappa|(X) = \int |f| \, \mathrm{d}|\mu|$ (and the same with n). □

Proposition 3.3. *Let X be a Polish space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of Borel functions on X locally uniformly converging to a continuous function f and satisfying the condition*

$$\inf_{N>0} \varliminf_{n\to\infty} \int |f_n| I\{|f_n| > N\} \mathrm{d}|\mu_n| < \infty. \tag{23}$$

Then inequality (6) *is valid.*

Proof. We consider, without loss of generality, that all the functions are $\mathbb{R}_+$-valued.

The evident inequality $\left|b_1^{[L]} - b_2^{[L]}\right| \le |b_1 - b_2|$ and the first assumption about (f_n) show that, for an arbitrary $L > 0$, the sequence $\left(f_n^{[L]}, n \in \mathbb{N}\right)$ locally uniformly converges to $f^{[L]}$. By construction it is uniformly bounded. Thus Proposition 3.2 asserts that

$$\int f^{[L]} \mathrm{d}|\mu| \le \varliminf_{n\to\infty} \int f_n^{[L]} \mathrm{d}|\mu_n|.$$

Herein for any positive L and N $f_n^{[L]} \le f_n \le N + f_n I\{f_n > N\}$, so that

$$\varliminf_{n\to\infty} \int f_n^{[L]} \mathrm{d}|\mu_n| \le N|\mu|(X) + \varliminf_{n\to\infty} \int f_n I\{f_n > N\} \mathrm{d}|\mu_n|.$$

Consequently,

$$\int f^{[L]} \mathrm{d}|\mu| \le N|\mu|(X) + \varliminf_{n\to\infty} \int f_n I\{f_n > N\} \mathrm{d}|\mu_n|.$$

By condition (23) the r.h.s. is finite for some N. It remains to note that, by the Beppo Levi theorem, $\int f \mathrm{d}|\mu| = \lim_{N\to\infty} \int f^{[N]} \mathrm{d}|\mu|$. □

Theorem 3.4. *Let X be a Polish space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of Borel functions on X locally uniformly converging to a continuous function f and satisfying condition* (15)*. Then relations* (6) *and* (8) *hold.*

Proof. Noting that condition (15) is stronger than (23), we obtain (6) from Proposition 3.3.

Condition (15) implies, as was already noted, that inequality (4) holds for eventually all n. It entails (5), as well $\left(\text{since } \left|f_n - f_n^{[N]}\right| \le |f_n| I\{|f_n| > N\}\right)$. The first assumption about (f_n) implies, as was shown in the proof of Proposition 3.2, that relation (21) holds for all $\varepsilon > 0$ and compacta $K \subset X$. So all the conditions of Theorem 3.1 are fulfilled and therefore its conclusion is valid. □

Juxtaposing Theorem 3.4 and Corollary 1.10, we obtain

Corollary 3.5. *Let X be a Polish space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of Borel functions on X pointwise converging to a function f and satisfying conditions* (13) *(for all $x \in X$) and* (15)*. Then relations* (6) *and* (8) *hold.*

4 Theorems for functions on a general topological space

In this section, X is a topological space. The first result is similar to Theorem 3.1, but without mentioning a set K.

Theorem 4.1. *Let X be a topological space and (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ. Let, further, (f_n) and f be a sequence of Baire functions on X and a continuous function on X such that conditions* (4) – (6) *are fulfilled and for any $\varepsilon > 0$*

$$\lim_{n\to\infty} |\mu_n|(H_n^\varepsilon) = 0, \tag{24}$$

where H_n^ε is defined by (14)*. Then relation* (8) *holds.*

Proof. Obviously,

$$\int_{X\setminus H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le \varepsilon |\mu_n|(X), \quad \int_{H_n^\varepsilon} \left|f_n^{[N]} - f^{[N]}\right| \mathrm{d}|\mu_n| \le 2N|\mu_n|(H_n^\varepsilon).$$

So condition (24) implies that for any positive N and ε

$$\varlimsup_{n\to\infty}\left|\int\left(f_n^{[N]}-f^{[N]}\right)\mathrm{d}\mu_n\right|\le\varepsilon\sup_n|\mu_n|(X).$$

Hence and from uniform boundedness of (μ_n) in variation asserted by Proposition 1.1 equality (7) follows. Thus all the conditions of Lemma 1.6 are fulfilled and therefore its conclusion is valid. ☐

Theorem 4.2. *Let X be a topological space and (μ_n) be a sequence in $\mathbb{M}(X)$ such that for any $h\in\mathrm{C_b}(X)$ the sequence $\left(\int h\mathrm{d}\mu_n\right)$ converges. Then the relation $\int|g_n|\ \mathrm{d}|\mu_n|\to 0$ holds for every uniformly bounded pointwise converging to zero sequence (g_n) of continuous functions such that all the functions $\sup\limits_{k\ge n}|g_k|,\ n\in\mathbb{N}$, are continuous.*

Proof. Proposition 1.1 asserts existence of a constant C such that

$$\sup_n|\mu_n|(X)\le C. \tag{25}$$

Let us fix $\varepsilon>0$ and denote $h_n=\sup\limits_{k\ge n}|g_k|,\ F_n=\{x: h_n(x)\ge\varepsilon/C\}$. By assumption each h_n is a continuous function, so each F_n is a functionally closed set. The sequence (F_n) decreases, since so does, by construction, (h_n). Obviously,

$$\bigcap_{n=1}^{\infty}F_n=\left\{x:\varlimsup_{n\to\infty}|g_n(x)|\ge\frac{\varepsilon}{C}\right\}.$$

Pointwise convergence of (g_n) to zero implies that the r.h.s. of this equality is the empty set. Thus $F_n\searrow\emptyset$. Hence and from the assumption about (μ_n) we have by Corollary 1.5

$$|\mu_n|(F_n)\to 0. \tag{26}$$

Writing

$$\begin{aligned}\int_{X\setminus F_n\cup F_n}|g_n|\mathrm{d}|\mu_n|&\le\int_{X\setminus F_n}h_n\mathrm{d}|\mu_n|+\|g_n\|_\infty\ |\mu_n|(F_n)\\&\le\varepsilon C^{-1}|\mu_n|(X)+\|g_n\|_\infty\ |\mu_n|(F_n)\end{aligned}$$

and taking to account (25), (26) and uniform boundedness of (g_n), we get $\varlimsup\int|g_n|\mathrm{d}|\mu_n|\le\varepsilon$. ☐

We will say that a set is an *additive lattice* if it is both a commutative group and a lattice with translation-invariant order.

Lemma 4.3. *Let a_1,a_2,b_1,b_2 be arbitrary members of an additive lattice. Then $|a_1\vee b_1-a_2\vee b_2|\le|a_1-a_2|\vee|b_1-b_2|$.*

Proof. It suffices to to prove that $a_1 \vee b_1 - a_2 \vee b_2 \le |a_1 - a_2| \vee |b_1 - b_2|$. To this end we write $a_1 \vee b_1 - a_2 \vee b_2 = (a_1 - (a_2 \vee b_2)) \vee (b_1 - (a_2 \vee b_2))$, $a_1 - (a_2 \vee b_2) \le a_1 - a_2 \le |a_1 - a_2|$, $\quad b_1 - (a_2 \vee b_2) \le b_1 - b_2 \le |b_1 - b_2|$. $\square$

Lemma 4.4. *Let x be a point in X and (φ_k) be a pointwise bounded sequence in $(\mathbb{C}^d)^X$ such that*

$$\lim_{U \in \tau(x)} \sup_k \sup_{x' \in U} |\varphi_k(x') - \varphi_k(x)| = 0. \tag{27}$$

Then the function $f \equiv \sup_k |\varphi_k|$ is continuous at x.

Proof. Denote $f_n = |\varphi_1| \vee \ldots \vee |\varphi_n|$. The sequence (f_n) pointwise converges to f. So, to deduce the desired conclusion from Lemma 1.9 it suffices to verify condition (13).

Writing on the basis of Lemma 4.3

$$|f_n(x') - f_n(x)| \le \bigvee_{k=1}^{n} |\varphi_k(x') - \varphi_k(x)|,$$

we get

$$\sup_{x' \in U} |f_n(x') - f_n(x)| \le \bigvee_{k=1}^{n} \sup_{x' \in U} |\varphi_k(x') - \varphi_k(x)|,$$

whence

$$\lim_{U \in \tau(x)} \sup_n \sup_{x' \in U} |f_n(x') - f_n(x)| = 0. \tag{28}$$

Thus (27) entails (13). $\square$

Theorem 4.5. *Let X be a topological space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ, (f_n) and f be a sequence of functions on X and a function on X satisfying conditions* (4) – (6)*. Suppose also that for any $x \in X$ and $N > 0$*

$$\lim_{U \in \tau(x)} \sup_n \sup_{x' \in U} \left| f_n^{[N]}(x') - f_n^{[N]}(x) \right| = 0 \tag{29}$$

and condition (12) *is fulfilled. Then relation* (8) *holds.*

Proof. Denote $g_n^N = f_n^{[N]} - f^{[N]}$. Condition (12) implies that the sequence $\left(f_n^{[N]}, n \in \mathbb{N} \right)$ pointwise converges to $f^{[N]}$, which together with (29) entails continuity of $f^{[N]}$ and the relation

$$\lim_{U \in \tau(x)} \sup_{k \ge n} \sup_{x' \in U} \left| \, \left| g_k^N(x') \right| - \left| g_k^N(x) \right| \, \right| = 0$$

for all $x \in X$ and $n \in \mathbb{N}$. By construction the sequence $(g_n^N, n \in \mathbb{N})$ is uniformly bounded. So Lemma 4.4 applied to $\varphi_k = \left|g_{k+n-1}^N\right|$ asserts that all the functions $\sup_{k \geq n} \left|g_k^N\right|$, $n \in \mathbb{N}$, are continuous, hereupon Theorem 4.2 asserts that $\int g_n^N \mathrm{d}\mu_n \to 0$, which together with weak convergence of (μ_n) to μ yields

$$\int f_n^{[N]} \mathrm{d}\mu_n \to \int f^{[N]} \mathrm{d}\mu \quad \text{as} \quad n \to \infty.$$

This relation together with condition (5) and relation (9) derived above from condition (6) entails (8). □

The next statement is obvious.

Lemma 4.6. *Relation* (28) *at a point $x \in X$ entails* (29) *(for any $N > 0$) and is tantamount to* (13) *at x plus continuity of all the functions at this point.*

The following three statements ensue from Theorem 4.5, with account of Lemma 4.6, in the same way (with obvious technical changes) as Proposition 3.2, Proposition 3.3 and Theorem 3.4 were derived from Theorem 3.1.

Proposition 4.7. *Let X be a topological space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a signed measure μ and (f_n) be a uniformly bounded sequence of continuous functions on X pointwise converging to a function f and satisfying condition* (13). *Then relations* (8) *and* (22) *hold.*

Proposition 4.8. *Let X be a topological space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of continuous functions on X pointwise converging to a function f and satisfying conditions* (13) *and* (23). *Then inequality* (6) *is valid.*

Theorem 4.9. *Let X be a topological space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of continuous functions on X pointwise converging to a function f and satisfying conditions* (13) *(for all $x \in X$) and* (15). *Then relations* (6) *and* (8) *hold.*

Theorem 4.9 is the analog, for functions on a general topological space, of Corollary 3.5 (with the additional condition of continuity of the pre-limit functions). The next four statements will allow us to modify, for a class of topological spaces, this theorem, making it more alike to Theorem 3.4.

Lemma 4.10. *Let x and (f_n) be a point in X and a sequence in $(\mathbb{C}^d)^X$ such that, firstly, relation* (13) *fails and, secondly,*

$$\bigcap_{U \in \tau(x)} U = \bigcap_{k=1}^{\infty} U_k \tag{30}$$

for some decreasing sequence (U_k) in $\tau(x)$. Then there exists a sequence $(x_n) \in$ cnvr$[x]$ *such that* (10) *fails.*

Proof. Note first that the limit of every subnet of a decreasing net is not less than the limit of the whole net. So the assumptions of the lemma imply existence of a positive number c such that

$$\overline{\lim_{n\to\infty}} \sup_{x'\in U_k} |f_n(x') - f_n(x)| > c$$

for all $k \in \mathbb{N}$. Consequently, there exist a strictly increasing sequence (n_k) of natural numbers and a sequence $(y_k) \in X^{\mathbb{N}}$ such that $y_k \in U_k$ (and therefore $y_k \to x$ by condition (30)) and $|f_{n_k}(y_k) - f_{n_k}(x)| > c$. It remains to note that $y_k = x_{n_k}$, where $x_n = y_k$ as $n_{k-1} < n \le n_k$ $(n_0 = 0)$, so that (x_n) also converges to x. □

Corollary 4.11. *Let X be a first-countable topological space and (f_n) be a sequence in $(\mathbb{C}^d)^X$ such that relation* (10) *holds for every $x \in X$ and $(x_n) \in$* cnvr$[x]$. *Then relation* (13) *holds for all $x \in X$.*

We will say that a topological space is *locally sequentially compact* if each its point has a sequentially precompact neighborhood. The following statement is obvious.

Lemma 4.12. *Let X be a first-countable locally sequentially compact topological space and (f_n) be a sequence in $(\mathbb{C}^d)^X$ locally uniformly converging to a function f. Then relation* (10) *holds for every $x \in X$ and $(x_n) \in$* cnvr$[x]$.

Juxtaposing Lemma 4.12 and Corollary 4.11, we get

Corollary 4.13. *Let X be a first-countable locally sequentially compact topological space and (f_n) be a sequence in $(\mathbb{C}^d)^X$ locally uniformly converging to a function f. Then for any $x \in X$ condition* (13) *is fulfilled.*

Theorem 4.9, Lemma 4.6 and Corollary 4.13 yield

Theorem 4.14. *Let X be a first-countable locally sequentially compact topological space, (μ_n) be a sequence in $\mathbb{M}(X)$ weakly converging to a measure μ and (f_n) be a sequence of continuous functions on X locally uniformly converging to a function f and satisfying condition* (15). *Then relations* (6) *and* (8) *hold.*

5 Sequential continuity of direct multiplication of signed measures

The following byproduct of Theorem 4.9 is interesting on its own right.

Theorem 5.1. *Let X and Y be topological spaces, (μ_n) and (ν_n) be sequences in $\mathbb{M}(X)$ and $\mathbb{M}(Y)$, respectively, weakly converging to measures μ and ν, respectively. Suppose also that for any positive ε there exists a compact set $K \in \mathcal{B}_0(Y)$ such that $\nu_n(Y \setminus K) < \varepsilon$ for all n. Then the sequence $(\mu_n \otimes \nu_n)$ weakly converges to $\mu \otimes \nu$.*

Proof. Let us fix $g \in \mathrm{C_b}(X \times Y)$ and denote

$$f_n(x) = \int g(x,y)\nu_n(\mathrm{d}y), \quad f(x) = \int g(x,y)\nu(\mathrm{d}y).$$

By Fubini's theorem $\iint g\mathrm{d}\mu_n\mathrm{d}\nu_n = \int f_n\mathrm{d}\mu_n$ and the same without n, so all we need is to establish relation (8).

By the choice of g and due to weak convergence of (ν_n) to ν the sequence (f_n) pointwise converges to f. Herein by construction $|f_n| \le M_n \equiv \|g\|_\infty|\nu_n|(Y)$ and therefore

$$|f_n|I\{|f_n| > N\} \le N^{-1}M_n^2, \quad \int |f_n|I\{|f_n| > N\}\mathrm{d}|\mu_n| \le N^{-1}M_n^2|\mu_n|(X).$$

These inequalities together with asserted by Proposition 1.1 uniform boundedness in variation of (μ_n) and (ν_n) entail (15).

By construction for any $B \in \mathcal{B}_0(Y)$

$$|f_n(x') - f_n(x)| \le 2\|g\|_\infty|\nu_n|(Y \setminus B) + |\nu_n|(Y)\sup_{y\in B}|g(x',y) - g(x,y)|.$$

So, to deduce (13) from the last assumption of the theorem and uniform boundedness in variation of (ν_n) it suffices to show that

$$\lim_{U\in\tau(x)}\sup_{x'\in U}\max_{y\in K}|g(x',y) - g(x,y)| = 0 \tag{31}$$

for every $x \in X$ and compact set $K \subset Y$.

Assume the contrary: there exist a point $x \in X$, a compact set $K \subset Y$, a positive number a and, for each $U \in \tau(x)$, points $x(U) \in U$ and $y(U) \in K$ such that

$$|g(x(U), y(U)) - g(x, y(U))| > a. \tag{32}$$

By the construction of the net $U \mapsto x(U)$

$$x(U) \to x \quad \text{as} \quad U \quad \text{runs through} \quad \tau(x). \tag{33}$$

Compactness of K implies existence of a cofinal subset $\mathcal{U} \subset \tau(x)$ and a point $y \in K$ such that

$$y(U) \to y \quad \text{as} \quad U \quad \text{runs through} \quad \mathcal{U}. \tag{34}$$

From (33) and cofinality of $\mathcal{U}$ we have $x(U) \to x$ as U runs through $\mathcal{U}$, which together with the previous relation and continuity of g yields $g(x(U), y(U)) \to g(x, y)$ as U runs through $\mathcal{U}$. Comparing this with (32), we get $|g(x, y(U)) - g(x, y)| \geq a$ for all $U \in \mathcal{U}$, which in view of (34) contradicts to continuity of g. Thus we have proved relation (31). Now Theorem 4.9 whose conditions we have verified asserts (8). $\square$

Corollary 5.2 (from Theorems 5.1 and 1.2)**.** *Let $X_1, \ldots, X_l$ be Polish spaces and let, for each $i \in \{1, \ldots, l\}$, $(\mu_n^i, n \in \mathbb{N})$ be a sequence of signed measures on $\mathcal{B}(X_i)$ weakly converging to a signed measure μ^i. Then the sequence $(\mu_n^1 \otimes \ldots \otimes \mu_n^l, n \in \mathbb{N})$ weakly converges to $\mu^1 \otimes \ldots \otimes \mu^l$.*

6 Passage to the limit in $\int \mathrm{d}\nu_n(s) \int f_n(s,x)\psi_n(s, \mathrm{d}x)$

Let $(X, \mathcal{X})$ be a measurable space (it will be implied tacitly that $\mathcal{X} = \mathcal{B}_0(X)$ in case X is a topological space). For a mapping $\psi : T \times \mathcal{X} \to \mathbb{C}$, where T is a nonvoid set, and for a point $s \in T$, we denote $\Psi(s) = \psi(s, \cdot)$, which will be not explained repeatedly. If X is a topological space, then, equipping $\mathbb{M}(X)$ with the weak convergence, we convert the latter into a convergence space. If herein T is a convergence space, then we substitute, for $\mathbb{M}(X)$-valued functions on T, the term "firm convergence" introduced in Section 1 by the more minute one "firm weak convergence".

The following statement is immediate from Corollary 5.2

Corollary 6.1. *Let $X_1, \ldots, X_l$ be Polish spaces and let, for each $i \in \{1, \ldots, l\}$, $(\Psi_n^i, n \in \mathbb{N})$ be a sequence of $\mathbb{M}(X_i)$-valued functions on some set (the same for all i) firmly weakly converging to some $\mathbb{M}(X_i)$-valued function Ψ^i. Then the sequence $(\Psi_n^1 \otimes \ldots \otimes \Psi_n^l, n \in \mathbb{N})$ weakly converges to $\Psi^1 \otimes \ldots \otimes \Psi^l$.*

Lemma 6.2. *Let X be a Polish space, T be a nonvoid set, s be a point in T, (s_n) be a sequence in T and $\Psi, \Psi_1, \Psi_2 \ldots$ be $\mathbb{M}(X)$-valued functions on T such that the sequence $(\Psi_n(s_n))$ weakly converges to $\Psi(s)$. Then the relation*

$$\int g(s_n, x)\psi_n(s_n, \mathrm{d}x) \to \int g(s, x)\psi(s, \mathrm{d}x) \tag{35}$$

holds for every bounded function g on $T \times X$ such that: $g(s_n, x_n) \to g(s, x)$ for any $x \in X$ and $(x_n) \in \mathrm{cnvr}[x]$; $g(s, \cdot) \in \mathrm{C}(X)$.

Proof. The first assumption about g implies, obviously, that

$$\lim_{n\to\infty} \sup_{x \in K} |g(s_n, x) - g(s, x)| = 0$$

for any sequentially precompact set $K \subset X$. Now, relation (35) follows from the second assumption about g and the assumption about (Ψ_n) by Proposition 3.2. $\square$

Referring in the proof of Lemma 6.2 to Theorem 4.14 instead of Proposition 3.2, we modify that lemma as follows.

Lemma 6.3. *Let X be a first-countable locally sequentially compact topological space, T be a nonvoid set, s be a point in T, (s_n) be a sequence in T and $\Psi, \Psi_1, \Psi_2 \ldots$ be $\mathbb{M}(X)$-valued functions on T such that the sequence $(\Psi_n(s_n))$ weakly converges to $\Psi(s)$. Then relation* (35) *holds for every bounded function g on $T \times X$ such that: $g(s_n, x_n) \to g(s, x)$ for any $x \in X$ and $(x_n) \in \mathrm{cnvr}[x]$; $g(s_n, \cdot) \in \mathrm{C}(X)$, $n \in \mathbb{N}$.*

Corollary 6.4. *Let X be either a Polish space or a first-countable locally sequentially compact topological space, T be a convergence space and (Ψ_n) be a sequence in $\mathbb{M}(X)^T$ firmly weakly converging to some $\Psi \in \mathbb{M}(X)^T$. Then the relation*

$$\sup_{s \in Q} \left| \int g(s,x)\psi_n(s, \mathrm{d}x) - \int g(s,x)\psi(s, \mathrm{d}x) \right| \to 0$$

holds for every sequentially precompact set $Q \subset T$ and bounded sequentially continuous function g on $T \times X$ such that for any $s \in T \quad g(s, \cdot) \in \mathrm{C}(X)$.

Proof. For any $x \in X, s \in T$ and $(s_n) \in \mathrm{cnvr}[s]$ we have $g(s_n, x) \to g(s, x)$ due to sequential continuity of g in the first argument. Hence and from boundedness of g we get by the dominated convergence theorem $\int g(s_n, x)\psi(s, \mathrm{d}x) \to \int g(s, x)\psi(s, \mathrm{d}x)$, which together with relation (35) asserted by Lemma 6.2 (if X is a Polish space) or by Lemma 6.3 yields

$$\int g(s_n, x)\psi_n(s_n, \mathrm{d}x) - \int g(s_n, x)\psi(s, \mathrm{d}x) \to 0.$$

And this is, since $s \in T$ and $(s_n) \in \mathrm{cnvr}[s]$ are arbitrary, tantamount to the conclusion of the corollary. □

Let $(T, \mathcal{T})$ and $(X, \mathcal{X})$ be measurable spaces (it will be implied tacitly that $\mathcal{T} = \mathcal{B}_0(T)$ in case T is a topological space). A mapping $\Psi : T \to \mathbb{M}(X)$ such that for each $h \in \mathrm{L}_\infty(X, \mathcal{X})$ the function $\int h(x)\psi(\cdot, \mathrm{d}x)$ is $\mathcal{T}$-measurable will be called a *transition measure* on $T \times \mathcal{X}$. Obviously, for any transition measure Ψ on $T \times \mathcal{X}$ and function $g \in \mathrm{L}_\infty(T \times X, \mathcal{T} \otimes \mathcal{X})$ the function $\int g(\cdot, x)\psi(\cdot, \mathrm{d}x)$ is $\mathcal{T}$-measurable, too.

Lemma 6.5. *Let X be either a Polish space or a first-countable locally sequentially compact topological space, T be a Polish space, ν_n be a sequence in $\mathbb{M}(T)$ weakly converging to some $\nu \in \mathbb{M}(T)$, (Ψ_n) be a sequence of transition measures on $T \times \mathcal{B}(X)$ firmly weakly converging to some transition measure Ψ and g be a bounded continuous function on $T \times X$. Suppose also that*

$$\sup_n \sup_{s \in T} \left| \int g(s,x)\psi_n(s, \mathrm{d}x) \right| < \infty \tag{36}$$

and the function $\int g(\cdot,x)\psi(\cdot,\mathrm{d}x)$ *is continuous. Then*

$$\int \nu_n(\mathrm{d}s)\int g(s,x)\psi_n(s,\mathrm{d}x) \to \int \nu(\mathrm{d}s)\int g(s,x)\psi(s,\mathrm{d}x). \tag{37}$$

Proof. Denote $h_n(s) = \int g(s,x)\psi_n(s,\mathrm{d}x)$, $h(s) = \int g(s,x)\psi(s,\mathrm{d}x)$. Boundedness of g and condition (36) imply uniform boundedness of the sequence (h_n). Corollary 6.4 asserts local uniform convergence of (h_n) to h. This function was assumed continuous. So (37) is, up to notation, the conclusion of Proposition 3.2. □

Lemma 6.6. *Let X be either a Polish space or a first-countable locally sequentially compact topological space, T be a sequentially compact convergence space and (Ψ_n) be a sequence in $\mathbb{M}(X)^T$ firmly weakly converging to some $\Psi \in \mathbb{M}(X)^T$. Then inequality (36) is valid for every bounded sequentially continuous function g on $T \times X$ such that for any $s \in T \quad g(s,\cdot) \in \mathrm{C}(X)$.*

Proof. If (36) is wrong, then, due to sequential compactness of T, there exist $g \in \mathrm{C_b}(T \times X)$, an infinite set $J \subset \mathbb{N}$ and a convergent sequence $(s_n, n \in J)$ in T such that $\left|\int g(s_n,x)\psi_n(s_n,\mathrm{d}x)\right| \to \infty$ as $n \to \infty$, $n \in J$. And this contradicts to relation (35) asserted by Lemma 6.2 (if X is a Polish space) or by Lemma 6.3. □

Lemmas 6.5 and 6.6 yield together

Corollary 6.7. *Let X be either a Polish space or a first-countable locally sequentially compact topological space, T be a compact metrizable topological space, ν_n be a sequence in $\mathbb{M}(T)$ weakly converging to some $\nu \in \mathbb{M}(T)$ and (Ψ_n) be a sequence of transition measures on $T \times \mathcal{B}_0(X)$ firmly weakly converging to a transition measure Ψ. Then relation (37) holds for every bounded continuous function g on $T \times X$ such that the function $\int g(\cdot,x)\psi(\cdot,\mathrm{d}x)$ is continuous.*

Theorem 6.8. *Assume the following: X is a Polish space, T is a compact metrizable topological space; (ν_n) is a sequence of measures on $\mathcal{B}(T)$ weakly converging to a measure ν; (Ψ_n) is a sequence of transition measures on $T \times \mathcal{B}(X)$ firmly weakly converging to a transition measure Ψ such that for any $g \in \mathrm{C_b}(T \times X)$ the function $\int g(\cdot,x)\psi(\cdot,\mathrm{d}x)$ is continuous; (f_n) is a sequence of Borel functions on $T \times X$ such that*

$$\lim_{N\to\infty}\varlimsup_{n\to\infty}\int |\nu_n|(\mathrm{d}s)\int |f_n(s,x)|I\{|f_n(s,x)|>N\}\;|\psi_n|(s,\mathrm{d}x) = 0; \tag{38}$$

f is a continuous function on $T \times X$ such that for any compactum $K \subset X$

$$\lim_{n\to\infty}\sup_{(s,x)\in T\times K}|f_n(s,x)-f(s,x)| = 0. \tag{39}$$

Then

$$\int |\nu|(\mathrm{d}s) \int |f(s,x)|\ |\psi|(s,\mathrm{d}x) < \infty \tag{40}$$

and

$$\int \nu_n(\mathrm{d}s) \int f_n(s,x)\psi_n(s,\mathrm{d}x) \to \int \nu(\mathrm{d}s) \int f(s,x)\psi(s,\mathrm{d}x). \tag{41}$$

Proof. Let us define the measures μ_n and μ on $\mathcal{B}(T) \otimes \mathcal{B}(X)$ by $\mu_n(S \times A) = \int_S \nu_n(\mathrm{d}s)\psi_n(s, A)$ (likewise without n), so that for every bounded Borel function g on $T \times X$

$$\int g(y)\mu_n(\mathrm{d}y) = \int \nu_n(\mathrm{d}s) \int g(s,x)\psi_n(s,\mathrm{d}x) \qquad (y = (s,x)).$$

Corollary 6.7 asserts weak convergence of (μ_n) to μ.

Relations (15), (6) and (8) for thus defined μ_n's turn into (38), (40) and (41), respectively. Condition (39) amounts to local uniform convergence of (f_n) to f. So (41) is asserted by Theorem 3.4, if one substitutes in it x by (s,x) and X by $T \times X$. □

Remark 6.9. *Referring to Theorem 4.14 instead of Theorem 3.4, we obtain the modification of Theorem 6.8, where X is a first-countable locally sequentially compact topological space, K in* (39) *is a sequentially precompact set and all the functions f_n are assumed continuous.*

Denote LBV the class of all real-valued right-continuous functions on $\mathbb{R}_+$ starting from zero and having finite variation in each interval. Every $F \in$ LBV uniquely determines the signed measure μ on $\mathcal{B}(\mathbb{R}_+)$ whose value on $[0,\, s]$ equals $F(s)$. Let $\mathrm{cont}(F)$ denote the set of continuity points of F. We will say that a sequence (F_n) in LBV *basically converges* to $F \in$ LBV if, firstly, $F_n(s) \to F(s)$ at every point $s \in \mathrm{cont}(F)$ and, secondly, $\sup\limits_n \int_0^t |\mathrm{d}F_n(s)| < \infty$ for all $t \in \mathbb{R}_+$. By Proposition 8.1.8 [2] these properties of (F_n) imply that for any $t \in \mathrm{cont}(F)$ the sequence of the corresponding signed measures on $\mathcal{B}([0,\, t])$ (not on $\mathcal{B}(\mathbb{R}_+)$!) weakly converges to μ.

The following statement is immediate from Theorem 6.8 and Corollary 6.1.

Theorem 6.10. *Let X be a Polish space, E be a topological vector space, $\varphi : X \to \mathbb{C}$, $\vartheta : X \to E'$ and $\chi : X^2 \to E'$ be continuous (w.r.t. the vague topology in E') mappings. Let, further, for each $n \in \mathbb{N}$ a transition measure Ψ_n on $\mathbb{R}_+ \times \mathcal{B}(X)$ satisfy the equation*

$$\begin{aligned}\int \varphi(x)\psi_n(t,\mathrm{d}x) = \int \varphi(x)\psi_n(0,\mathrm{d}x) + \int_0^t \mathrm{d}F_n(s) \int \vartheta(x)a_n(s,x)\psi_n(s,\mathrm{d}x)\\ + \int_0^t \mathrm{d}G_n(s) \iint \chi(x,y)b_n(s,x,y)\psi_n(s,\mathrm{d}x)\psi_n(s,\mathrm{d}y),\end{aligned} \tag{42}$$

where $F_n \in \mathrm{LBV}$, $G_n \in \mathrm{LBV}$, $a_n : \mathbb{R}_+ \times X \to E$ *is a* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(X)|\mathcal{B}_0(E)$-*measurable mapping and* $b_n : \mathbb{R}_+ \times X^2 \to E$ *is a* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(X) \otimes \mathcal{B}(X)|\mathcal{B}_0(E)$-*measurable mapping. Suppose that there exist a transition measure* Ψ *on* $\mathbb{R}_+ \times \mathcal{B}(X)$, *continuous functions* $F \in \mathrm{LBV}$, $G \in \mathrm{LBV}$ *and continuous mappings* $a : \mathbb{R}_+ \times X \to E$, $b : \mathbb{R}_+ \times X^2 \to E$ *such that: the sequence* (Ψ_n) *firmly weakly converges to* Ψ; *the sequences* (F_n) *and* (G_n) *basically converge to* F *and* G, *respectively; for any* $t > 0$ *and compactum* $K \subset X$

$$\lim_{n\to\infty} \sup_{s\le t,\, x\in K} |\vartheta(x)a_n(s,x) - \vartheta(x)a(s,x)| = 0, \tag{43}$$

$$\lim_{n\to\infty} \sup_{\substack{s\le t \\ x,\,y\in K}} |\chi(x,y)b_n(s,x,y) - \chi(x,y)b(s,x,y)| = 0. \tag{44}$$

Then Ψ *satisfies the equation*

$$\int \varphi(x)\psi(t,\mathrm{d}x) = \int \varphi(x)\psi(0,\mathrm{d}x) + \int_0^t \mathrm{d}F(s) \int \vartheta(x)a(s,x)\psi(s,\mathrm{d}x) + \int_0^t \mathrm{d}G(s) \iint \chi(x,y)b(s,x,y)\psi(s,\mathrm{d}x)\psi(s,\mathrm{d}y). \tag{45}$$

Remark 6.11. *If the topology in* E *is induced by some metric* ρ, *then, obviously, equalities*

$$\lim_{n\to\infty} \sup_{s\le t,\, x\in K} \rho(a_n(s,x), a(s,x)) = 0, \qquad \lim_{n\to\infty} \sup_{\substack{s\le t \\ x,\,y\in K}} \rho(b_n(s,x,y) - b(s,x,y)) = 0$$

imply (43) *and* (44).

To make use of Theorem 6.10 one has to prove relative compactness of (Ψ_n) w.r.t. the firm weak convergence and to show that the class of triplets $(\varphi, \vartheta, \chi)$ such that equality (42) (and therefore (45)) holds for all t is wide enough to ensure uniqueness of the solution of equation (45) (more exactly, of the family, parametrized by φ, ϑ and χ, of these equations). Both tasks are quite feasible, but they require another ample article, with its own concepts and techniques.

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