Mathematica Aeterna, Vol. 2, 2012, no. 9, 827 - 831

An Introduction on Weakly TR-Contraction in Metric Spaces

Z. Bahmani

Islamic Azad University of Genaveh-Branch , Iran

e-mail: Bahmani.math@gmail.com

Abstract

In this paper , we introduce the concepts of weakly TR-contraction in metric spaces . Then we prove some results.

Keywords : Weakly TR-contraction , Coincide point , Sequentially contraction .

2010 MSC : 47H09 , 47H10 ,54E40 , 54E50 .

1 Introduction

Contractions are one of the important class of mappings .Many of authors have studied fixed, periodic and coincide points of them . Recently , A . Beiranvand , S . Moradi , M . omid and H . Pazandeh [1] introduced a new class of contractive mappings :T-contraction and T-contractive extending the Banach's contraction principle and the Edeistein's fixed point theorems $(see\ [2])$, respectively . Authors in [3] considered various extensions of classic contraction type of mappings [more specifically : Kannan , Zanfirescu , weak contraction and also the so-called $D(a\ ,b)$ class] . For these classes of contractions , conditions for existence and uniqueness of fixed points , as well for its asymptotic behavior is given [1,4,5] . The goal of this paper is to introduce the new class of contractions in metric spaces .

First , we have the following definitions and results .

Definition 1 .1 . Let (X,d) be a metric space . A map said to be contraction when there exists $0 \leq \alpha < 1$ so that

$$d(Sx, Sy) \leq \alpha d(x,y)$$

Theorem 1 . 2(Banach's contraction principle) [6] . Let (X,d) be a complete space and $T: X \to X$ a contraction . Then T has an unique fixed point.

Definition 1 . 3 [7] . Let (X,d) be a metric space and $S,T,R: X \to X$ three functions. A mapping S is said to be a TR-contraction if there is $\alpha \in [0,1)$ constant such that

$$d(TSx, RSy) \leq \alpha d(Tx, Ty)$$

For all $x, y \in X$.

Example 1 .4 . Let $X = R$ with usual metric . We consider the functions $S,T,R: X \to X$ defined by $Tx = \frac{x}{2}, Rx = \frac{x}{8}$ and $Sx = \frac{x}{4}$. If $\alpha \in \left[\frac{1}{4},1\right)$, then S is a TR-contraction.

Notation 1 .5 . In this paper , the space of all linear bounded mapping on a normed space X is denoted by $BL(X)$. It is a normed space with the following norm :

$$\|T\| = \{sup\|Tx\|: x \in X, \|x\| = 1\}$$

2 Main Results

Throughout this section, X denotes a metric space.

Definition 2 .1. Let (X,d) be a metric space and $S,T,R: X \to X$ three functions. A mapping S is said to be a weakly TR-contraction if there are $\alpha, \beta \in [0,1)$ with $0 \leq \alpha + \beta < 1$ such that

$$d(TSx, RSy) \leq \alpha d(x,y) + \beta d(Tx, Ry)$$

For all.

Remark 2 .2 . Clearly , any TR-contraction is weakly TR-contraction but the converse need not be hold; for example , consider $X = [0, \infty)$ with usual metric and $S,T,R : X \to X$ by $Tx = x^2$, $Rx = x$ and $Sx = \sqrt{x}$.

Set $x = \frac{1}{2}$ and $y = 0$, then we have

$$|x - \sqrt{y}| = d(TSx\, , RSy) \xRightarrow{x=\frac{1}{2}, y=0} d\left(TS\left(\frac{1}{2}\right), RS(0)\right) = \frac{1}{2}$$

And

$$|x^2 - \sqrt{y}| = d(Tx\, , Ry) \xRightarrow{x=\frac{1}{2}, y=0} d\left(T\left(\frac{1}{2}\right), R(0)\right) = \frac{1}{4}$$

Which $\left(TS\left(\frac{1}{2}\right), RS(0)\right) > d\left(T\left(\frac{1}{2}\right), R(0)\right)$. Hence S isnt a TR-contraction. It is easy to show that S is a weakly TR-contraction.

Theorem 2 .3 . Let $S : X \to X$ be a weakly TR-contraction. Also, suppose that X be complete. Further, Let be a contraction. Then T and R have a coincide point.

Proof . By Banach's contraction principle , has an unique fixed point , say x^* .

Now, set $x = y = x^*$ in [2 . 1 . 1] . Then we have

$$d(TSx^*\, , RSx^*) \le \alpha d(x^*\, , x^*) + \beta d(Tx^*\, , Rx^*)$$

Which implies . Since $0 \le \beta < 1$, so $d(Tx^*\, , Rx^*) = 0$ which means $x^* = Rx^*$. This completes the proof. □

Definition 2 .4 . Let $S,T : X \to X$. We say that is T-contraction if there is a $\alpha \in [0,1)$ such that

$$d(TSx\, , TSy) \le \alpha d(Tx\, , Ty)$$

For all $x, y \in X$.

Remark 2 .5 . Indeed, the previous definition is the concept of a TR-contraction in which $R = T$.

Definition 2 .6 .[8] Let $S : X \to X$ be a mapping . We say that S is sequentially contraction , if there exist a sequence as $(T_n) : X \to X$ such that S be a T_n – contraction for each n .

Example 2 .7. Consider $X = [1, \infty)$ with usual metric. Also, let $S, T_n : X \to X$ by $Sx = \sqrt{x}$ and $T_n x = \frac{x^2}{n}$ for all $n \in \mathbf{N}$. Then

$$d(T_n Sx , T_n Sy) = \frac{1}{n}|x - y|$$

And

$$d(T_n x , T_n y) = \frac{1}{n}|x - y||x + y|$$

So ,S is sequentially contraction .

Finally we prove the following result.

Theorem 2 .8. Let X be a normed space and $S : X \to X$ a mapping. Suppose that there exists $T \in BL(X)$ such that S be a T –contraction. Then S is sequentially contraction.

Proof. $BL(X)$ is closed with sup-norm . Hence there exists a sequence, say (T_n), such that $T_n \to T$ as $n \to \infty$. On the other hand, S is T-contraction, i .e,

$$d(TSx , TSy) \leq \alpha d(Tx , Ty)$$

For some $0 \leq \alpha < 1$. We have

$$d\left(\lim_{n\to\infty} T_n Sx , \lim_{n\to\infty} [T_n Sy) \leq \alpha d\left(\lim_{n\to\infty} [T_n x . T_n y)\right]\right]$$

So

$$\lim_{n\to\infty} d(T_n Sx , T_n Sy) \leq \alpha \lim_{n\to\infty} d(T_n x , T_n y)$$

Therefore , We can choose $N \in \mathbf{N}$ so that

$$d(T_n Sx , T_n Sy) \leq \alpha d(T_n x , T_n y)$$

For all .Set $\tilde{T}_n = T_{n+(N-1)}$ for each . Obviously , is $\tilde{T}_n$-contraction for all n , as desired . □

References

[1] A . Beiranvand , S . Moradi , M . Omid and H . Pazandeh , Two fixed point theorem for special mapping , arXiv :0903 . 1504V1 [math.FA] .

[2] K . Goebel and W . A . Kirk , Topics in Metric fixed point theory , Cambringe University Press , NY , 1990 .

[3] J . R .Morales and E . Rojas ,Fixed point theorems for a class of mappings depending of another function and defined on cone metric spaces , arXiv : 0906.2106V1 [math.FA].

[4] J . R .Morales and E . Rojas , T-Zamfirescu and T-weak contraction mapping on cone metric spaces , arXiv :0909.1255V1 [math.FA] .

[5] J . R .Morales and E . Rojas , Cone metric spaces and fixed point theorems of T-Kannan contractive mappings , Int . Jounal of Math . Analysis , Vol 4.4 , (2010) , 175-184 .

[6] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons,Canada,1989.

[7] J . R .Morales and E . Rojas ,on the existence of fixed points of contraction mappings depending of two functions on cone metric spaces , arXiv : 0910.4921V1[math.FA] .

[8] Z. Bahmani, Sequentially Contractions in Cone Metric Spaces, Math. Aeterna,Vol.2, N0.08,(2012),715-722.

Received: November, 2012